Properly Improper

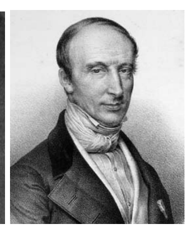
$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Agnesi

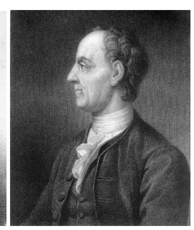
Serret

Dirichlet

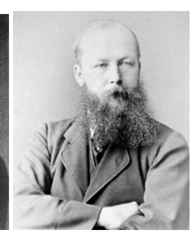
Gauss



Cauchy

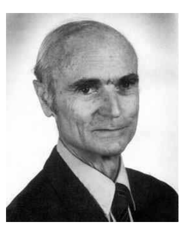


Euler

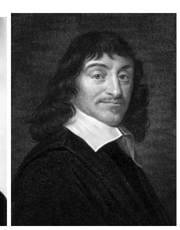


Wolstenholm

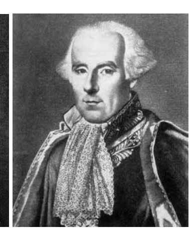
Leibniz



Coxeter



Descartes



Laplace

By

Donald A. Cole

“What’s one and one and one and one and one and one and one and one and one and one and one and one?” “I don’t know,” said Alice, “I lost count.” “She can’t do addition,” said the Red Queen.

—Lewis Carroll

PREFACE

As a College Freshman Calculus student (many long years ago), I found myself absolutely dumbfounded when in some book containing a table of definite integrals I saw the following:

$$\boxed{\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}}$$

At that time, I was knowledgeable enough to know that the function in the integrand of that integral did not have an anti-derivative — so how could someone calculate the value of that integral? Well, I pursued the answer to that question as best I could and when I finally stumbled upon a solution, I was hooked – I've been a fan of definite integrals that do not integrate in the conventional sense ever since. I subsequently learned that I'm evidently in good company. G.H. Hardy (1877-1947), the greatest English mathematician of the first half of the 20th century was quoted as saying, "I could never resist an integral"; and, his reputation for doing non-conventional integration (the kind of integrals that I'm a fan of) was reputed to be phenomenal. In this regard, Hardy brought Srinivasa Ramanujan (the genius and self-taught Indian mathematician) to Cambridge all the way from India based on a letter that Ramanujan sent to Hardy in January of 1913. Attached to that letter were about 120 theorems, many of which involved the solution of definite integrals that completely astounded Hardy. Up until this time, Ramanujan was unknown by the mathematical community. Here is an example of just one of the integrals that was included in the letter received by Hardy:

$$\int_0^\infty \prod_{k=0}^{\infty}\left[\frac{1+\left(\frac{x}{b+k+1}\right)^2}{1+\left(\frac{x}{a+k}\right)^2}\right]dx = \frac{1}{2}\pi^{1/2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}.$$

What do we mean by non-conventional integration? To answer that question, one has to agree on what we mean by conventional integration. We learn in elementary calculus that a definite integral $I = \int_a^b f(x)dx = F(b) - F(a)$ where $F'(x) = f(x)$. In-other-words, to solve for the value of *I*, one is taught to find the anti-derivative of $f(x)$ and then evaluate that anti-derivative at both limits of integration and then compute their difference. However, the value of the definite integral between certain specific limits can sometimes be obtained by some other techniques, even in cases when the integrand does not have an anti-derivative. That is, it is sometimes possible to arrive at the value of $I = F(b) - F(a)$ without finding the form of $F(x)$ at all. Such a case was that of $I = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ mentioned at the beginning of this preface. So, this is what is meant by non-conventional integration—finding the value of $F(b) - F(a)$ without determining $F(x)$. It sounds like a really good trick if one can do it. Well, in many cases, it can be done and the techniques for doing so are numerous and often of great cleverness and ingenuity. It is not possible to give an exhaustive list of such techniques, but those used most often are compiled in a table in Chapter 1 (see Table 4 of Properties/Techniques for Evaluation of Integrals).

So that's what this book is about, the evaluation of definite integrals that do not integrate in the conventional sense but whose value can be determined. Actually, it's not the value of the integral that matters so much, but rather the method used to obtain that value. Most of the definite integrals we shall examine are improper (but not all of them). An improper integral is a definite integral that has either or both limits of integration that are infinite or an integrand that approaches infinity at one or more points within its range of integration. Improper integrals that **converge** and whose value can be determined exactly are termed "Properly Improper," and I've always thought that term would make a good title for a book (a subtitle might be "A Mathematical Oxymoron"). I also think that it is a good subject for a book because it is a subject that contemporary professional mathematicians do not seem to be overly concerned about. For some unknown reason, the methods for evaluating these properly improper integrals do not seem to be taught in most math curriculums anymore. The subject is difficult to teach because there is no theory or systematic methodology for evaluating properly improper integrals. Generally, each such integral is a new challenge or puzzle and as such requires techniques of creative manipulation for a solution; no two such techniques being necessarily similar (with one or two exceptions that we shall study). The cleverness and creativity of the solution is what I find ingenious, interesting, and very intriguing (albeit, in my opinion, when the integral can be equated to a value, that expression often gives the integral a rather exotic or mysterious appearance and that in itself I find fascinating). I also think that such solutions should be documented so that they do not become a lost art. So, in a "nutshell" that's what this book deals with, if you enjoy integrals and enjoy computation, this book is for you!

I emphasized the word converge in the previous paragraph because the convergence of the integrals that we are going to be dealing with is very important. Apply some of the methods detailed in this book to an integral that does not converge (i.e., diverges) and get a finite answer—and your answer would be wrong! However, this book is not about convergence of integrals—that's a subject for another book. Hopefully, I have not included any divergent integrals here, but if I have, I apologize ahead of time; keep in mind, the methodologies of manipulating the integrals is being stressed in this book and is much more important and interesting than their ultimate value. Improper integrals whose interval of integration includes a singularity of the function in the integrand can be thought of as analogous to an astronomer's use of the word "singularity," i.e., black hole. Hopefully, none of us, as we deal with properly improper integrals, will fall into the "Mathematical Black Hole."

I would like to say a word or two about how I've attempted to organize this book and it hasn't been an easy organization. I have endeavored to base each chapter on a specific elementary definite integral property or technique with examples of how that property/technique is used to evaluate the integral. Since more than one technique is often involved in the evaluation of any given integral, the chapters tend to intersect with one another. Chapter 1 is an exception as it deals with the notation used, contains a list of recognizable (elementary) integral forms, and contains a list of the integral properties and techniques alluded to above. The last chapter is also an exception. It is the detailed solution of properly improper integrals whose solution I consider to be the *crème de la crème* of methodologies/derivations. In my opinion, the solutions are so clever that every time I study one, I find myself wishing that I had been clever enough to have discovered it. Further, I've included an Appendix. The appendix contains material that really doesn't fit in with the other chapters, although the material is derived from the integral that opens this preface and therefore seems appropriate for this book.

Additionally, I've tried to include historical vignettes or tidbits of information about the men and women of science/mathematics who are responsible for the creative solutions to the integrals we will be encountering or whose methodology has been responsible for the solution (these are the mathematicians pictured on the title page). I've always maintained that mathematics would be a much more popular subject to a greater number of people if the history of math were taught along with the math itself. Mathematics through the centuries has been densely populated with crazy stories, zany geniuses, and clever anecdotes. Further, in some of the derivations, I've attempted to identify the so-called "aha" moment (that moment of sudden realization, inspiration, insight, recognition, or comprehension) that the mathematician responsible for the derivation (or proof) must have had. Of course, it's just my opinion, but in some cases, it's pretty obvious. In some cases, it's my "aha" moment—the moment I realized where the problem was going to take me. Why am I including this sort of revelation in this book? By revealing what I, a professional mathematician, think while I'm working on a problem seems to me to be important—maybe an insight for the student (maybe not). So, in an effort to make this a text book that teaches better and makes the subject of computational mathematics come alive with humor, a little bit of "pizzazz", and convey the "spirit of mathematics", I have done the following whenever I sit down to write or work problems for this book: I have imagined that sitting next to me is a colleague who is interested in learning what I am doing. I try to explain to this colleague what I am doing and why, and this explanation is what I write—hopefully giving the book a one-on-one tutorial aspect. As I said before, "what a professional mathematician thinks as the problems are worked" (or at least, what one professional mathematician thinks).

Finally, I would like to dedicate this book to three important women, Cynthia Richey, Debra Mairs, and Julia Mitchell; to Cindy, Debbie, and Julie, daughters of my heart.

Don Cole, June 2015

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction.

—Tolstoy

Chapter 1. Preliminaries

Notation To Be Used

$\mathbb{N} =$ The set of integers

$\mathbb{N}^+$ = the set of positive integers

$\mathbb{R} =$ the set of real numbers

$\mathbb{R}^+ =$ the set of positive real numbers

$\log(x) =$ the natural logarithm

$\log^2(x) = [\log(x)]^2$

$\epsilon =$ is a member of

I_n = general notation for a properly improper integral (subscripts distinguish one from another).

$(a, b) \to (c, d)$ = the integration interval (a, b) is transformed to (c, d).

$\Rightarrow$ = implies

$(2n - 1)!! = 1{\cdot}3{\cdot}5{\cdot}7{\cdot} \cdots (2n - 1)$, where $n \epsilon \mathbb{N}^+$

Recognizable Forms (Elementary Integral Table)

Since the creative manipulation of a properly improper integral will hopefully lead one to the recognizable form of an elementary integral for the eventual solution of the improper integral, the following table of recognizable forms is included below. Unless otherwise noted, a is considered to be a constant and, $a \epsilon \mathbb{R}$.

Table 1: Recognizable Forms/Elementary Integrals	
1. $\int a \cdot dx = ax$	8. $\int \cos(ax)dx = \frac{1}{a}\sin(ax)$
2. $\int a \cdot f(x)\, dx = a\int f(x)\, dx$	9. $\int \tan(ax)dx = -\frac{1}{a}\log[\cos(ax)]$
3. $\int x^n dx = \frac{x^{n+1}}{n+1},\ n \epsilon \mathbb{N}, n \neq -1$	10. $\int \sec(ax)dx = \frac{1}{a}\log[\sec(ax) + \tan(ax)]$
4. $\int \frac{f'(x)}{f(x)} dx = \log[f(x)]$	11. $\int \csc(ax) = -\frac{1}{a}\log[\csc(ax) - \cot(ax)]$
5. $\int \frac{dx}{x} = \log(x)$	12. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
6. $\int e^{ax}dx = \frac{1}{a}e^{ax}$	13. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$
7. $\int \sin(ax)dx = -\frac{1}{a}\cos(ax)$	14. $\int a^x dx = \frac{a^x}{\log(a)}$

Some Useful Trigonometric Identities

The process of evaluating definite integrals often involves manipulation of the integral's integrand in order to obtain a recognizable form. Knowledge of trigonometric identities can be very helpful with many of the definite integrals that will be encountered in this book. As a result, a table of useful identities is included below.

Table 2: Useful Trigonometric Identities	
1. $\sin^2(x) + \cos^2(x) = 1$	7. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
2. $1+\tan^2(x) = \sec^2(x)$	8. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
3. $1 + \cot^2 x = \csc^2 x$	9. $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
4. $\sin x = \cos\left(\frac{\pi}{2} - x\right)$	10. $\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{1+\cos(x)}} = \frac{1-\cos(x)}{\sin(x)} = \frac{\sin(x)}{1+\cos(x)}$
5. $\sin(2x) = 2\sin x \cos x$	11. $\tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \tan^{-1}(x)$
6. $\cos(2x) = \cos^2 x - \sin^2 x$	

Some Useful Infinite Series

One of the techniques that can be used for evaluating definite integrals involves expanding the integrand (or a portion thereof) into a power series and then integrating the power series term by term. The table included below contains a number of known convergent series that may be helpful in this context.

Table 3: Useful Infinite Series	
1. $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$	6. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$
2. $\sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$	7. $\sum_{k=0}^{\infty} \frac{1}{(k+1)^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$
3. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$	8. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}$
4. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$	9. $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log(2)$
5. $\sum_{k=0}^{\infty} \frac{1}{(2k+2)^2} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \cdots = \frac{\pi^2}{24}$	10. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots = G^*$

*G is known as Catalan's constant ($G \doteq 0.915965594177 \cdots$)

Integral Properties/Evaluation Techniques

Some simple elementary properties of definite integrals frequently come into play during the efforts to evaluate properly improper integrals. I address them here, just to ensure that we are all on the same page; they are delineated without proof. Unless otherwise specified, *a*, *b*, c and *d* are finite constants and $\epsilon \mathbb{R}$. These are the properties alluded to in the preface, some of which constitute the subject of the ensuing chapters. By no means do the properties delineated in this table constitute all of the integral properties used in the evaluation of properly improper integrals. They do, however, constitute those used most frequently. As I stated above, these properties are delineated without proof, however, they are explained in more detail at the beginning of each chapter that deals with them.

Table 4: Properties/Techniques for Evaluation of Integrals
1. The variable of integration is merely a dummy variable. It may be denoted by x, y, z, or any other symbol we choose to use; the value of the integral will remain unchanged. Therefore, $\int_a^b f(x)dx = \int_a^b f(symbol)d(symbol)$.
2. Change of variable (CV): $\int_a^b f(x)dx = \int_{h^{-1}(a)}^{h^{-1}(b)} f[h(u)]h'(u)du$ where $x = h(u)$ so $u = h^{-1}(x)$.
3. Negation: $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
4. Integration by Parts (IBP): $\int_a^b f(x)dx = [uv]_a^b - \int_a^b vdu$ where $f(x) = u(x)v(x)$.
5. Interval Subdivision: $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_n}^b f(x)dx \quad n\epsilon\mathbb{N}^+$
6. Interval Preservation (IP): $\int_a^b f(x)dx = \int_a^b f(a+b-u)du$ where $x = a+b-u$
7. Symmetry: $\int_a^b f(x)dx = 2\int_a^{(a+b)/2} f(x)dx$ if $f(x)$ is symmetric about $x = (a+b)/2$
8. Odd Function: $\int_{-a}^a f(x) = 0$ if $f(-x) = -f(x)$
9. Interval Normalization (IN): $\int_0^\infty f(x)dx = \int_0^1 \frac{x^2 f(x) + f(1/x)}{x^2} dx$.
10. Differentiation (DUI)*: If $I = \int_a^b f(x,q)dx$ then $\frac{dI}{dq} = \int_a^b \frac{\partial f(x,q)}{\partial q} dx$, a,b not functions of q.
11. Interchange of Operations (IO): $\int_a^b \left[\int_c^d f(x,y)dx\right] dy = \int_c^d \left[\int_a^b f(x,y)dy\right] dx$. Also applies to the operations of summation and integration.

* DUI stands for differentiation under the integral sign

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Do not worry about your difficulties in mathematics. I can assure you mine are still greater

—Albert Einstein

Chapter 2. Change of Variable (CV)

This chapter is devoted to the property of definite integrals that allows the integrand to be altered in anticipation that an anti-derivative for the altered integrand can be found. This is property #2 from the table of integral properties delineated in Chapter 1 (Table 4). Let's address this property in a bit more detail before we resort to seeing how it works with a few examples. Here is the property as stated in the table

$$I = \int_a^b f(x)dx = \int_{h^{-1}(a)}^{h^{-1}(b)} f[h(u)]h'(u)du \text{ where } x = h(u) \text{ so that } u = h^{-1}(x).$$

This property as shown above looks complicated, but it is not—it's not even something new. Every college freshman Calculus student learns about this property when being taught to integrate elementary integrals. The notation we have used makes it look complex; here is what this all means. In an attempt to evaluate I, you decide to make a substitution for the variable of integration that will, hopefully, change the integrand to a form that is easier to evaluate, i.e., a recognizable form. In terms of the notation above, you have decided to use the function $x = h(u)$. No problem, however after substituting $h(u)$ for x {which is the $f[h(u)]$ above}, before you can claim the resulting integral is equivalent to the original one, I, you must also substitute the appropriate term for dx and for the limits of integration, (a, b). Well, based on your choice of $h(u)$, $dx = h'(u)du$, and the integration interval, (a, b), will become 2 new values that I have designated as $h^{-1}(a)$ and $h^{-1}(b)$, where $u = h^{-1}(x)$. There is one small problem here. The function $x = h(u)$ must be able to be uniquely inverted on the interval $a \leq x \leq b$, meaning the inverse function, which we have called $h^{-1}(x)$ must be single valued on (a, b). Generally, this is not of concern, but it is worth pointing out because there is a way around this dilemma, if your $h^{-1}(x)$ is multiple valued on the original integration interval. We will explain this with the following example: Suppose the integral that you wish to evaluate is $I = \int_{-1}^{1} x^2 dx = \left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{2}{3}$, you certainly wouldn't need a change of variable (CV) to evaluate this simple integral, it's already in a recognizable form; however, it illustrates the problem we are trying to address. Let's further suppose that the CV you wish to use is $u = x^2$. Under this transformation $du = 2xdx$ or $dx = du/(2x) = (½du)u^{-½}$, but, $(-1, 1) \to (?, ?)$. If $x = -1$, u becomes 1 and when $x = 1$, u becomes 1 again and an integration interval of (1, 1) is preposterous. The solution to this dilemma is to subdivide the original interval, e.g., $I = \int_{-1}^{0} x^2\, dx + \int_0^1 x^2 dx$ (see property #5 in Table 4 of integral properties of Chapter 1). In the first integral (the one that goes from –1 to 0) let $x = -u^{½}$ so that $dx = -½u^{-½}du$ and $(-1, 0) \to (1, 0)$. In the second integral (the one that goes from 0 to 1) let $x = +u^{½}$ so that $dx = +½u^{-½}du$ and $(0, 1) \to (0, 1)$. Therefore,

$$I = -\frac{1}{2}\int_1^0 \frac{u}{\sqrt{u}}du + \frac{1}{2}\int_0^1 \frac{u}{\sqrt{u}}du = \frac{1}{2}\int_0^1 u^{1/2}du + \frac{1}{2}\int_0^1 u^{1/2}du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_{-1}^{0} + \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^1 = \frac{2}{3}.$$

So, the dilemma is solved and we get the right answer. Let's do a few examples of meaningful CV applications.

Example 2-1. $\boldsymbol{I_1 = \int_0^{\pi/2} \frac{\sin(x)\cos(x)}{1+\sin^2(x)} dx}$

In my opinion, the aha moment for this integral is right at the beginning when one decides that a CV is called for and that it should be $u = \sin(x)$. Under that CV, $x = \sin^{-1}(u)$ so that $dx = 1/(1-u^2)^{1/2}du$, and $(0, \pi/2) \to (0, 1)$. Further, the $\cos(x)$ function in the numerator will become $(1-u^2)^{1/2}$ and that will cancel with the like term that will be in the denominator due to the dx calculation. If you are worried about the $\sin^{-1}(u)$ being multi-valued, your worries are unnecessary in this case. The function $\sin^{-1}(u)$ is a multi-valued function but not on the interval of 0 to $\pi/2$. Continuing, we have

$$I_1 = \int_0^1 \frac{u\sqrt{1-u^2}}{1+u^2} \cdot \frac{1}{\sqrt{1-u^2}} du = \int_0^1 \frac{u}{1+u^2} du = \frac{1}{2}\int_0^1 \frac{2u}{1+u^2} du = \left[\frac{1}{2}\log(1+u^2)\right]_0^1 = \frac{1}{2}\log(2)$$

Therefore,

$$\boxed{\boldsymbol{I_1 = \int_0^{\pi/2} \frac{\sin(x)\cos(x)}{1+\sin^2(x)} dx = \frac{1}{2}\log(2)} \quad \textbf{Q.E.D.}}$$

Example 2-2. $\boldsymbol{I_2 = \int_1^{\infty} \frac{1}{(x+a)\sqrt{x-1}} dx\,, a} \in \mathbb{R}$

Again, in my opinion, the aha moment comes at the beginning from the following chain of thought. It would probably be helpful to get rid of that radical sign in the denominator of the integrand. I can do that by substituting a variable squared for that term under the radical, and it will eradicate the radical (a little humor—yes, I know, very little). So, let $u^2 = x - 1$ so that $dx = 2udu$, and $(1, \infty) \to (0, \infty)$. Our integral becomes under this CV the elementary form of the inverse tangent.

$$I_2 = \int_0^{\infty} \frac{2udu}{(1+a+u^2)u} = 2\int_0^{\infty} \frac{du}{(1+a)+u^2} = \left[\frac{2}{\sqrt{1+a}}\tan^{-1}\left(\frac{u}{\sqrt{1+a}}\right)\right]_0^{\infty} = \frac{\pi}{\sqrt{1+a}}$$

And our final value is

$$\boxed{\boldsymbol{I_2 = \int_1^{\infty} \frac{1}{(x+a)\sqrt{x-1}} dx = \frac{\pi}{\sqrt{1+a}}} \quad \textbf{Q.E.D.}}$$

Example 2-3. $\boldsymbol{I_3 = \int_0^{\infty} \frac{1}{1+e^{ax}} dx,\ a} \in \mathbb{R}$

Aha, this is an easy one. If one makes a CV of $u = e^{ax}$ we are going to have an integrand whose denominator will be the product of two linear expressions in the variable u, one from the direct substitution and the other from calculation of the dx term. A partial fraction expansion should then make the resulting integral duck soup to integrate. Therefore, let $u = e^{ax}$ so that $dx = du/(au)$ and $(0, \infty) \to (1, \infty)$.

$$I_3 = \int_1^{\infty} \frac{1}{1+u} \cdot \frac{du}{au} = \frac{1}{a}\int_1^{\infty} \frac{du}{u(1+u)} = \frac{1}{a}\int_1^{\infty} \left(\frac{1}{u} - \frac{1}{1+u}\right) du = \frac{1}{a}\int_1^{\infty} \frac{du}{u} - \frac{1}{a}\int_1^{\infty} \frac{du}{1+u}\,.$$

Duck soup indeed! The last two terms are recognizable forms for the log function. Thus,

$$I_3 = \left[\frac{1}{a}\log(u) - \frac{1}{a}\log(1+u)\right]_1^\infty = \frac{1}{a}\left[\log\left(\frac{u}{1+u}\right)\right]_1^\infty = \frac{1}{a}\log(1) - \frac{1}{a}\log\left(\frac{1}{2}\right).$$

And, our final result is

$$\boxed{\boldsymbol{I_3 = \int_0^\infty \frac{1}{1+e^{ax}}dx = \frac{1}{a}\log(2)} \quad \textbf{Q.E.D.}}$$

Example 2-4. $\boldsymbol{I_4 = \int_{-\infty}^\infty \frac{1}{\cosh(ax)}dx\,, a \in \mathbb{R}}$

By definition, $\cosh(ax) = \frac{1}{2}(e^{ax} + e^{-ax})$, so let's substitute that for the denominator and see what thought the result evokes.

$$I_4 = \int_{-\infty}^\infty \frac{2}{e^{ax}+e^{-ax}}dx = 2\int_{-\infty}^\infty \frac{1}{e^{ax}+\frac{1}{e^{ax}}}dx.$$

Aha, if we do a CV of $u = e^{ax}$, the denominator is going to become something that looks like an inverse tangent form. So let $u = e^{ax}$ which implies $x = (1/a)\log(u)$ which in turn implies $dx = du/(au)$ and $(-\infty, \infty) \to (0, \infty)$. Our integral becomes

$$I_4 = 2\int_0^\infty \frac{1}{u+\frac{1}{u}}\cdot\frac{du}{au} = \frac{2}{a}\int_0^\infty \frac{u}{u^2+1}\cdot\frac{du}{u} = \frac{2}{a}\int_0^\infty \frac{du}{1+u^2} = \left[\frac{2}{a}\tan^{-1}(u)\right]_0^\infty = \frac{2}{a}\cdot\frac{\pi}{2}$$

It's exactly an inverse tangent! Our final result

$$\boxed{\boldsymbol{I_4 = \int_{-\infty}^\infty \frac{1}{\cosh(ax)}dx = \frac{\pi}{a}} \quad \textbf{Q.E.D.}}$$

Example 2-5. $\boldsymbol{I_5 = \int_0^1 \frac{1-x}{1+x+x^2}dx}$

It is worth remembering that when a quadratic expression appears in the integrand, completing the square on that expression is often helpful. It is a technique that is certainly worth trying. You will recall from high school algebra that in order to complete the square, you take the square of one-half the coefficient of the first power term and both add and subtract it from the quadratic expression. Thus,

$$1 + x + x^2 = 1 + x + x^2 + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \left(x + \frac{1}{2}\right)^2$$

Substituting this for the expression in the denominator, we have

$$I_5 = \int_0^1 \frac{1-x}{\frac{3}{4}+\left(x+\frac{1}{2}\right)^2}dx = \int_0^1 \frac{dx}{\frac{3}{4}+\left(x+\frac{1}{2}\right)^2} - \int_0^1 \frac{xdx}{\frac{3}{4}+\left(x+\frac{1}{2}\right)^2}$$

Aha! It's that moment! If the integrand is broken into two integrals as shown above, one will be an inverse tangent form and the other will be a log function form. If you don't see that, make a CV of $u = x + ½$, $dx = du$, and $(0, 1) \to (1/2, 3/2)$.

$$I_5=\int_{1/2}^{3/2}\frac{\frac{3}{2}-u}{\frac{3}{4}+u^2}du=\frac{3}{2}\int_{1/2}^{3/2}\frac{du}{\frac{3}{4}+u^2}-\int_{1/2}^{3/2}\frac{u}{\frac{3}{4}+u^2}du=\frac{3}{2}\int_{1/2}^{3/2}\frac{du}{\frac{3}{4}+u^2}-\frac{1}{2}\int_{1/2}^{3/2}\frac{d\left(\frac{3}{4}+u^2\right)}{\frac{3}{4}+u^2}$$

As advertised, the first integral on the right of the last equal sign is a recognizable form of the inverse tangent while the second integral on the right of the last equal sign is a recognizable form of the natural logarithm function. Continuing, we obtain

$$I_5=\frac{3}{2}\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)\right]_{1/2}^{3/2}-\frac{1}{2}\left[\log\left(\frac{3}{4}+u^2\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}=\frac{3}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)-\frac{1}{2}[\log(3)-\log(1)].$$

And so, our final value is

$$\boxed{\boldsymbol{I_5=\int_0^1\frac{1-x}{1+x+x^2}dx=\frac{\sqrt{3}\pi}{6}-\frac{1}{2}\log(3)}\quad \textbf{Q.E.D.}}$$

Example 2-6. $\boldsymbol{I_6=\int_0^{\pi/2}\frac{1}{a\sin^2(x)+b\cos^2(x)}dx\;\; a,b\in\mathbb{R}^+}$

It takes a bit of experimentation here to find the winning approach, but once found, the integral literally falls apart. By dividing both numerator and denominator by $\cos^2(x)$, we get

$$I_6=\int_0^{\pi/2}\frac{\frac{1}{\cos^2(x)}}{a\frac{\sin^2(x)}{\cos^2(x)}+b}dx=\int_0^{\pi/2}\frac{\sec^2(x)}{a\tan^2(x)+b}dx.$$

Now divide both numerator and denominator by the parameter b. Thus, we obtain

$$I_6=\int_0^{\pi/2}\frac{\frac{1}{b}\sec^2(x)}{\frac{a}{b}\tan^2(x)+1}dx=\frac{1}{b}\int_0^{\pi/2}\frac{\sec^2(x)}{\frac{a}{b}\tan^2(x)+1}dx=\frac{1}{b}\int_0^{\pi/2}\frac{1+\tan^2(x)}{\frac{a}{b}\tan^2(x)+1}dx\,.$$

Now make the following CV. Let $u=\tan(x)$ so that $du=\sec^2(x)dx$ or $dx=du/(1+u^2)$ and $(0,\pi/2)\to(0,\infty)$. Hence,

$$I_6=\frac{1}{b}\int_0^{\infty}\frac{1+u^2}{\frac{a}{b}u^2+1}\cdot\frac{du}{1+u^2}=\frac{1}{b}\int_0^{\infty}\frac{1}{\frac{a}{b}u^2+1}du.$$

And yet again, make another CV. Let $z=\sqrt{\frac{a}{b}}u$. Then, $du=\sqrt{\frac{b}{a}}dz$ and $(0,\infty)\to(0,\infty)$. We then have

$$I_6=\frac{1}{b}\int_0^{\infty}\frac{1}{z^2+1}\cdot\sqrt{\frac{b}{a}}dz=\frac{1}{\sqrt{ab}}\int_0^{\infty}\frac{dz}{z^2+1}=\left[\frac{1}{\sqrt{ab}}\tan^{-1}(z)\right]_0^{\infty}=\frac{\pi}{2\sqrt{ab}}.$$

This last integral is the recognizable form of the inverse tangent function and our so final result is

$$\boxed{\boldsymbol{I_6=\int_0^{\pi/2}\frac{1}{a\sin^2(x)+b\cos^2(x)}dx=\frac{\pi}{2\sqrt{ab}}}\quad \textbf{Q.E.D.}}$$

Example 2-7. $\boldsymbol{I_7 = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx}$

As another example illustrating the use of CV to solve integrals, let's take a look at I_7, an integral that was lifted from the lifting theory of aerodynamics (some more humor). Actually, this is a simple one to solve if you have the least bit of familiarity with trigonometric identities. Make the following CV. Let $x = \cos(2u)$ so that $dx = -2\sin(2u)du$ and $(-1, 1) \rightarrow (\pi/2, 0)$. Under this CV, we have

$$I_7 = -\int_{\pi/2}^{0} \frac{2\sin(2u)}{\tan(u)}\, du = \int_{0}^{\pi/2} \frac{4\sin(u)\cos(u)}{\tan(u)}\, du = 4\int_{0}^{\pi/2} \cos^2(u)\, du$$

Now, using another trigonometric identity, we obtain

$$I_7 = 4\int_{0}^{\pi/2} \left[\frac{1}{2} + \frac{1}{2}\cos(2u)\right] du = 2\int_{0}^{\pi/2} du + 2\int_{0}^{\pi/2} \cos(2u)du = 2\left(\frac{\pi}{2}\right) + [\sin(2u)]_{0}^{\pi/2}$$

Our final answer is therefore

$$\boxed{\boldsymbol{I_7 = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx = \pi} \quad \textbf{Q.E.D.}}$$

Example 2-8. $\boldsymbol{I_8 = \int_{0}^{a} \frac{dx}{\sqrt{x+a}+\sqrt{x}},\ a \in \mathbb{R}^+}$

This is another pretty simple one to solve. However, before trying to determine a CV that is appropriate, this integral requires one to rationalize the denominator. One does that by multiplying both numerator and denominator by the quantity $\sqrt{x+a} - \sqrt{x}$. What right do we have to do this? As long as you multiply both numerator and denominator by the same quantity, you are really multiplying by 1 which, of course, does not change the value of the original fraction—only it's appearance. Therefore,

$$I_8 = \int_{0}^{a} \frac{dx}{\sqrt{x+a}+\sqrt{x}} \cdot \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}} = \int_{0}^{a} \frac{\sqrt{x+a}-\sqrt{x}}{x+a-x}\, dx = \frac{1}{a}\int_{0}^{a} \left(\sqrt{x+a} - \sqrt{x}\right) dx$$

This last expression can be split into two integrals as

$$I_8 = \frac{1}{a}\int_{0}^{a} (x+a)^{1/2} dx - \frac{1}{a}\int_{0}^{a} x^{1/2} dx$$

Now we are ready for the CV which is applicable to the 1st integral above (the 2nd integral is already a recognizable form). Let $x + a = u$ so that $dx = du$ and $(0, a) \rightarrow (a, 2a)$. We then have

$$I_8 = \frac{1}{a}\int_{a}^{2a} u^{1/2} du - \frac{1}{a}\int_{0}^{a} x^{1/2} dx = \left[\frac{2}{3a} u^{3/2}\right]_{a}^{2a} - \left[\frac{2}{3a} x^{3/2}\right]_{0}^{a}$$

This last expression evaluates to

$$I_8 = \frac{2}{3a}\left[2^{3/2}a^{3/2} - a^{3/2}\right] - \frac{2}{3a} a^{3/2}$$

Then, upon simplification, we have our final result

$$\boxed{I_8 = \int_0^a \frac{dx}{\sqrt{x+a}+\sqrt{x}} = \frac{4}{3}\sqrt{a}(\sqrt{2}-1) \quad \text{Q.E.D}}$$

Example 2-9. $I_9 = \int_{-\infty}^{\infty} \frac{dx}{a+2bx+cx^2},\ ac - b^2 > 0\ \ a, b, c \in \mathbb{R}$

Immediately, upon seeing this problem, I have two different thoughts about it. First, when a problem appears that gives a condition (or conditions) on parameters that appear in the problem (in this case, the $ac - b^2 > 0$) the question of why always occurs to me. Here is one possible explanation. The fact that $ac - b^2$ is not allowed to be zero <u>might</u> imply that the final result will have that expression in a fraction's denominator (that is, if it were zero, it would be undefined). Additionally, the fact that $ac - b^2$ is not allowed to be negative <u>might</u> imply that the final result will contain the square root of that expression (that is, if it were negative, it would be complex and that certainly seems unlikely). Second, the integrand's denominator contains a quadratic expression while the integrand's numerator does not contain the integration variable at all—this implies to me that the final result is going to involve the recognizable form of the inverse tangent function although to get there is going to involve manipulation of said denominator. In-point-of-fact, this gives us a clue on how to proceed. Let's complete the square on the denominator. I'm going to call the denominator D, so $D = cx^2 + 2bx + a$. Completing the square, we get

$$D = c\left(x^2 + \frac{2bx}{c} + \frac{a}{c}\right) = c\left(x^2 + \frac{2bx}{c} + \frac{b^2}{c^2} + \frac{a}{c} - \frac{b^2}{c^2}\right) = c\left(x + \frac{b}{c}\right)^2 + a - \frac{b^2}{c}$$

As a result,

$$I_9 = \int_{-\infty}^{\infty} \frac{dx}{a - \frac{b^2}{c} + c\left(x + \frac{b}{c}\right)^2}$$

Now is the time for a CV of $u = x + b/c$, $dx = du$, and $(-\infty, \infty) \to (-\infty, \infty)$. We then obtain

$$I_9 = \int_{-\infty}^{\infty} \frac{du}{a - \frac{b^2}{c} + cu^2} = c\int_{-\infty}^{\infty} \frac{du}{ac - b^2 + c^2u^2}$$

Now, another change of variable gives us the recognizable form that we expected, namely, the inverse tangent form. Let $cu = z$ so that $du = dz/c$ and $(-\infty, \infty) \to (-\infty, \infty)$. Thus,

$$I_9 = \int_{-\infty}^{\infty} \frac{dz}{ac - b^2 + z^2} = \frac{1}{\sqrt{ac-b^2}}\left[\tan^{-1}\left(\frac{z}{\sqrt{ac-b^2}}\right)\right]_{-\infty}^{\infty} = \frac{1}{\sqrt{ac-b^2}}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

And, our final result is

$$\boxed{I_9 = \int_{-\infty}^{\infty} \frac{dx}{a + 2bx + cx^2} = \frac{\pi}{\sqrt{ac - b^2}} \quad \text{Q.E.D.}}$$

Note that our two thoughts about the final result at the beginning of the derivation were correct (and helpful).

Example 2-10. $I_{10} = \int_{\beta}^{\alpha} \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}\ \ \alpha, \beta \in \mathbb{R}$

The derivation required to solve this problem is not much different than the previous example except for the recognizable form that results from completing the square on the expression in the denominator. Nevertheless, there is something fascinating about this problem and its solution. I'm not sure that I can articulate what the fascination is. See if you agree with me. Again, calling the denominator (sans the radical sign) *D*, we complete the square as follows

$$D = (x-\alpha)(\beta - x) = x(\alpha+\beta) - x^2 - \alpha\beta = \left(\frac{\alpha+\beta}{2}\right)^2 - \alpha\beta - \left(x - \frac{\alpha+\beta}{2}\right)^2$$

Our integral then becomes

$$I_{10} = \int_\beta^\alpha \frac{dx}{\sqrt{\left(\frac{\alpha+\beta}{2}\right)^2 - \alpha\beta - \left(x-\frac{\alpha+\beta}{2}\right)^2}}$$

To change the variable, let $u = x - (\alpha+\beta)/2$ so that $dx = du$ and $(\beta, \alpha) \to \left(\frac{\beta-\alpha}{2}, \frac{\alpha-\beta}{2}\right)$.

Under that change of variable, our integral becomes

$$I_{10} = \int_{\frac{\beta-\alpha}{2}}^{\frac{\alpha-\beta}{2}} \frac{du}{\sqrt{\left(\frac{\alpha+\beta}{2}\right)^2 - \alpha\beta - u^2}} = \int_{\frac{\beta-\alpha}{2}}^{\frac{\alpha-\beta}{2}} \frac{du}{\sqrt{\frac{(\alpha-\beta)^2}{4} - u^2}} = \sin^{-1}\left[\frac{u}{\frac{\alpha-\beta}{2}}\right]_{\frac{\beta-\alpha}{2}}^{\frac{\alpha-\beta}{2}} = \sin^{-1}(1) - \sin^{-1}(-1)$$

The recognizable form, as you can see, is the inverse sine function, giving us the fascinating result of, but completely independent of the two parameters α and β. Exotic—Yes!

$$\boldsymbol{I_{10} = \int_\beta^\alpha \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = \pi} \quad \textbf{Q.E.D.}$$

Example 2-11. $\boldsymbol{I_{11} = \int_0^\infty \frac{dx}{1+2x\cos(\alpha)+x^2} \quad \alpha \in \mathbb{R}}$

Not to belabor the technique of completing the square, but here is another integral that lends itself to said technique.

$$D = 1 + 2x\cos(\alpha) + x^2 = x^2 + 2x\cos(\alpha) + \cos^2(\alpha) + 1 - \cos^2(\alpha) = (x+\cos\alpha)^2 + \sin^2(\alpha)$$

Hence,

$$I_{11} = \int_0^\infty \frac{dx}{\sin^2(\alpha) + (x+\cos\alpha)^2}$$

Now make the change of variable such that $u = x + \cos(\alpha)$, $dx = du$, and $(0, \infty) \to (\cos\alpha, \infty)$. Thus,

$$I_{11} = \int_{\cos\alpha}^\infty \left[\frac{du}{\sin^2(\alpha)+u^2}\right]$$

This last integral is, of course, the recognizable form of the inverse tangent. Therefore, we have

$$I_{11} = \frac{1}{\sin(\alpha)}\left[\tan^{-1}\left(\frac{u}{\sin\alpha}\right)\right]_{\cos\alpha}^\infty = \frac{1}{\sin(\alpha)}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right] = \frac{1}{\sin(\alpha)}\left[\frac{\pi}{2} - \tan^{-1}(\cot\alpha)\right]$$

However, the cotangent of an angle is the same as the tangent of the angle's compliment. We can therefore write

$$I_{11} = \frac{1}{\sin(\alpha)}\left[\frac{\pi}{2} - \tan^{-1}\left(\tan\left\{\frac{\pi}{2} - \alpha\right\}\right)\right] = \frac{1}{\sin(\alpha)}\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right)\right] = \frac{\alpha}{\sin(\alpha)}$$

That is, our final result is

$$\boxed{\boldsymbol{I_{11} = \int_0^\infty \frac{dx}{1 + 2x\cos(\alpha) + x^2} = \frac{\alpha}{\sin(\alpha)}} \quad \textbf{Q.E.D.}}$$

This integral can also be evaluated on the interval (0, 1). It is interesting to do so, so let's do that. The derivation is exactly the same as above until we get to the change of variable step, that is, the change of variable is the same, but obviously, the interval for the change is different, i.e., here is the change. Let $u = x + \cos(\alpha)$, $dx = du$, and $(0, 1) \to (\cos\alpha, 1 + \cos\alpha)$. The integral then becomes

$$\int_{\cos(\alpha)}^{1+\cos(\alpha)} \frac{du}{\sin^2(\alpha)+u^2} = \frac{1}{\sin(\alpha)}\tan^{-1}\left[\frac{u}{\sin(\alpha)}\right]_{\cos(\alpha)}^{1+\cos(\alpha)} = \frac{1}{\sin(\alpha)}\left[\tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right) - \tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right]$$

Now, using the appropriate trigonometric identities, the last expression becomes

$$\frac{1}{\sin(\alpha)}\left[\tan^{-1}\left(\cot\frac{\alpha}{2}\right) - \tan^{-1}(\cot\alpha)\right] = \frac{1}{\sin(\alpha)}\left\{\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right] - \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\alpha\right)\right]\right\}$$

So we obtain

$$\frac{1}{\sin(\alpha)}\left[\frac{\pi}{2} - \frac{\alpha}{2} - \left(\frac{\pi}{2} - \alpha\right)\right] = \frac{\alpha}{2\sin(\alpha)}$$

This value is exactly ½ the value that the integral has on the interval 0 to ∞.

Example 2-12. $\boldsymbol{I_{12} = \int_0^a \frac{x^3 dx}{\sqrt{a^8-x^8}}}$ $\boldsymbol{a \in \mathbb{R}^+}$

This is actually a very easy integral to evaluate, but it illustrates a couple of mathematical points that I thought it would be good to emphasize. I'll do so after we do the evaluation. The aha moment comes right at the beginning when you realize that the expression under the radical in the denominator is the difference of two squares which readily factors into $a^4 + x^4$ and $a^4 - x^4$. Therefore, a change of variable $u = x^4$ so that $du = 4x^3dx$ and $(0, a) \to (0, a^4)$ should produce a nice result. Thus,

$$I_{12} = \frac{1}{4}\int_0^{a^4} \frac{du}{\sqrt{(a^4+u)(a^4-u)}} = \frac{1}{4}\int_0^{a^4} \frac{du}{\sqrt{a^8-u^2}}$$

However, the last integral is the recognizable form of the inverse sine function, and we have

$$I_{12} = \frac{1}{4}\sin^{-1}\left[\frac{u}{a^4}\right]_0^{a^4} = \frac{1}{4}\sin^{-1}(1) = \frac{1}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{8}$$

Our final result is therefore

$$\boxed{\boldsymbol{I_{12} = \int_0^a \frac{x^3 dx}{\sqrt{a^8 - x^8}} = \frac{\pi}{8}} \quad \textbf{Q.E.D.}}$$

The points I wish to illustrate come about when we change the original integral by modifying the integration interval from (0, a) to ($-a$, a), i.e., $\int_{-a}^{a} \frac{x^3 dx}{\sqrt{a^8-x^8}}$. Given this new integral, based on entry # 8 in Table 4, Chapter 1, one can immediately say that the value of this new integral is zero, that is, $\int_{-a}^{a} \frac{x^3 dx}{\sqrt{a^8 - x^8}} = 0$. The integrand is an odd function and one can see this by replacing the variable x with $-x$ and obtaining the negation of the original integrand or by examining the graph of the integrand as depicted in the accompanying figure. The graph is symmetric about the origin and therefore the area between the curve and below the x-axis cancels with the area between the curve and above the x-axis, as can be seen in figure 2-1. The second point worth observing is that if one does not recognize the odd function property and attempts to evaluate the integral by making the change of variable $u = x^4$, as we did earlier, we note that the inverse function for the change of variable is not single valued on $(-a, a)$ and we must therefore break the integral up into two pieces, ala, $\int_{-a}^{0} \frac{x^3 dx}{\sqrt{a^8 - x^8}} + \int_{0}^{a} \frac{x^3 dx}{\sqrt{a^8 - x^8}}$ (as addressed at the beginning of this chapter).

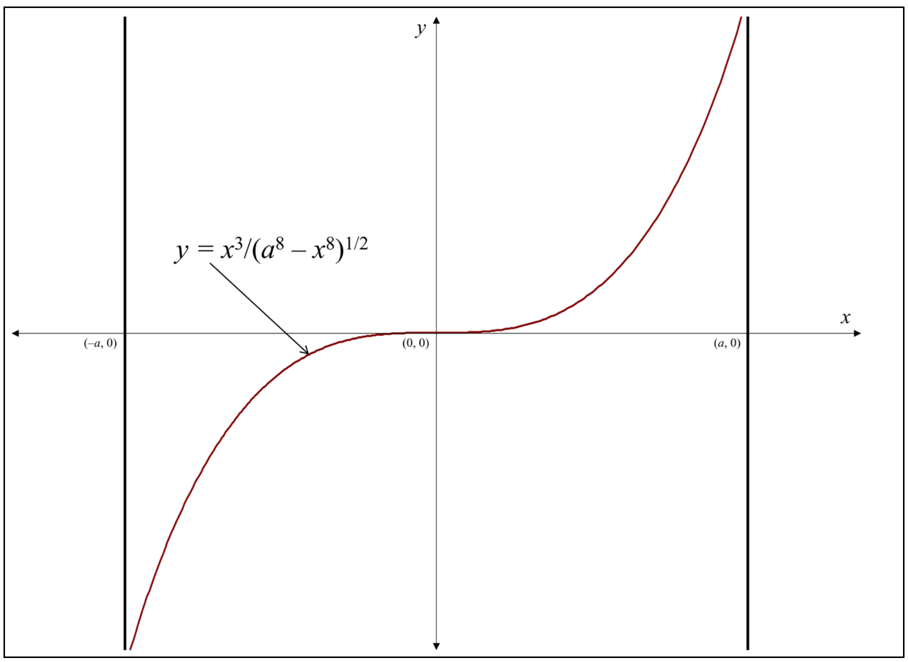


Figure 2-1. Graph of $x^3/(a^8 - x^8)$ between –a and +a

Example 2-13. $I_{13} = \int_{-1/2}^{1} \frac{dx}{\sqrt{1+x+x^2}}$

This example is a bit strange—the integration interval looks a little weird, but you will subsequently see that it is configured to make life easy later in the derivation. I, your author, like this problem as it illustrates the CV property/technique quite well. Anyway, here it is! One starts by completing the square on the expression under the radical sign in the denominator of the integrand. This gives us

$$1 + x + x^2 = \frac{3}{4} + \frac{1}{4} + x + x^2 = \frac{3}{4} + \left(\frac{1}{2} + x\right)^2$$

Our integral then becomes

$$I_{13} = \int_{-½}^{1} \frac{dx}{\sqrt{\frac{3}{4}+\left(\frac{1}{2}+x\right)^2}}$$

Aha, I would suggest a CV of the following: $u = ½ + x$, so that $du = dx$ and $(–½ , 1) \rightarrow (0, 3/2)$.

That CV simply transforms our integral to

$$I_{13} = \int_{0}^{3/2} \frac{du}{\sqrt{\frac{3}{4}+u^2}}$$

Look what happens if we now make another CV. Let $u = \frac{\sqrt{3}}{2}\tan(\theta)$ so that $du = \frac{\sqrt{3}}{2}\sec^2(\theta)d\theta$ and $(0, 3/2) \rightarrow (0, \pi/3)$. Wait a minute, you might say. This is making things worse, isn't it? A knowledge of trigonometric identities is what drives this aha moment. As I said, look what happens.

$$I_{13} = \int_{0}^{\pi/3} \frac{\frac{\sqrt{3}}{2}\sec^2(\theta)d\theta}{\sqrt{\frac{3}{4}+\frac{3}{4}\tan^2(\theta)}} = \int_{0}^{\pi/3} \frac{\frac{\sqrt{3}}{2}\sec^2(\theta)d\theta}{\frac{\sqrt{3}}{2}\sqrt{1+\tan^2(\theta)}} = \int_{0}^{\pi/3} \sec(\theta)d\theta = [\log(\sec\theta) + \tan\theta]_0^{\pi/3} = \log\left(2+\sqrt{3}\right)$$

Of course, this last integral is a recognizable form and we have our final solution, namely

$$\boldsymbol{I_{13} = \int_{-½}^{1} \frac{dx}{\sqrt{1+x+x^2}} = \log\left(2+\sqrt{3}\right)} \quad \textbf{Q.E.D.}$$

Example 2-14. $I_{14} = \int_{-\infty}^{\infty} \frac{dx}{\left(a^2+x^2\right)^2} \quad a \epsilon \mathbb{R}$

Figure 2-2. Italian Mathematician Maria Gaetana Agnesi (1718 – 1799)

"Analytics is the Art of resolving all kinds of Mathematical Questions, by finding or computing unknown numbers, or quantities, by the means of others that are known or given."—Maria Gaetana Agnesi

Mathematician Maria Agnesi is not responsible for the evaluation of I_{14} (it's probably too simple a problem for anybody to claim responsibility for its solution), however, she has a much more interesting connection to this integral. I will explain that connection shortly.

Appointed to the University of Bologna by Pope Benedict XIV at the age of 32, Maria Agnesi became the first female professor of mathematics on a faculty anywhere in the world. Evidently, she displayed her mathematical ability at a very early age; she was a gifted and precocious child. Her father, Don Pietro Agnesi, actively encouraged his daughter in her studies and, proud of her, exhibited her in the kinds of academic meetings fashionable at that time.

Maria Agnesi was the author of a famous two-volume work on the methods of Calculus. This work by Agnesi is the first surviving mathematical work by a woman. Although Newton and Leibniz had invented Calculus many years before, communication of new ideas and discoveries was slow to spread in that time; as a result, Calculus was a relatively new mathematical field to Agnesi and her Italian contemporaries. Her book on the subject was translated into French and published under the license of the Royal Academy of Sciences and declared to be the most complete and the best done in this field. The book includes a discussion of the curve now known as "the Witch of Agnesi." It is fate that this curve has come down to us through the years with this name, for it is certainly not the name that Ms. Agnesi intended, or for that matter, the name which anybody else intended. The curve was first discussed by Fermat and a construction for the curve was given by Grandi in 1703. In 1718 Grandi gave the curve the Latin name *versoria* which means *turning curve*, so named because of its shape (see the graph in figure 2-3). Grandi also gave the Italian *versiera* for the Latin *versoria* and indeed, Agnesi quite correctly states in her book that the curve was called *la versiera*. However, an Englishman by the name of John Colson translated Agnesi's book from Italian into English and Colson mistook *la versiera* for *laversiera* which means ungodly woman or she-devil. Hence, today we know the curve as the Witch. By-the-way, John Colson (1680-1760) was the Lucasian Professor of Mathematics at Cambridge. In my opinion, Colson, through his mis-translation, actually did Agnesi a favor. If the name *turning curve* had caught on, Agnesi's name would undoubtedly not have been associated with the curve and this "romantic" name *Witch of Agnesi* would have been lost forever.

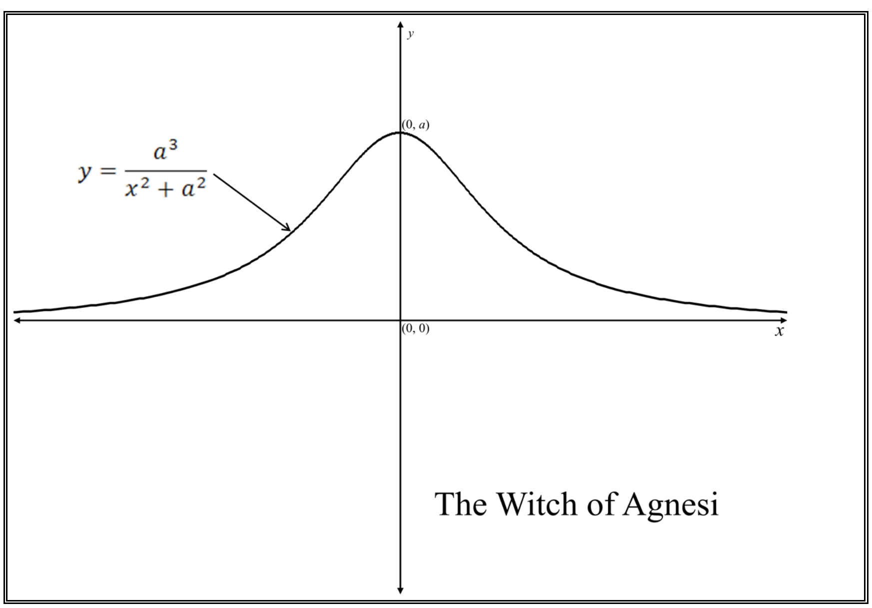


Figure 2-3. Graph of the Witch of Agnesi

Before we discuss Maria Agnesi's connection with the integral I_{14}, let's attempt to solve the integral first and then we will address her association with it.

$$I_{14} = \int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^2}$$

Whenever you have an integral that exhibits a constant plus the variable of integration squared, as I_{14} does in the denominator (i.e., $a^2 + x^2$) it is a good idea to consider a change of variable in which the variable of integration is set equal to the square root of the constant times the tangent of the new variable. In this case that would mean let $x = a\tan(\theta)$. The reason for this is that when you form the term $a^2 + x^2$ in terms of the new variable, you will get $a^2 + a^2\tan^2(\theta)$ which reduces to $a^2\sec^2(\theta)$ by means of a trigonometric identity and that may be very helpful, particularly since dx will equal $a\sec^2(\theta)d\theta$ and the two $\sec^2(\theta)$ terms will often annihilate one-another (i.e., cancel) and, in this case, they do, leaving a simple $\sec^2(\theta)$ term in the denominator. Continuing, $(-\infty, \infty) \rightarrow (-\pi/2, \pi/2)$, thus

$$I_{14} = \int_{-\pi/2}^{\pi/2} \frac{a\sec^2(\theta)d\theta}{a^4\sec^4(\theta)} = \frac{1}{a^3}\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sec^2(\theta)} = \frac{1}{a^3}\int_{-\pi/2}^{\pi/2} \cos^2(\theta)d\theta$$

Now, using another trigonometric identity, namely $\cos^2(\theta) = ½ + ½\cos(2\theta)$, we have

$$I_{14} = \frac{1}{a^3}\int_{-\pi/2}^{\pi/2} \left[\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right] d\theta = \frac{1}{2a^3}\int_{-\pi/2}^{\pi/2} d\theta + \frac{1}{2a^3}\int_{-\pi/2}^{\pi/2} \cos(2\theta)d\theta$$

And, of course, these last two integrals can be integrated in the conventional sense and we have

$$I_{14} = \frac{1}{2a^3}[\theta]_{-\pi/2}^{\pi/2} + \frac{1}{4a^3}[\sin(2\theta)]_{-\pi/2}^{\pi/2} = \frac{\pi}{2a^3}$$

So, as the final value

$$\boxed{\boldsymbol{I_{14} = \int_{-\infty}^{\infty} \frac{1}{(a^2+x^2)^2}dx = \frac{\pi}{2a^3}} \quad \textbf{Q.E.D.}}$$

Finally, here is the connection between this integral and Ms. Agnesi. If you take Ms. Agnesi's curve, the Witch of Agnesi, and form a solid-of-revolution (SOR) from it by rotating it about the x-axis, and then compute the volume of that solid of revolution, this integral will be involved in that volume calculation. Quite interesting! (See figure 2-4 for an explanation.)

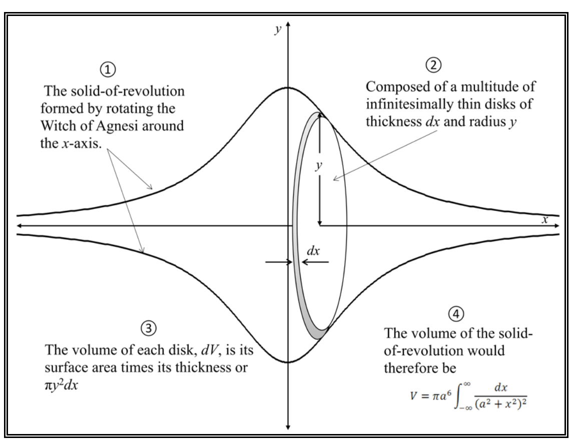


Figure 2-4. The Volume of the SOR formed from the Witch of Agnesi

Example 2-15. $\boldsymbol{I_{15} = \int_0^{2a} \frac{x^{\frac{3}{2}}}{\sqrt{2a-x}} dx \;\; a \in \mathbb{R}^+}$

Evaluation of this integral will allow us to immediately write down the value of another integral. The other integral is $\int_0^{\infty}(3a-x)\sqrt{\frac{x}{2a-x}}\,dx$; I will explain after we solve I_{15}.

The CV needed here is the following. Let $x = 2a\sin^2(\theta)$ so that $dx = 4a\sin(\theta)\cos(\theta)d\theta$ and $(0, 2a) \rightarrow (0, \pi/2)$. Further, $x^{3/2} = (2a)^{3/2}\sin^3(\theta)$ and $(2a-x)^{½} = (2a)^{½}\cos(\theta)$. Under this CV, our integral becomes

$$I_{15} = \int_0^{\pi/2} \frac{(2a)^{3/2}\sin^3(\theta)4a\sin(\theta)\cos(\theta)}{(2a)^{1/2}\cos(\theta)} d\theta = 8a^2\int_0^{\pi/2}\sin^4(\theta)d\theta$$

Now, using the trigonometric identity $\sin^2(\theta) = ½ - ½\cos(2\theta)$, this becomes

$$I_{15} = 8a^2\int_0^{\pi/2}[½ - ½\cos(2\theta)]^2 d\theta = 8a^2\int_0^{\pi/2}[¼ - ½\cos(2\theta) + ¼\cos^2(2\theta)]\,d\theta$$

This last expression can, of course, be written as three separate integrals. Hence

$$I_{15} = 2a^2\int_0^{\pi/2}d\theta - 4a^2\int_0^{\pi/2}\cos(2\theta)d\theta + 2a^2\int_0^{\pi/2}\cos^2(2\theta)\,d\theta$$

Obviously, the first two integrals above are tractable; to integrate the last integral above, we must use the trigonometric identity $\cos^2(2\theta) = ½ + ½\cos(4\theta)$, which gives us

$$I_{15} = 2a^2[\theta]_0^{\pi/2} - 2a^2[\sin(2\theta)]_0^{\pi/2} + a^2[\theta]_0^{\pi/2} + \left[\frac{a^2}{4}\sin(4\theta)\right]_0^{\pi/2}$$

The 1st term above is πa^2, the 2nd term is zero, the 3rd is $\pi a^2/2$ and the 4th is also zero. So we obtain as a final value

$$\boxed{\boldsymbol{I_{15} = \int_0^{2a}\frac{x^{\frac{3}{2}}}{\sqrt{2a-x}}dx = \frac{3\pi a^2}{2}} \quad \textbf{Q.E.D.}}$$

We said, at the beginning of this example that we would be able to immediately write down the value of another integral upon evaluating I_{15}, and the other integral was the following one:

$$\int_0^{\infty}(3a-x)\sqrt{\frac{x}{2a-x}}\,dx = \frac{3\pi a^2}{2}.$$

As you can plainly see, we did immediately write down a value for it, namely the same value that we obtained for I_{15}. How do we know that this value is correct? To answer that question, we need to examine the integrand of I_{15} and the Cartesian equation of a classic curve known as the Cissoid of Diocles. Figure 2-5 shows a graph of the Cissoid of Diocles along with its equation, namely, $y^2 = x^3/(2a-x)$. Note that the integrand of I_{15} is simply the square root of the right side of the equation representing the Cissoid of Diocles, or y. Now, suppose that we wished to calculate the area in the first quadrant between the Cissoid of Diocles and its asymptote. One way to do that is to consider that the area is composed of many, many vertical rectangles of height y and of an infinitely small width dx. Of course the area of one of those rectangles will be the height y times the width dx which we have set equal to dA in figure 2-5 (see ① in the figure). Integrating that from $x = 0$ to $x = 2a$ will then produce the desired area. So, the value of I_{15} represents the area between the Cissoid of Diocles and its asymptote in the first quadrant.

However, one could also compute that same area by considering it to be composed of many horizontal rectangles of width $2a - x$ and infinitely small height dy. In this case, $dA = (2a - x)dy$, and the first quadrant area would be as shown in the figure (see ② in figure 2-5). Wait-a-minute, you might say. That integral is different than the one claiming to also be equal to $3\pi a^2/2$. Yes, but if you calculate dy from the equation for the Cissoid of Diocles and then substitute that value for dy in the integral, you obtain the proper properly improper integral (humor, again?). Note, that we have demonstrated the value of this integral through a purely geometric argument. Quite interesting and certainly a case of non-conventional integration!!

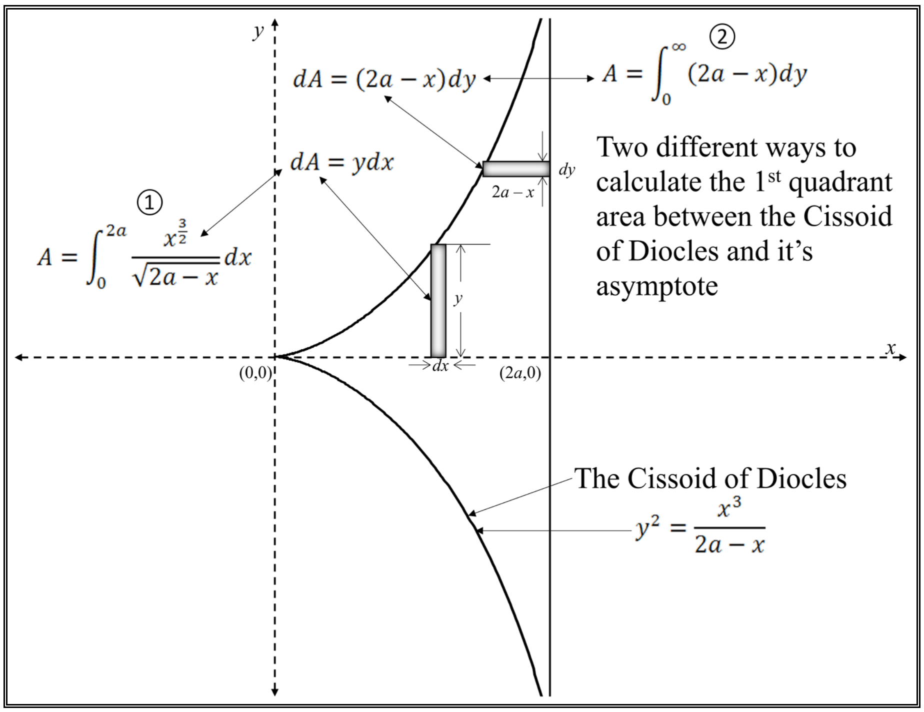


Figure 2-5. The area encompassed by the Cissoid of Diocles and its asymptote

Example 2-16. $I_{16} = \int_0^a \frac{dx}{\sqrt{ax - x^2}}$ $a \in \mathbb{R}^+$

We will end this chapter with a very simple integral to compute; we do so not to necessarily show another example of "Change of Variable", but to echo what was said in the preface in that the value of the integral sometimes seems intriguing and/or exotic. Completing the square of the expression inside the radical gives $I_{16} = \int_0^a \frac{dx}{\sqrt{\frac{a^2}{4} - \left(x - \frac{a}{2}\right)^2}}$. Now, a CV of $u = x - a/2$ so that $du = dx$ and $(0, a) \rightarrow (-a/2, +a/2)$. We therefore have, $\int_{-a/2}^{a/2} \frac{du}{\sqrt{a^2/4 - u^2}} = \sin^{-1}\left(\frac{2u}{a}\right)_{-a/2}^{+a/2} = \sin^{-1}(1) - \sin^{-1}(-1) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

So, our final result is π—<u>completely independent of the value of *a*</u>. In other words, no matter what value we assign to the parameter *a* (under the constraint of it being positive and real), the integral's value is always π. That seems to me to be quite magical!

$$\boldsymbol{I_{16} = \int_0^a \frac{dx}{\sqrt{ax - x^2}} = \pi \quad \text{Q.E.D.}}$$

I must study politics and war that my sons may have the liberty to study mathematics and philosophy

—John Adams

Chapter 3. Interval Preservation (IP)

This chapter is devoted to the property that we designated as "Interval Preservation" in Table 4 of integral properties of Chapter 1 (property #6). That doesn't mean that the solution of the integrals that we will attempt in this chapter can't also embody some of the other properties from that table, in point-of-fact, they more than likely will. However, every integral evaluation in this chapter will definitely use the IP property to arrive at a value since the object of this chapter is to understand how this particular property can be used. Before we look at any integrals, however, let's address the property in a bit more detail. Here is the property as stated in the table:

$$\int_a^b f(x)dx = \int_a^b f(a+b-u)du \text{ where } x = a+b-u$$

First of all, this property is a special case of property #2, i.e., "Change of Variable" but a change of variable to accomplish the specific purpose of leaving the integral's interval of integration unchanged. If we go through the usual process of seeing what happens when we substitute $a + b - u$ for our variable of integration, x, we get $dx = -du$ and $(a, b) \rightarrow (b, a)$. It flips the interval and we obtain

$$\int_a^b f(x)dx = -\int_b^a f(a+b-u)du = \int_a^b f(a+b-u)du.$$

But, of course, the leading negative sign flips it right back (property #3—Negation) and our integration interval is preserved albeit the original integrand is now changed. This property can be quite helpful in the evaluation of integrals, particularly integrals involving trigonometric functions. Let's do such an integral.

Example 3-1. $I_1 = \int_0^{\pi/2} \frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)}+\sqrt{\cos(x)}} dx$

This is quite a formidable looking integral. My first thought upon seeing this integral was to rationalize the denominator, that is, get rid of the radical signs in the denominator. One can do that by multiplying both numerator and denominator by $\sqrt{\sin(x)} - \sqrt{\cos(x)}$. It doesn't take long before one realizes that this approach gets you nowhere. However, by applying the IP change of variable $x = \pi/2 - u$, the integral literally solves itself. Under that change of variable we get $dx = -du$ and $(0, \pi/2) \rightarrow (\pi/2, 0)$ and we have

$$I_1 = -\int_{\pi/2}^{0} \frac{\sqrt{\sin(\frac{\pi}{2}-u)}}{\sqrt{\sin(\frac{\pi}{2}-u)}+\sqrt{\cos(\frac{\pi}{2}-u)}} du = \int_0^{\pi/2} \frac{\sqrt{\cos(u)}}{\sqrt{\cos(u)}+\sqrt{\sin(u)}} du.$$

This last step is so because the sine of an angle is the same as the cosine of its compliment and vice-versa (this is the "aha moment"). We can therefore write

$$2I_1 = \int_0^{\pi/2} \frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)}+\sqrt{\cos(x)}}dx + \int_0^{\pi/2} \frac{\sqrt{\cos(x)}}{\sqrt{\cos(x)}+\sqrt{\sin(x)}}dx = \int_0^{\pi/2} \frac{\sqrt{\sin(x)}+\sqrt{\cos(x)}}{\sqrt{\sin(x)}+\sqrt{\cos(x)}}dx = \int_0^{\pi/2} dx = \frac{\pi}{2}.$$

Don't forget, the variable u is just a dummy variable and we can use any symbol we choose in its place (property #1). If we use x, the fact that our interval of integration has been preserved allows us to combine the two integrals into one and viola, our integral solves itself and our final result is

$$\boxed{\boldsymbol{I_1 = \int_0^{\pi/2} \frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)}+\sqrt{\cos(x)}}dx = \frac{\pi}{2^2}} \quad \text{Q.E.D}.}$$

Now that you've seen an example of how the IP property can be used let's discuss a further use of this interval preserving change of variable. Suppose that we have a definite integral of known value. Let's call that value V_1, that is $V_1 = \int_a^b f(x)dx$. Also suppose that we are confronted with solving $V_2 = \int_a^b xf(x)dx$. Interestingly enough, the IP property can always be used to crack V_2 given that we already know V_1. Let's do it. Let $x = a + b - u$, $dx = -du$, and $(a, b) \rightarrow (b, a)$ and we have

$$V_2 = -\int_b^a (a+b-u)f(a+b-u)du = \int_a^b (a+b)\,f(a+b-u)du - \int_a^b uf(a+b-u)du.$$

Now, focus on the two terms of the above expression to the right of the 2nd equal sign. The first term is simply $(a + b)V_1$ and the second term is V_2. So, the IP property gives us

$$2V_2 = (a+b)V_1 \quad \text{or} \quad V_2 = \left(\frac{a+b}{2}\right)V_1.$$

This result is basically a theorem, so let's call it the IP theorem and see how it works using the previous example, meaning we will attempt to solve the following:

Example 3-2. $\boldsymbol{I_2 = \int_0^{\pi/2} \frac{x\sqrt{\sin(x)}}{\sqrt{\sin(x)}+\sqrt{\cos(x)}}dx}$.

We know from working example 3-1 that $I_1 = \int_0^{\pi/2} \frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)}+\sqrt{\cos(x)}}dx = \frac{\pi}{4}$. Our IP theorem tells us that the value of I_2 is simply $I_2 = \frac{1}{2}\left(\frac{\pi}{2}\right)I_1$. We can simply write down the result without doing any work at all. That is

$$\boxed{\boldsymbol{I_2 = \int_0^{\pi/2} \frac{x\sqrt{\sin(x)}}{\sqrt{\sin(x)}+\sqrt{\cos(x)}}dx = \frac{\pi^2}{2^4}} \quad \text{Q.E.D.}}$$

Let's carry this little theorem a bit further. Once we know the value $\int_a^b xf(x)dx$, we can then use that result to obtain the value of $\int_a^b x^2 f(x)dx = \frac{a+b}{2}\int_a^b xf(x)dx = \left(\frac{a+b}{2}\right)^2 \int_a^b f(x)dx$. Generalizing this idea, we have the very delicious result that

$$\int_a^b x^n f(x)dx = \left(\frac{a+b}{2}\right)^n \int_a^b f(x)dx \quad n \,\epsilon\, \mathbb{N}^+$$

For the particular integrals in examples 3-1 and 3-2, this gives us

$$\int_0^{\pi/2} \frac{x^n\sqrt{\sin(x)}}{\sqrt{\sin(x)}+\sqrt{\cos(x)}}dx = \frac{\pi^{n+1}}{2^{2(n+1)}}$$

Example 3-3. $\boldsymbol{I_3 = \int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}dx}$.

Here we can solve this integral by solving the simpler integral (the integrand sans the numerator x-term) and then use the IP theorem to arrive at the final result. That is,

$$\int_0^{\pi} \frac{\sin(x)}{1+\cos^2(x)}dx = -\int_0^{\pi} \frac{d[\cos(x)]}{1+cos^2(x)} = -[\tan^{-1}(\cos x)]_0^{\pi} = -[\tan^{-1}(-1) - \tan^{-1}(1)] = \frac{\pi}{2}.$$

Thus, by our IP theorem, $I_3 = (\pi/2)(\pi/2) = \pi^2/4$. Formally, we write

$$\boxed{I_3 = \int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}dx = \frac{\pi^2}{4} \quad \text{Q.E.D.}}$$

Example 3-4. $\boldsymbol{I_4 = \int_0^{\pi/2} \frac{\sin^2(x)}{\sin(x)+\cos(x)}dx}$.

Again, let's use the IP property to see where it leads with I_4. Letting $x = \pi/2 - u$, so that $dx = -du$, and $(0, \pi/2) \rightarrow (\pi/2, 0)$, we obtain

$$I_4 = -\int_{\pi/2}^{0} \frac{\sin^2(\frac{\pi}{2}-u)}{\sin(\frac{\pi}{2}-u)+\cos(\frac{\pi}{2}-u)}du = \int_0^{\pi/2} \frac{\cos^2(u)}{\cos(u)+\sin(u)}du.$$

Therefore, we can write

$$2I_4 = \int_0^{\pi/2} \frac{\sin^2(x)}{\sin(x)+\cos(x)}dx + \int_0^{\pi/2} \frac{\cos^2(x)}{\cos(x)+\sin(x)}dx = \int_0^{\pi/2} \frac{dx}{\sin(x)+\cos(x)}.$$

Well, we certainly have a simpler integral, but this one still requires a bit more work before we can write down the final value. Another change of variable should help; let $y = \tan(x/2)$ meaning that $x = 2\tan^{-1}(y)$, $dx = 2dy/(1+y^2)$, and $(0, \pi/2) \rightarrow (0, 1)$. We now need to figure out what $\sin(x)$ and $\cos(x)$ are equal to in terms of our new variable, y. If we take the trigonometric identity $\tan(\frac{x}{2}) = \pm\sqrt{\frac{1-\cos(x)}{1+\cos(x)}}$ and solve it for $\cos(x)$, we get $\cos(x) = \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} = \frac{1-y^2}{1+y^2}$. Now we need a similar conversion for $\sin(x)$. We do know that $\sin(x) = 2\sin(\frac{x}{2})\cos(\frac{x}{2})$. If we both multiply and divide the right side of this expression by $\cos(x/2)$ we get $\sin(x) = 2\tan(\frac{x}{2})\cos^2(\frac{x}{2}) = \frac{2\tan(\frac{x}{2})}{\sec^2(\frac{x}{2})} = \frac{2\tan(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} = \frac{2y}{1+y^2}$. So, under this new variable, our integral becomes

$$2I_4 = \int_0^1 \frac{\frac{2}{1+y^2}}{\frac{2y}{1+y^2}+\frac{1-y^2}{1+y^2}}dy = 2\int_0^1 \frac{1}{1+2y-y^2}dy.$$

If we now complete the square on the denominator of the integrand we get $2 - (y - 1)^2$. And this is the "aha moment". A further expansion of the denominator into partial fractions should allow us to actually perform the integration and get a value for the integral. However, this "aha moment" is a bit uncommon. What I mean by that is It seems to me that the "aha moment" should have been when one sees that the change of variable of $y = \tan(x/2)$ is going to lead to a solution. Personally, I didn't see that. Making that change of variable was, to me, more of an experiment, i.e., is this change of variable going to lead anywhere? But, I didn't see my way clear to a solution until the last expression above. The denominator of the last integral above can be factored as the difference of two squares. That then gives us the idea of further expansion into partial fractions. Anyway, let's continue—factoring and then expanding the last integrand into partial fractions gives us

$$I_4 = \int_0^1 \frac{1}{2-(y-1)^2} dy = \int_0^1 \frac{1}{(\sqrt{2}+1-y)(\sqrt{2}-1+y)} dy = \frac{1}{2\sqrt{2}} \int_0^1 \frac{1}{\sqrt{2}+1-y} dy + \frac{1}{2\sqrt{2}} \int_0^1 \frac{1}{\sqrt{2}-1+y} dy.$$

Both integrals on the right side of the equal sign in the above expression can easily be doctored into recognizable forms that constitute the natural log function. That is,

$$I_4 = -\frac{1}{2\sqrt{2}} \int_0^1 \frac{d(\sqrt{2}+1-y)}{\sqrt{2}+1-y} + \frac{1}{2\sqrt{2}} \int_0^1 \frac{d(\sqrt{2}-1+y)}{\sqrt{2}-1+y}.$$

And, of course, we can now see that integration gives us

$$I_4 = \left[-\frac{1}{2\sqrt{2}} \log(\sqrt{2}+1-y)\right]_0^1 + \left[\frac{1}{2\sqrt{2}} \log(\sqrt{2}-1+y)\right]_0^1 = \frac{1}{2\sqrt{2}} \left[\log(\sqrt{2}+1) - \log(\sqrt{2}-1)\right].$$

By the property of logarithms, we combine these last two terms into one and we have the final value for the original integral—all as a result of that simple IP change of variable at the very beginning.

$$\boldsymbol{I_4 = \int_0^{\pi/2} \frac{\sin^2(x)}{\sin(x)+\cos(x)} dx = \frac{1}{2\sqrt{2}} \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)} \quad \textbf{Q.E.D.}$$

Example 3-5. $\boldsymbol{I_5 = \int_0^1 \frac{\log(x+1)}{x^2+1} dx}$

So far, in all of the examples showing how the IP property can lead to a solution, the IP change of variable was always the first step taken in the derivation. That is not necessarily always going to be the case, and this example illustrates that fact. In I_5, the $x^2 + 1$ in the denominator looks suspect to me. If we were to make a change of variable of $x = \tan(\theta)$, the denominator becomes $\sec^2(\theta)$ and so does dx. So the two cancel and that's just so convenient. Let's try it, remembering to also change the integration interval, that is, (0, 1) → (0, π/4).

$$I_5 = \int_0^{\pi/4} \frac{\log[\tan(\theta)+1]}{\tan^2(\theta)+1} [\sec^2(\theta) d\theta] = \int_0^{\pi/4} \log[\tan(\theta)+1] d\theta.$$

Now we've got an integrand that is composed of a trig function and we know that often the IP property can be helpful, so let's try that. Let $\theta = \pi/4 - u$, $d\theta = -du$, and (0, π/4) → (π/4, 0).

$$I_5 = -\int_{\pi/4}^0 \log\left[\tan\left(\frac{\pi}{4} - u\right) + 1\right] du = \int_0^{\pi/4} \log\left[\tan\left(\frac{\pi}{4} - u\right) + 1\right] du$$

Now, recall, from entry #9 in the table of useful trigonometric identities in Chapter 1, we have

$$\tan\left(\frac{\pi}{4} - u\right) = \frac{\tan\left(\frac{\pi}{4}\right) - \tan(u)}{1+\tan\left(\frac{\pi}{4}\right)\tan(u)} = \frac{1-\tan(u)}{1+\tan(u)}.$$

Thus, our integral becomes

$$I_5 = \int_0^{\pi/4} \log\left[\frac{1-\tan(u)}{1+\tan(u)} + 1\right] du = \int_0^{\pi/4} \log\left[\frac{1-\tan(u)+1+\tan(u)}{1+\tan(u)}\right] du = \int_0^{\pi/4} \log\left[\frac{2}{1+\tan(u)}\right] du.$$

Aha! It's that moment! Because of the property of logarithms, this last integral becomes

$$I_5 = \log(2) \int_0^{\pi/4} du - \int_0^{\pi/4} \log[1+\tan(u)] du = \frac{\pi}{4} \log(2) - I_5.$$

Magnificent!!! We have our final solution which is

$$I_5 = \int_0^1 \frac{\log(x+1)}{x^2+1} dx = \frac{\pi}{8}\log(2) \quad \text{Q.E.D.}$$

The magnificent derivation above (example 3-5) is due to Joseph Alfred Serret and the integral is known as Serret's integral. He published this solution in 1844 at the age 25 (Oh, to be 25 again). Following is a brief bio of him.

Figure 3-1. French Mathematician Joseph Serret (1819-1885)

Serret was admitted to the École Polytechnique in Paris in 1838 and, after two years of study, graduated in 1840. Some years later, he was appointed as an examiner at the College Sainte Barbe. At the same time he undertook research for a doctorate in mathematical sciences at the Faculty of Sciences in Paris. On 25 October 1847 he submitted two theses to the Faculty of Science for his doctorate; for his two theses, following an oral examination, Serret was awarded his doctorate in 1847 and, in the following year, he was appointed as an entrance examiner for the École Polytechnique; he held this position until 1862.

Serret did important work in Differential Geometry (and, in my opinion, Differential Geometry is one of the most difficult mathematical subjects!!!!); he made major advances in this topic. The fundamental formulae in the theory of space curves are the Frenet-Serret formulae. In Differential Geometry, the Frenet-Serret formulas describe the kinematic properties of a particle moving along a continuous differentiable curve in three-dimensional Euclidean space R_3 (see what I mean!!!!). Serret also published papers on number theory, calculus, the theory of functions, group theory, mechanics, differential equations and astronomy. However, he was best known during his lifetime as the author of a number of extremely well-received textbooks. He published *Cours d'algèbre supérieure* in 1849. The book, based on lectures he gave to the Faculty of Science in Paris. It contains a presentation of classical Galois theory (named after Evariste Galois—another French mathematician), which today forms much of the basis of modern cryptography. During his lifetime Serret was honored with election to the Paris Academy of Sciences and, after his death, he was honored with a Paris street named for him.

Example 3-6. $\boldsymbol{I_6 = \int_0^{\pi} \frac{x}{1+\cos(\alpha)\sin(x)} dx \;\; \alpha \in \mathbb{R}}$

Here is an example of an integral that can be evaluated using the IP tool as one step in a very long and complicated series of trigonometric manipulations to eventually arrive at a final result. Let's begin! Make the interval preserving change of variable right here at the beginning, that is, let $x = \pi - u$ so that $dx = -du$ and $(0, \pi) \rightarrow (\pi, 0)$. We then have

$$I_6 = -\int_{\pi}^{0} \frac{\pi - u}{1+\cos(\alpha)\sin(\pi-u)} du = \pi \int_0^{\pi} \frac{1}{1+\cos(\alpha)\sin(u)} du - I_6$$

Hence, transposing we obtain

$$2I_6 = \pi \int_0^{\pi} \frac{1}{1+\cos(\alpha)\sin(u)} du = \pi \int_0^{\pi} \frac{1}{1+2\cos(\alpha)\sin(\frac{u}{2})\cos(\frac{u}{2})} du\,.$$

Replacing $\sin(u)$ with $2\sin(u/2)\cos(u/2)$ in this last integral is a very non-intuitive step. This must have been the aha moment for whoever first did this derivation, however, it's difficult to be sure because wherever this approach is leading is certainly not clear, at least not yet. The same thing can also be said about the next few steps also. Nevertheless, the next step is to divide both the numerator and denominator of the integrand by $\cos^2(u/2)$. Doing that, we obtain

$$2I_6 = \pi \int_0^{\pi} \frac{\sec^2(\frac{u}{2})}{\sec^2(\frac{u}{2})+2\cos(\alpha)\tan(\frac{u}{2})} du = \pi \int_0^{\pi} \frac{\sec^2(\frac{u}{2})}{1+2\cos(\alpha)\tan(\frac{u}{2})+\tan^2(\frac{u}{2})} du.$$

Aha! Now I see it! And it's clever as can be. The 1 that is the first term in the denominator can be replaced by $\sin^2(\alpha) + \cos^2(\alpha)$ and then we will have the perfect square of binomial term in the denominator plus the square of a constant term and that makes the integrand start to look like the recognizable form of an inverse tangent function. This has promise! Continuing, we get

$$2I_6 = \pi \int_0^{\pi} \frac{\sec^2(\frac{u}{2})}{\sin^2(\alpha)+\cos^2(\alpha)+2\cos(\alpha)\tan(\frac{u}{2})+\tan^2(\frac{u}{2})} du = \pi \int_0^{\pi} \frac{\sec^2(\frac{u}{2})}{\sin^2(\alpha)+\left[\cos(\alpha)+\tan(\frac{u}{2})\right]^2} du\,.$$

Yes, indeed! If we now make the following change of variable $z = \cos(\alpha) + \tan(u/2)$ so that $dz = \frac{1}{2}\sec^2(u/2)$ and $(0, \pi) \rightarrow (\cos(\alpha), \infty)$. This CV gives us

$$2I_6 = \pi \int_{\cos(\alpha)}^{\infty} \frac{2dz}{\sin^2(\alpha)+z^2} = 2\pi \left[\frac{1}{\sin(\alpha)} \tan^{-1}\left(\frac{z}{\sin\alpha}\right)\right]_{\cos\alpha}^{\infty} = \frac{2\pi}{\sin(\alpha)}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right] = \frac{2\pi}{\sin(\alpha)}\left[\frac{\pi}{2} - \frac{\pi}{2} + \alpha\right]$$

As a result, our final answer is

$$\boxed{\boldsymbol{I_6 = \int_0^{\pi} \frac{x}{1+\cos(\alpha)\sin(x)} dx = \frac{\pi\alpha}{\sin(\alpha)}} \quad \text{Q.E.D.}}$$

There is something fascinating about science (and mathematics). One gets such wholesale returns of conjecture out of such a trifling investment of fact

—Mark Twain

Chapter 4. Integration By Parts (IBP)

This chapter is devoted to the integral property/technique termed integration by parts. Again, this is not a subject that should be foreign to the reader. It is generally taught in beginning Calculus, but is also a very valuable tool or technique in the evaluation of properly improper integrals. Here is the property as stated in the table of integral properties (Chapter 1—Table 4, entry #4):

$$\int_a^b f(x)dx = [uv]_a^b - \int_a^b vdu \quad \text{where } f(x) = u(x)v(x).$$

This equation is merely derived from the equation from differential calculus that deals with differentiating the product of two functions. In-other-words, if we start with the product of two functions designated as $u(x)$ and $v(x)$ and differentiate, here is what we get

$$\frac{df(x)}{dx} = \frac{d[u(x)v(x)]}{dx} = u'(x)v(x) + u(x)v'(x)$$

We can write this more simply as

$$d[f(x)] = d(uv) = ud(v) + vd(u)$$

If we now transpose and integrate, we get

$$\int_a^b udv = \int_a^b d(uv) - \int_a^b vdu$$

However, the middle integral above is simply the product of u and v evaluated at the integrals limits of integration, e.g.,

$$\int_a^b udv = [uv]_a^b - \int_a^b vdu.$$

So that's where that integration by parts formula comes from. To use this formula, how one divides the integrand up into a $u(x)$ and a $v(x)$ usually has to be decided. Often, there is more than one way to do it. If IBP is a viable methodology for a particular integral, the choice can very well impact whether or not one is able to evaluate the original integral or not. We will certainly see how it is done in the examples of this chapter.

Example 4-1. $I_1 = \int_0^\infty \frac{\log(1+x)}{x^{3/2}} dx$

Here, if IBP is the proper way to proceed, there are only two logical choices for u, that is, either u will be set equal to the numerator and dv the denominator (times dx), or visa-versa. Since the choice of dv must ultimately be integrable, it seems prudent to let $dv = x^{-3/2}dx$ and

$u = \log(1 + x)$. Under that assumption, let's continue. Letting $u = \log(1 + x) \Rightarrow du = dx/(1 + x)$ and $dv = x^{-3/2}dx \Rightarrow v = -2x^{-1/2}$. Therefore,

$$I_1 = \left[\frac{-2\log(1+x)}{\sqrt{x}}\right]_0^\infty + 2\int_0^\infty \frac{1}{\sqrt{x}(1+x)}dx = 2\int_0^\infty \frac{1}{\sqrt{x}(1+x)}dx$$

If you have trouble seeing that the first term in I_1 above approaches zero in both limits, use L'Hopital's rule to perform the evaluation. The remaining integral can easily be evaluated with a simple CV. Let $x = z^2$ so that $dx = 2zdz$ and $(0, \infty) \rightarrow (0, \infty)$. Hence,

$$I_1 = 2\int_0^\infty \frac{2z}{z(1+z^2)}dz = 4\int_0^\infty \frac{1}{1+z^2}dz = 4[\tan^{-1}(z)]_0^\infty = 2\pi.$$

Thus, our final result is

$$\boxed{\boldsymbol{I_1 = \int_0^\infty \frac{\log(1+x)}{x^{3/2}}dx = 2\pi} \quad \text{Q.E.D.}}$$

Example 4-2. $\boldsymbol{I_2 = \int_0^\infty \log\left(1 + \frac{a^2}{x^2}\right)dx \;\; a \in \mathbb{R}}$

In this very nice example, there is really only one choice for u, namely, let $u = \log(1 + a^2/x^2)$ and $dv = dx$. Proceeding,

$$u = \log\left(1 + \frac{a^2}{x^2}\right) = \log\left(\frac{x^2+a^2}{x^2}\right) = \log(x^2 + a^2) - \log(x^2) = \log(x^2 + a^2) - 2\log(x)$$

$$du = \frac{2xdx}{a^2+x^2} - \frac{2dx}{x} = -\frac{2a^2dx}{x(a^2+x^2)} \quad \text{and} \quad dv = dx \Rightarrow v = x$$

Therefore,

$$I_2 = \left[x\log\left(1 + \frac{a^2}{x^2}\right)\right]_0^\infty + 2a^2\int_0^\infty \frac{x}{x(a^2+x^2)}dx = 2a\left[\tan^{-1}\left(\frac{x}{a}\right)\right]_0^\infty = \pi a.$$

Our final result being,

$$\boxed{\boldsymbol{I_2 = \int_0^\infty \log\left(1 + \frac{a^2}{x^2}\right)dx = \pi a} \quad \text{Q.E.D.}}$$

Example 4-3. $\boldsymbol{I_3 = \int_0^1 x\log\left(1 + \frac{x}{2}\right)dx}$

Again, for IBP we have our choice for u and dv, and again the prudent choice is $u = \log(1 + x/2)$ and $dv = xdx$. With that choice, $du = dx/(2 + x)$ and $v = \frac{1}{2}x^2$. This gives us

$$I_3 = \left[\frac{1}{2}x^2\log\left(1 + \frac{x}{2}\right)\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{2+x}dx = \frac{1}{2}\log\left(\frac{3}{2}\right) - \frac{1}{2}\left[\int_0^1 \left(x - 2 + \frac{4}{x+2}\right)dx\right]$$

This last remaining integral is "duck soup." Upon integration and evaluation, we get

$$I_3 = \frac{1}{2}\log\left(\frac{3}{2}\right) - \frac{1}{2}\left[\frac{1}{2}x^2 - 2x + 4\log(x+2)\right]_0^1 = \frac{1}{2}\log\left(\frac{3}{2}\right) - \frac{1}{2}\left[\frac{1}{2} - 2 + 4\log(3) - 4\log(2)\right]$$

And finally, upon simplification, our final result is

$$\boxed{I_3 = \int_0^1 x \log\left(1 + \frac{x}{2}\right) dx = \frac{3}{4}\left[1 - 2\log\left(\frac{3}{2}\right)\right] \quad \text{Q.E.D.}}$$

Example 4-4. $\boldsymbol{I_4 = \int_0^{1/\sqrt{2}} \frac{\sin^{-1}(x)}{(1-x^2)^{3/2}} dx}$

This example looks complex, but it's a dead giveaway! Integrating by parts with $u = \sin^{-1}(x)$ will result in, upon differentiating to get du, a term with a square root in the denominator and integrating with $dv = (1 - x^2)^{3/2}dx$ will also put an equal square root term in the denominator. Upon setting up the IBP equation, those two terms will multiply together thereby rationalizing the denominator and giving us an integral that is manageable—aha! Therefore, let $u = \sin^{-1}(x)$ so that $du = \frac{1}{\sqrt{1-x^2}}dx$ and let $dv = \frac{dx}{(1-x^2)^{3/2}}$ so that $v = ????$ Whoops, my aha moment has turned to an "uh-huh". Can't do it that way—can't integrate that dv! The problem is not such a dead giveaway after all! This problem comes from a massive 1900 page, two volume set of books by Joseph Edwards entitled "A Treatise on the Integral Calculus". It was an unworked exercise problem at the end of the chapter on integration by parts. After re-thinking the problem, I've had a second aha! Look at the upper limit of integration—it's the dead giveaway and I should have realized it immediately. The upper limit, $1/\sqrt{2}$, is the ratio that the side of a 45-45-90 right triangle would have to the hypotenuse. That is just too much of a coincidence—it implies to me that the integral requires a CV before any IBP is involved for if we let $x = \sin(\theta)$ so that $dx = \cos(\theta)d\theta$, then when we calculate the change to the integration interval we get a nice mapping, i.e., the interval becomes $(0, 1/\sqrt{2}) \to (0, \pi/4)$. We also have $\sin^{-1}(x) = \theta$ and $(1 - x^2)^{3/2} = \cos^3(\theta)$. Thus,

$$I_4 = \int_0^{\pi/4} \frac{\theta\cos(\theta)}{\cos^3(\theta)} d\theta = \int_0^{\pi/4} \theta\sec^2(\theta)d\theta.$$

Now it's time for the IBP. Let $u = \theta$ so that $du = d\theta$. Let $dv = \sec^2(\theta)$ so that $v = \tan(\theta)$. So, our IBP equation becomes

$$I_4 = [\theta\tan(\theta)]_0^{\pi/4} - \int_0^{\pi/4} \tan(\theta)d\theta = \frac{\pi}{4} + [\log\{\cos(\theta)\}]_0^{\pi/4} = \frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right)$$

Hence, our final result is

$$\boxed{I_4 = \int_0^{1/\sqrt{2}} \frac{\sin^{-1}(x)}{(1-x^2)^{3/2}} dx = \frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} - \frac{1}{2}\log(2) \quad \text{Q.E.D.}}$$

Example 4-5. $\boldsymbol{I_5 = \int_0^{\infty} e^{-ax}\cos(bx)dx \;\; a, b \in \mathbb{R}}$

Most students should already be familiar with this integral. It is usually solved in elementary calculus class. To solve it, one does an integration by parts twice which results in a linear equation in I_5, which can then be solved for I_5, giving us the integral's value. (By-the-way, this methodology of obtaining a linear equation which when solved, gives the integral's value is certainly an example of non-conventional integration, which, of course, is the subject of this book.) Continuing, let $u = e^{-ax}$ so that $du = -ae^{-ax}dx$ and let $dv = \cos(bx)dx$ so that $v = (1/b)\sin(bx)$. Therefore,

$$I_5 = \left[\frac{e^{-ax}\sin(bx)}{b}\right]_0^{\infty} + \frac{a}{b}\int_0^{\infty} e^{-ax}\sin(bx)dx = \frac{a}{b}\int_0^{\infty} e^{-ax}\sin(bx)dx.$$

Now, a second integration by parts with $u = e^{-ax}$ and $dv = \sin(bx)$, we get

$I_5 = \frac{a}{b}\left\{\left[\frac{-e^{-ax}\cos(bx)}{b}\right]_0^{\infty} - \frac{a}{b}\int_0^{\infty} e^{-ax}\cos(bx)dx\right\} = \frac{a}{b}$In an analogous manner, the integral $I_6 = \int_0^{\infty} e^{-ax}\sin(bx)dx$ can be solved, that is, d$\left[\frac{1}{b} - \frac{a}{b}I_5\right]$

As a result, the final value is

$$\boldsymbol{I_5 = \int_0^{\infty} e^{-ax}\cos(bx)dx = \frac{a}{a^2+b^2}} \quad \textbf{Q.E.D.}$$

Doing two integrations by parts and then solving the resulting equation for I_6.

$$\boldsymbol{I_6 = \int_0^{\infty} e^{-ax}\sin(bx)dx = \frac{b}{a^2+b^2}} \quad \textbf{Q.E.D.}$$

Example 4-6. $\boldsymbol{I_7 = \int_0^1 \sin^{-1}(x)dx}$

This pesky little integral gave me a week's worth of trouble. I had mistakenly thought that its solution illustrated the technique of IP (Interval Preservation) so well that I was willing to use it in the IP chapter regardless of the fact that the damn integral was not even improper. How wrong I was. My first mistake was that I immediately thought change of variable, i.e., $x = \sin(u)$. Well, that got me in all kinds of trouble; I won't go into the details, however having spent so much time working on this simple problem, I'll be damned if I'll leave it out of the book now.

Let $u = \sin^{-1}(x)$ and $dv = dx$. Then, $du = dx(1 - x^{2)-1/2}$ and $v = x$. Hence,

$$I_7 = \int_0^1 \sin^{-1}(x)\,dx = [x\cdot\sin^{-1}x]_0^1 - \int_0^1 \frac{xdx}{\sqrt{1-x^2}} = \frac{\pi}{2} - \int_0^1 \frac{xdx}{\sqrt{1-x^2}}$$

Now, a CV of $x = \sin(\theta)$ with $dx = \cos(\theta)d\theta$ and $(0, 1) \to (0, \pi/2)$ should take care of this final integral, that is

$$I_7 = \frac{\pi}{2} - \int_0^{\pi/2} \frac{\sin\theta\cos\theta d\theta}{\sqrt{1-\sin^2(\theta)}} = \frac{\pi}{2} - \int_0^{\pi/2} \frac{\sin\theta\cos\theta d\theta}{\cos\theta} = \frac{\pi}{2} - \int_0^{\pi/2} \sin\theta d\theta = \frac{\pi}{2} - [-\cos\theta]_0^{\pi/2} = \frac{\pi}{2} - 1$$

And our final solution is

$$\boldsymbol{I_7 = \int_0^1 \sin^{-1}(x)\,dx = \frac{\pi}{2} - 1} \quad \textbf{Q.E.D.}$$

It is a mathematical fact that fifty percent of all doctors graduate in the bottom half of their class.

−Unknown

Chapter 5. Differentiation Under the Integral (DUI)

This chapter is devoted to the property designated as differentiation in Table 4 of chapter 1. Here is the property as stated in the table: If $I = \int_a^b f(x,q)dx$ then $\frac{dI}{dq} = \int_a^b \frac{\partial f(x,q)}{\partial q} dx,\ a,b$ not functions of q. It looks very complex and mysterious, but it is not. We will subsequently address just what this means, why we want to do it, and how it works. But first let me digress slightly before we get into it. I have heard that this technique is sometimes referred to as Feynman Integration—named after the late physics Nobel Prize recipient Richard Feynman (1918 – 1988). Here is a passage from his book, "Surely You're Joking Mr. Feynman" that explains why some refer to it as Feynman Integration.

I had learned to do integrals by various methods shown in a book that my high school physics teacher Mr. Bader had given me. [It] showed how to differentiate parameters under the integral sign. It turns out that it's not taught very much in the universities; they don't emphasize it. But I caught on how to use that method and I used that one damn tool again and again. [If] guys at MIT or Princeton had trouble doing a certain integral, [then] I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools before giving the problem to me. — Richard Feynman

Later, in the same book, Feynman writes of an incident that occurred during his work on the Atomic Bomb project in New Mexico. One of his co-workers was stumped by an integral that needed to be solved and this co-worker and his colleagues had been trying to solve it for some time without success.

When one of the guys was explaining [his] problem, I said, "Why don't you do it by differentiating under the integral sign?" In half an hour he had it solved, and they'd been working on it for three weeks. So I did something, using my different box of tools.
– Richard Feynman

Actually, this technique predates Feynman by approximately 3 centuries. It was developed by Leibniz (1646 – 1716) in the late 17^{th} century. We will state the theorem that deals with it, verbally describe how it can be used, and then give a number of examples of the way it works.

Theorem: Let $f(x,y)$ be a function such that both $f(x,y)$ and its partial derivatives with respect to x are continuous in both x and y in some region of the (x,y)-plane, including $a(x) \leq y \leq b(x)$, and $x_0 \leq x \leq x_1$. Also suppose that the functions $a(x)$ and $b(x)$ are both continuous and both have continuous derivatives for $x_0 \leq x \leq x_1$. Then, for the interval $x_0 \leq x \leq x_1$, we have

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,y)dy\right) = f[x,b(x)]\cdot b'(x) - f[x,a(x)]\cdot a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,y)dy$$

where x is the parameter we are trying to differentiate with respect to, and y is the variable of integration. This statement is the general and modern form of the Leibniz integral rule and can be derived using the fundamental theorem of calculus. It looks very daunting and complicated, but it assumes that the limits of the interval of integration are also a function of the parameter that the derivative is to be taken with respect to. In our use of this theorem, the limits of the interval of integration will almost always be constants. As a result, the first two terms on the right side of the equation will be zero and the rule can be stated simply as:

$$\frac{d}{dx}\left(\int_{a}^{b} f(x,y)\,dy\right) = \int_{a}^{b} \frac{\partial}{\partial x} f(x,y)dy$$

In other words, in order to differentiate the integral with respect to the parameter x, simply take the partial derivative with respect to x of the integrand, thereby treating the variable of integration, y, as a constant. In still other words, it is perfectly alright to interchange the order of the operations of differentiation and integration, that is, to differentiate the integral, we simply differentiate the integrand. SIMPLE!

Wait a minute you may say! We are trying to integrate—not differentiate, what's going on? How can we solve integrals by differentiating? It sounds paradoxical! Now, let me verbally explain why we want to differentiate a definite integral in order to determine its ultimate value. Consider the definite integral $I = \int_a^b f(x)dx$. If this integral converges, it represents a constant value and the variable x has no meaning; it is a dummy variable as we explained in Chapter1. We can turn the expression into a function however by inserting a new variable into the integrand. Let us call this new variable q; when we do this, we basically have $I(q) = \int_a^b f(x,q)dx$. The whole idea now, is to differentiate this expression with respect to the new variable, q, that is, $I'(q) = \int_a^b \frac{\partial}{\partial q} f(x,q)\,dx$, under the assumption that when we do, the resulting integral can then be integrated (with respect to the variable x). If it can, it will result in a differential equation in the variable q. If we can then solve that differential equation, it will give us $I(q)$ (sans an integration operation), but with an unknown constant, C. If C can be determined by some initial condition (such as when q has some value that we know or can determine), then we will have a solution to the original integral. Sounds simple, but of course there can be pitfalls. However, the technique also has great versatility and hopefully you will see later in the book how often this technique can be used. One more thing before we take a look at some examples. Some definite integrals may have integrands that are already a function of a parameter and insertion of a parameter is not necessary. However, if insertion is required to use this technique, it can be expedient to use some foresight when inserting the variable q into the integrand. Once inserted, the next step is to differentiate with respect to q with the expectation of being able to evaluate the resulting integral. The insertion can, more than likely, be done in a variety of ways, i.e., by multiplying some argument in the integrand by q, replacing some exponent in the integrand with q, adding q as the argument of some other function, etc. Foresight can assure one of being able to perform the integration after the differentiation. This will hereinafter be referred to as "insertion insight." Examples follow and hopefully will make this all clear. Oh again, one more thing before we turn to the examples. You differentiate with respect to the integral's parameter with the expectation that the result can be integrated. If it cannot, there is no reason why you cannot differentiate again; maybe then the result will integrate. Of course the resulting differential equation would then be of second order, but if it can be solved—do it!

Example 5-1. $\boldsymbol{I_1 = \int_0^1 \frac{x^2-1}{\log x} dx}$.

It would be very convenient if after insertion of q and differentiating with respect to q we get a log(x) term in the numerator of the integrand. Of course that would cancel with the like term in the denominator giving us a relatively simple integral to evaluate (this is our insertion insight). Sure enough, replacing the exponent 2 in the numerator with q gives us exactly what we want. Hence,

$$I_1(q) = \int_0^1 \frac{x^q-1}{\log x} dx.$$

First note that $I_1(2)$ is the value we are ultimately interested in and that $I_1(0) = 0$, can be used to determine the constant of integration that results from solving the subsequent differential equation. Now, performing the differentiation with respect to q, gives us the following

$$\frac{dI_1}{dq} = \int_0^1 \frac{x^q \log x}{\log x} dx = \int_0^1 x^q dx = \left[\frac{x^{q+1}}{q+1}\right]_0^1 = \frac{1}{q+1}$$

Notice that the next to the last term above comes from recognizable form 3 and evaluates to the $1/(q+1)$ term. The last term on the right when we integrate to solve the differential equation will be recognizable form 4. So, solving this differential equation, we have

$$\int dI_1 = \int \frac{dq}{q+1} \text{ or } I_1(q) = \log(q+1) + C$$

where C is the constant of integration. We know that $I_1(0) = 0$ and this allows us to evaluate the constant C, namely $C = 0$. Further, we are interested in the value of $I_1(2)$, the value of the exponent in the original integral, and hence, $I_1(2) = \log(2+1)$. Therefore, we can now state with assurance that

$$\boxed{\boldsymbol{I_1 = \int_0^1 \frac{x^2-1}{\log x} dx = \log 3 \quad \text{Q.E.D.}}}$$

The steps necessary to arrive at the value of I_1 above were rather straight forward and simple. Keep in mind that to successfully apply this method, some experimentation may be required to discover the appropriate insertion—in other words insertion insight is helpful but not always obvious. Additionally, after insertion and differentiation with respect to q, some further manipulation of the integrand may be necessary in order integrate $I'(q)$. Also keep in mind that in some cases it is possible to dream up an insertion that does indeed allow the resulting integral (after differentiation) to be solved, but the resulting differential equation cannot be solved. Even more frustrating is when the differential equation can be solved but no initial conditions can be found in order to determine the constant of integration (so close and yet not close enough). These are some of the pitfalls I alluded to previously. By-the-way, note that by inserting q and substituting it for the exponent 2 in the previous example, a much more general integral results, whose evaluation we have also obtained, namely, $\log(q + 1)$.

Example 5-2. $\boldsymbol{I_2 = \int_0^\infty \cos(ax)e^{-b^2x^2}dx \;\; a, b \in \mathbb{R}}$

Here we have an integral that already has a parameter, so no insertion is required if DUI is going to be the proper methodology. And, aha, I see that it is. When we differentiate with respect to the parameter, a, it will result in the integrand being multiplied by x and that should make an integration by parts quite applicable. Let's give it a try and let us call I_2, $I_2(a)$, to remind us that we are going to differentiate with respect to a (i.e., that I_2 is a function of a).

$$\frac{dI_2(a)}{da} = -\int_0^\infty x\sin(ax)e^{-b^2x^2}dx$$

For the integration by parts, let $u = \sin(ax)$ so that $du = a\cos(ax)dx$ and let $dv = -xe^{-b^2x^2}dx$ so that $v = \frac{1}{2b^2}e^{-b^2x^2}$. We then have

$$\frac{dI_2(a)}{da} = \left[\frac{1}{2b^2}\sin(ax)e^{-b^2x^2}\right]_0^\infty - \frac{a}{2b^2}\int_0^\infty \cos(ax)e^{-b^2x^2}dx$$

The first term on the right above vanishes and the integral is our original $I_2(a)$. So, we have the simple differential equation

$$\frac{dI_2(a)}{da} = -\frac{a}{2b^2}I_2(a) \quad \text{or} \quad \frac{dI_2(a)}{I_2(a)} = -\frac{a}{2b^2}da$$

This equation is easily solved as

$$\log[I_2(a)] = -\frac{a^2}{4b^2} + C$$

where C is the constant of integration, yet to be determined. But first, let us solve for $I_2(a)$.

$$I_2(a) = e^{-\frac{a^2}{4b^2}+C} = e^{-\frac{a^2}{4b^2}}e^C = Ke^{-\frac{a^2}{4b^2}}$$

Where $K = e^C$, just a constant, yet to be determined. I said at the beginning of this chapter, that after the differential equation is solved, one needs an initial condition in order to determine the constant of integration. Look at $I_2(0)$ for an initial condition in this case, that is,

$$I_2(0) = \int_0^\infty e^{-b^2x^2}dx.$$

Does this look familiar? It should, except for the b^2 term in the exponent of e, it's the integral in the preface that dumbfounded me as a college freshman. The value given in the preface is $\sqrt{\pi}/2$ and the value with the addition of the b^2 in the exponent gives this a value of $\sqrt{\pi}/2b$. Therefore we have our needed initial condition. Of course, we haven't derived this result, we have just stated it. Let's accept it for now, and we will derive it in a subsequent chapter. Anyway, with that as our initial condition, solving for K gives us $K = \sqrt{\pi}/2b$ and our final integral value is

$$\boldsymbol{I_2 = \int_0^\infty \cos(ax)e^{-x^2}dx = \frac{\sqrt{\pi}}{2b}e^{-a^2/4b^2}} \quad \textbf{Q.E.D.}$$

Isn't this DUI technique fantastic—it makes for relatively easy solutions to integrals that would, by other means, be very difficult? I can't imagine why it's not taught in school curriculums. Further, in the preface I mentioned that sometimes the solutions to these integrals took on a rather exotic or mysterious quality. Look what happens if we assign the value of $b = ½$ and $a = \sqrt{2}/2$ to the two parameters in I_2 above. $I_2 = \sqrt{\pi/e}$. WOW! What could be more exotic than the square root of the ratio of two of the most fundamental constants of mathematics?

Example 5-3. $\boldsymbol{I_3 = \int_0^\infty \frac{\sin(ax)}{x}dx \quad a \in \mathbb{R}}$

In this example, we are going to study an integral known as the Dirichlet Discontinuous Integral. Evidently, there are a number of integrals that Dirichlet studied in relation to his research into the attraction of ellipsoids and each of them is referred to as Dirichlet's Discontinuous Integral. The one we are going to address is I_3 above. This integral, however, appeared earlier in the works of Fourier, Poisson, and Legendre. But first let's discuss a little bit about the man himself, Johann Peter Gustav Lejeune Dirichlet (pronounced "Dear-a-clay).

Lejeune Dirichlet's family was from Belgium, although Dirichlet was born in Germany and was therefore a German citizen. By the age of 16, Dirichlet completed his school qualifications

German Mathematician Johann Peter Gustav Lejeune Dirichlet (1805-1859)

and was ready to enter university. However, the standards in German universities were not high at this time so Dirichlet decided to study in Paris; this was a very fortuitous decision on Dirichlet's part because it brought him in contact with some of the leading mathematicians of the time; this included Fourier, Laplace, Legendre and Poisson, to name just a few. Not too many years later, the standards in German universities became the best in the world, thanks in part, to Dirichlet himself.

Dirichlet was a hit in Paris. His first original research brought him instant fame for it concerned the famous Fermat's Last Theorem. This theorem states that no three positive integers a, b, and c can satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than two. This theorem was first conjectured by Pierre de Fermat in 1637 in the margin of his copy of *Arithmetica* where he claimed he had a proof that was too large to fit in the margin. There is no doubt today that Fermat was mistaken because the first successful proof was not released until 1994 by Andrew Wiles, and formally published in 1995, after 358 years of effort by many mathematicians; Wiles' proof makes use of mathematics that were unknown in Fermat's time. However, back in Dirichlet's time, it was an unsolved problem that was already approximately 200 years old. The cases $n = 3$ and $n = 4$ had been proved by Euler, and Dirichlet attacked the theorem for $n = 5$, which he subsequently proved. As alluded to above, it brought Dirichlet immediate fame as it was the first advance in the theorem since Euler's proof. This theretofore unsolved problem stimulated the development of algebraic number theory in the 19th century and the proof of the modularity theorem in the 20th century. It is among the most notable theorems in the history of mathematics and prior to its proof it was in the Guinness Book of World Records for "most difficult mathematical problems".

Dirichlet's fame spread and as a result, in 1831, Dirichlet was appointed to the Berlin Academy and an improving salary from the university put him in a position to marry. He married Rebecca Mendelssohn, one of the sisters of composer Felix Mendelssohn. Today, there is a crater on the moon named for Dirichlet and also an asteroid.

Dirichlet is credited with being the first mathematician to give the modern formal definition of a function; while trying to gauge the range for which convergence of the Fourier series can be shown, Dirichlet defines a function by the property that "to any x there corresponds a single finite y." This work undoubtedly led him to dream up a function which today is known as the (what else) Dirichlet Function (not to be confused with his integral). There are functions that can be made-up that do not bound an area and are therefore not integrable in the Riemannian sense. Probably, the most famous of these functions is this one due to Dirichlet which is defined as

$$D(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$$

This function is discontinuous everywhere! You cannot graph it, for in any interval of finite length there are an infinite quantity of both rational and irrational numbers and so Dirichlet's function is an extremely busy one, ping-ponging back and forth from 0 to 1 like, as a knowledgeable professor I know of once said, "an over-caffeinated frog on a sugar-high"; I prefer as a caffeinated cricket on cocaine (same idea but the alliteration is appealing).

Enough about this famous mathematician; let's get back to his discontinuous integral:

$$I_3 = \int_0^\infty \frac{\sin(ax)}{x} dx.$$

The first thing one might ask oneself is "why is this integral termed discontinuous?" Examine the integrand and realize that if the parameter $a = 0$, the value of the integral is zero over the entire range of integration. If it should turn out that when we eventually evaluate the integral that its value is non-zero at non-zero values of a and independent of a, then obviously a discontinuity exists at zero; and, indeed, this does turn out to be the case as we shall soon see.

There are many ways to attack this integral, and we shall examine a few of them. If we were to attempt to use the DUI methodology by differentiating with respect to the parameter a and attempt to take that calculation to its logical conclusion, we would end up having to evaluate $\int_0^\infty \cos(ax)dx$. This integral is indeterminate in the sense that it cannot be evaluated at the upper limit of integration. Therefore differentiating with respect to a appears to be a dead end. So if the DUI methodology is to be used, it becomes obvious that an entirely new parameter (i.e., q) will have to be somehow inserted into the integrand. Trying to practice a little insertion insight, it would surely be less complicated if the differentiation with respect to q after insertion creates a numerator with a term that cancels with the denominator. Here is the winning insertion insight; multiply the integrand by e^{-qx}. When we differentiate with respect to q, the x term in the denominator will cancel with an x term in the numerator and we will be left with an integrand of $-e^{-qx} \cdot \sin ax$ and that is an integral that we have already dealt with in the chapter on IBP. Further, the integrated result will have an e^{-qx} term in the numerator and this will go to zero when evaluated at the upper integration limit. We will still have to solve the resulting differential equation and then find an initial condition so we can evaluate the constant of integration. Nevertheless, let's give it a try.

$$I_3(q) = \int_0^\infty \frac{e^{-qx}\sin(ax)}{x} dx$$

$$\frac{dI_3(q)}{dq} = -\int_0^\infty e^{-qx}\sin(ax)\, dx$$

And, as stated above, we know the value from our IBP chapter, which is

$$\frac{dI_3(q)}{dq} = -\frac{a}{a^2+q^2}$$

Solving this differential equation, we have

$$I_3(q) = -a\left[\frac{1}{a}\tan^{-1}\left(\frac{q}{a}\right)\right] + C = C - \tan^{-1}\left(\frac{q}{a}\right)$$

where C is the arbitrary constant that results from solving the differential equation. We can calculate C by realizing that in the original formulation, $I_3(q) \to 0$ as $q \to \infty$ (i.e., the e^{-qx} factor in the integrand goes to zero everywhere as $q \to \infty$ because over the entire interval of integration $x \geq 0$). Thus,

$$C = \tan^{-1}(\pm\infty) = \pm\frac{\pi}{2}$$

where we use the plus sign if $a > 0$ and the negative sign if $a < 0$. And, since we already know that when $a = 0$, I_3 also equals zero, we have the stunning result,

$$\boxed{I_3 = \int_0^\infty \frac{\sin(ax)}{x} dx = \begin{cases} \pi/2 & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -\pi/2 & \text{if } a < 0 \end{cases} \quad \text{Q.E.D.}}$$

Example 5-4. $I_4 = \int_0^\pi \log[a + b\cos(x)]dx, \quad a > b \quad a \,\&\, b \in R^+$

Here's an integral that already has two parameters. If DUI is applicable, which parameter do we differentiate with respect to? Well, it will certainly be simpler if the answer to that question is the parameter a. Let's see what happens—we will call the integral $I_4(a,b)$.

$$\frac{dI_4(a,b)}{da} = \int_0^\pi \frac{1}{a+b\cos(x)} dx$$

A change of variable is now called for in order to evaluate the above. Let $u = \tan(x/2)$ so that $du = ½\sec^2(x/2)$ and $(0, \pi) \to (0, \infty)$. Additionally, $\cos(x) = (1 - u^2)/(1 + u^2)$ while $dx = 2du/(1 + u^2)$.

$$\frac{dI_4(a,b)}{da} = \int_0^\infty \frac{1}{a+b\left(\frac{1-u^2}{1+u^2}\right)} \left(\frac{2}{1+u^2}\right) du = 2\int_0^\infty \frac{1}{a(1+u^2)+b(1-u^2)} du$$

Rearranging the denominator of this last expression gives us

$$\frac{dI_4(a,b)}{da} = 2\int_0^\infty \frac{1}{a+b+u^2(a-b)} du = \frac{2}{a-b}\int_0^\infty \frac{1}{\frac{a+b}{a-b}+u^2} du$$

This last integral is merely the recognizable form of the inverse tangent function. Hence,

$$\frac{dI_4(a,b)}{da} = \frac{2}{a-b}\frac{1}{\sqrt{\frac{a+b}{a-b}}}\tan^{-1}\left[\frac{u}{\sqrt{\frac{a+b}{a-b}}}\right]_0^\infty = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[u\sqrt{\frac{a-b}{a+b}}\right]_0^\infty = \frac{\pi}{\sqrt{a^2-b^2}}$$

So, our differential equation is

$$dI_4(a,b) = \frac{\pi da}{\sqrt{a^2-b^2}} \quad \text{or} \quad I_4(a,b) = \pi\int \frac{da}{\sqrt{a^2-b^2}} + C.$$

So, to arrive at a final value, all we have to do is integrate this last integral and then find an initial condition to figure out the value of C, the constant of integration. To do this last integral requires a change of variable. Let $a = b\csc(\theta)$ so that $da = -b\csc(\theta)\cot(\theta)$. Under this change of variable the integral becomes

$$-\pi\int \frac{b\csc(\theta)\cot(\theta)}{b\cot(\theta)} d\theta = -\pi\int \csc(\theta)\, d\theta = -\pi\log[\csc(\theta) - \cot(\theta)]$$

Because this is an indefinite integration, we must change back to the original variable. Of course, csc(θ) = a/b and $\cot(\theta) = \frac{1}{b}\sqrt{a^2 - b^2}$. Hence,

$$I_4(a,b) = -\pi \log\left[\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b}\right] + C = -\pi \log\left[\frac{a-\sqrt{a^2-b^2}}{b}\right] + C$$

This last term can be algebraically manipulated to obtain

$$I_4(a,b) = \pi \log\left(a + \sqrt{a^2 - b^2}\right) + K$$

where K is just a different constant then in the previous expression. And now we need to see if we can figure out the value of K. When $b = 0$, the original integral becomes $I_4(a,0) = \int_0^\pi \log(a) dx$. This integrates to πlog(a) and can therefore be used to determine K. That is,

$$\pi \log(a) = \pi \log(2a) + K \quad \text{or} \quad K = \pi \log\left(\frac{1}{2}\right).$$

After all this calculation, we can write down the final result

$$\boldsymbol{I_4 = \int_0^\pi \log[a + b\cos(x)]dx = \pi \log\left(\frac{a + \sqrt{a^2 + b^2}}{2}\right)} \quad \textbf{Q.E.D}$$

Earlier in the chapter I said how easy this DUI technique could solve difficult integrals. That doesn't imply that all difficult integrals will be easy.

Example 5-5. $\boldsymbol{I_5 = \int_0^\infty \frac{\log(1+a^2x^2)}{b^2+x^2} dx, \quad a, b \in \mathbb{R}^+}$

Aha! This integral is going to be tough algebraically, but simple in terms of methodology. By differentiating with respect to the parameter a, we will end up with an integrand whose denominator is the product of two quadratic terms, namely, $(1 + a^2x^2)$ and $(b^2 + x^2)$. Partial fractions can then be used to give us recognizable inverse tangent forms or recognizable logarithm forms, depending on what is in the respective numerators of the partial fractions. Well, we will see—

$$\frac{dI_5(a)}{da} = \int_0^\infty \frac{2ax^2}{(1+a^2x^2)(b^2+x^2)} dx = \frac{2}{1-a^2b^2}\int_0^\infty \left(\frac{a}{1+a^2x^2} - \frac{ab^2}{b^2+x^2}\right) dx$$

Since the numerators contain no integration variables, inverse tangent forms shall govern. The best way to see this is to divide both numerator and denominator of the first fraction by a^2 and to leave the second fraction just as it is. Thus

$$\frac{dI_5(a)}{da} = \frac{2}{1-a^2b^2}\int_0^\infty \left(\frac{1}{a} \cdot \frac{1}{\frac{1}{a^2}+x^2} - ab^2 \cdot \frac{1}{b^2+x^2}\right) dx.$$

Now, upon integration, we have

$$\frac{dI_5(a)}{da} = \left[\frac{2}{1-a^2b^2}\tan^{-1}(ax)\right]_0^\infty - \left[\frac{2ab}{1-a^2b^2}\tan^{-1}\left(\frac{x}{b}\right)\right]_0^\infty .$$

This, upon evaluation at the upper and lower limits, becomes the differential equation that we still have to solve

$$\frac{dI_5(a)}{da} = \frac{2}{1-a^2b^2}\left(\frac{\pi}{2}\right) - \frac{2}{1-a^2b^2}\left(\frac{ab\pi}{2}\right) = \frac{\pi(1-ab)}{1-a^2b^2} = \frac{\pi}{1+ab}.$$

Solving, we obtain

$$I_5(a) = \frac{\pi}{b}\log(1+ab) + C$$

where C is the constant of integration. Note that $I(0) = 0$. (Look back at the original statement of the integral and let $a = 0$. The value of the integral is obviously zero.) Thus, $C = 0$ and we have our final result.

$$\boxed{\boldsymbol{I_5 = \int_0^\infty \frac{\log(1+a^2x^2)}{(b^2+x^2)}dx = \frac{\pi}{b}\log(1+ab)} \quad \textbf{Q.E.D.}}$$

Example 5-6. $\boldsymbol{I_6 = \int_0^\infty \frac{\tan^{-1}\left(\frac{x}{a}\right)}{x(x^2+b^2)}dx \;\; a, b \in \mathbb{R}^+}$

This example looks extremely complex, however, the aha moment occurs immediately. If we differentiate with respect to the parameter a, we will get an integrand denominator that is merely the product of two quadratic binomial expressions. The technique of partial fractions should then give us two elementary recognizable forms. Hopefully, the resulting differential equation can be solved. Let's give it a go—

$$\frac{dI_6(a)}{da} = \int_0^\infty \frac{x\left(-\frac{1}{a^2}\right)dx}{x\left(\frac{x^2}{a^2}+1\right)(x^2+b^2)} = -\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\int_0^\infty \left(\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right)dx$$

Sure enough, the two recognizable forms are, of course, inverse tangent functions. We therefore have

$$\frac{dI_6(a)}{da} = \frac{1}{a^2-b^2}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) - \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right]_0^\infty = \frac{1}{a^2-b^2}\left[\frac{\pi}{2a} - \frac{\pi}{2b}\right] = -\frac{\pi}{2ab(a+b)}$$

We now have the differential equation, but, at first glance, it doesn't look solvable—but then, a second aha moment. If we write the last expression as the following, it's "duck soup" to solve the equation.

$$\frac{dI_6(a)}{da} = \frac{\pi}{2b^2}\left(\frac{1}{a+b} - \frac{1}{a}\right)$$

Solving, we obtain

$$I_6(a) = \frac{\pi}{2b^2}\log\left(\frac{a+b}{a}\right) + C$$

where C is the constant of integration. We now need an initial condition and we can get it from the fact that when $a \to \infty$, $I(a) \to 0$ and

$$\lim_{a\to\infty} \log\left(\frac{a+b}{a}\right) = \lim_{a\to\infty} \log\left(\frac{1+b/a}{1}\right) \to \log(1) = 0$$

We can therefore conclude that $C = 0$ and our final result is

$$\boxed{\boldsymbol{I_6 = \int_0^\infty \frac{\tan^{-1}\left(\frac{x}{a}\right)}{x(x^2+b^2)} dx = \frac{\pi}{2b^2} \log\left(\frac{a+b}{a}\right)} \quad \textbf{Q.E.D.}}$$

Example 5-7. $\boldsymbol{I_7 = \int_0^{\pi/2} \frac{\log[1+\cos(\alpha)\cos(x)]}{\cos(x)} dx \quad \alpha \in \mathbb{R}}$

Since the current chapter is about DUI, one can surmise that I_7 can probably be solved by using DUI and differentiating with respect to the parameter α, and that would be correct. However, suppose you encountered this integral out of context of this DUI tutorial; in-other-words, in a setting where there is no clue as to which properly improper technique you should use to crack this integral. How do you decide which technique(s) to use? That is a difficult question and one that I don't have a nice patent answer for. I can only answer by saying "based upon 46 years of experience, here is what I would do!" DUI is one of the many integral solving tools that I possess in my bag of tricks. But, before deciding upon DUI as the methodology I would use, I exercise a little insight into what will happen to this integrand if I differentiate with respect to α. The argument of the log function will appear in the integrand's denominator multiplied by the cos(x) function that is already in the denominator and the numerator of the integrand will also contain a cos(x) due to differentiating the log's argument. Aha, those two cosine functions will cancel and that is very promising; the odds for DUI being applicable here have just increased. The next thing I would do is look for an initial condition that could be used to evaluate the constant of integration that will result from solving the differential equation if I use DUI. And indeed, there is such a condition, namely $I_7 = 0$ when $\alpha = \pi/2$; the odds for DUI being applicable here have just increased even more. Let's go ahead and try DUI!

$$\frac{dI(\alpha)}{d\alpha} = \int_0^{\pi/2} \frac{-\sin(\alpha)\cos(x)dx}{[1+\cos(\alpha)\cos(x)]\cos(x)} = -\sin(\alpha)\int_0^{\pi/2} \frac{dx}{1+\cos(\alpha)\cos(x)}$$

Aha, we've seen an integrand very similar to what we now have and it calls for a CV of $u = \tan(x/2)$. That is, $x = 2\tan^{-1}(u)$, $dx = 2du/(1+u^2)$, and $(0, \pi/2) \to (0, 1)$. We now have to figure out what cos(x) equals as a function of our new variable, namely, u. Note the following trigonometric identity

$$\cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \frac{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right)} = \frac{1 - \frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}}{1 + \frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}} = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \frac{1-u^2}{1+u^2}$$

As a result, we can now write

$$\frac{dI_7(\alpha)}{d\alpha} = -\sin(\alpha)\int_0^1 \frac{\frac{2du}{1+u^2}}{1+\cos(\alpha)\left(\frac{1-u^2}{1+u^2}\right)} = \frac{-2\sin(\alpha)}{1-\cos(\alpha)}\int_0^1 \frac{du}{\frac{1+\cos(\alpha)}{1-\cos(\alpha)}+u^2}$$

This last integral is now the recognizable form of the inverse tangent function. We therefore have

$$\frac{dI_7(\alpha)}{d\alpha} = \frac{-2\sin(\alpha)}{1-\cos(\alpha)}\cdot\frac{\sqrt{1-\cos(\alpha)}}{\sqrt{1+\cos(\alpha)}}\left[\tan^{-1}\left(\frac{u\sqrt{1-\cos(\alpha)}}{\sqrt{1+\cos(\alpha)}}\right)\right]_0^1 = \frac{-2\sin(\alpha)}{\sqrt{1-\cos(\alpha)}\sqrt{1+\cos(\alpha)}}\left[\tan^{-1}\left(\frac{\sqrt{1-\cos(\alpha)}}{\sqrt{1+\cos(\alpha)}}\right)\right]$$

Upon further simplification, we finally obtain a differential equation that when solved, will yield the solution we seek (and quite a simple differential equation it turns out to be).

$$\frac{dI_7(\alpha)}{d\alpha} = -2\tan^{-1}\left[\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}}\right] = -2\tan^{-1}\left[\tan\left(\frac{\alpha}{2}\right)\right] = -\alpha$$

Solving, we get

$$I_7(\alpha) = -\frac{1}{2}\alpha^2 + C$$

where *C* is an arbitrary constant of integration. But, our initial condition allows us to evaluate *C*.

$$I_7\left(\frac{\pi}{2}\right) = 0 \Longrightarrow C = \frac{\pi^2}{8}$$

We have our final value

$$\boldsymbol{I_7 = \int_0^{\pi/2} \log[1+\cos(\alpha)\cos(x)]\frac{dx}{\cos(x)} = \frac{1}{2}\left(\frac{\pi^2}{4}-\alpha^2\right)} \quad \textbf{Q.E.D.}$$

At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined there was anything so delicious in the world. From that moment, mathematics was my chief interest and my chief source of happiness.

—Bertrand Russell

Chapter 6. Interchange of Operations (IO)

This chapter is devoted to solving integrals by using the property designated "interchange of operations" in table 4 of chapter 1 (property #11). Here is what was shown in the table as an illustration of this property:

$$\int_a^b \left[\int_c^d f(x,y)dx\right] dy = \int_c^d \left[\int_a^b f(x,y)\, dy\right] dx$$

Clearly, the interchange we are alluding to here is the order in which a double integral is integrated—standard notation (see the left side of the above equation) dictates first integrating with respect to the variable inside the square brackets (in this case x) and only when that has been done, then integrate with respect to the outside variable (in this case y). However, in some cases it is legitimate to reverse that order and reversing that order can, sometimes, allow for the integration to be tractable when it was not to begin with. When is it legitimate? Fubini's theorem, introduced by Guido Fubini (1907), is a result that gives conditions under which it is possible to compute a double integral using iterated integrals. One may switch the order of integration if the double integral yields a finite answer when the integrand is replaced by its absolute value. As a consequence it allows the order of integration to be changed in iterated integrals.

We are dealing with properly improper integrals, none of which are double integrals. Why do we need this property of switching the order of integration? The answer is that a factor of the function to be integrated (i.e., a factor in the integrand) may, itself, be the result of a known integration between certain constant limits, that is, this factor is the result of a known definite integral. Upon substituting this known integral for the factor, a double integral is created. Sometimes it is possible to either interchange the order of integration in the double integral or transform the entire double integral itself to a new system of coordinates (such as from rectangular to polar) and obtain the sought-after value. The upcoming examples will certainly illustrate this concept and clarify the process.

Before doing so however, there is one more thing to address about this idea of interchange of operations. Entry #11 in the aforementioned table additionally has the statement "also applies to operations of summation and integration". What's that all about? One of the techniques that can be used for evaluating properly improper integrals involves expanding the integrand (or a portion thereof) into a power series and then integrating the power series term-by-term. This can be expressed in the following way:

$$\int_a^b f(x)dx = \int_a^b \sum_{k=0}^{\infty} f_k(x)dx = \sum_{k=0}^{\infty} \int_a^b f_k(x)dx$$

The mere fact that we wish to integrate term-by-term means we need to interchange the order of the integration operation with that of the summation operation. One of the elementary properties of integrals says that the integral of a sum is the sum of the integrals, however, when the sum itself is infinite (as all power series expansions are), how legitimate is this interchange? Fubini comes to the rescue again. The same theorem that applies to the order of integration of a double integral also applies to this situation. Now let's examine a few examples.

Example 6-1. $\boldsymbol{I_1 = \int_0^1 \frac{\log(1+x)}{x} dx}$

For a power series expansion of log(1 + x), consider the following: From the recognizable forms table, we know that

$$\int_0^x \frac{du}{1+u} = \log[1+u]_0^x = \log(1+x).$$

However, if you just divide 1 + u into 1 (do the actual long division) you obtain

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots = \sum_{k=0}^{\infty} (-1)^k u^k$$

Now, if we integrate from 0 to x both sides of the above equation, we can set whatever we get equal to log(1 + x) and viola, we have the sought after power series for the function log(1 + x). So, doing that we obtain

$$\log(1+x) = \int_0^x \frac{du}{1+u} = \int_0^x \left[\sum_{k=0}^{\infty} (-1)^k u^k\right] du$$

And now you should see why we need to change the order of operations. Mathematically, I want to integrate term-by-term, but that means interchange the summation operation with the integration operation. The mathematical question here is, if I do it in the order shown, will I get the same result if I interchange those two operations. If the answer is yes, and in this case it is, then I'm not making an error and I won't come to an incorrect evaluation when I attempt I_1 (at least not for reasons of the interchange). Continuing,

$$\log(1+x) = \sum_{k=0}^{\infty} (-1)^k \int_0^x u^k du = \sum_{k=0}^{\infty} (-1)^k \left[\frac{u^{k+1}}{k+1}\right]_0^x = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}.$$

Now we are prepared to attempt the solution to I_1. I'm going to substitute the power series I've just developed for the numerator of the integrand of I_1.

$$I_1 = \int_0^1 \frac{\log(1+x)}{x} dx = \int_0^1 \frac{1}{x} \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}\, dx$$

Now I need to do the interchange operation again, and doing so gives me

$$I_1 = \sum_{k=0}^{\infty} \int_0^1 \frac{(-1)^k}{x(k+1)} x^{k+1} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \int_0^1 x^k dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \left[\frac{x^{k+1}}{k+1}\right]_0^1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2}$$

Now examine Table 3 in Chapter 1, entry #3. The series that we have just shown is identical to the value of I_1 and converges to $\pi^2/12$. We therefore have our final result, namely

$$\boxed{\boldsymbol{I_1 = \int_0^1 \frac{\log(1+x)}{x} dx = \frac{\pi^2}{12}} \quad \textbf{Q.E.D.}}$$

Example 6-2. $\boldsymbol{I_2 = \int_0^1 \frac{\log^2(x)}{1+x^2}dx}$

The solution of this integral is required for one of my selections in the final chapter (the crème de la crème chapter), so we will be seeing this integral again. The best way of attacking this integral is to first make a change of variable. That is, let $x = e^{-t} \Rightarrow t = -\log(x)$, $dx = -e^{-t}dt$ and $(0, 1) \to (\infty, 0)$. Under this change of variable, our integral becomes

$$I_2 = -\int_\infty^0 \frac{t^2}{1+e^{-2t}}e^{-t}dt = \int_0^\infty \frac{t^2e^{-t}}{1+e^{-2t}}dt.$$

If we now expand the denominator into a power series (by simply performing the long division) we obtain

$$I_2 = \int_0^\infty t^2e^{-t}(1 - e^{-2t} + e^{-4t} - e^{-6t} + \cdots)dt = \int_0^\infty t^2\left[\sum_{k=0}^\infty e^{-(2k+1)t}\right]dt$$

Guess what's next—swapping the order of summation and integration, of course—and we obtain

$$I_2 = \sum_{k=0}^\infty \left[\int_0^\infty t^2e^{-(2k+1)t}dt\right]$$

The integral we are now left with can easily be evaluated using the integration by parts technique. Let $u = t^2$ so that $du = 2tdt$ and let $dv = e^{-(2k+1)t}dt$ so that $v = -e^{-(2k+1)t}/(2k+1)$. Therefore

$$I_2 = \sum_{k=0}^\infty \left\{\left[-\frac{t^2}{2k+1}e^{-(2k+1)t}\right]_0^\infty + \frac{2}{2k+1}\int_0^\infty te^{-(2k+1)t}dt\right\} = \sum_{k=0}^\infty \left[\frac{2}{2k+1}\int_0^\infty te^{-(2k+1)t}dt\right]$$

A second integration by parts with $u = t$ and $dv = e^{-(2k+1)t}dt$ gives us

$$I_2 = \sum_{k=0}^\infty \left\{\frac{2}{2k+1}\left[-\frac{t}{2k+1}e^{-(2k+1)t}\right]_0^\infty + \frac{1}{2k+1}\int_0^\infty e^{-(2k+1)t}dt\right\} = \sum_{k=0}^\infty \frac{2}{(2k+1)^2}\int_0^\infty e^{-(2k+1)t}dt$$

Finally, we can make the final integration to arrive at

$$I_2 = 2\sum_{k=0}^\infty \left\{\frac{1}{(2k+1)^2}\left[-\frac{1}{2k+1}e^{-(2k+1)t}\right]_0^\infty\right\} = 2\sum_{k=0}^\infty \frac{1}{(2k+1)^3}$$

Table 3, entry #7 of Chapter 1 tells us that this final sum is $\pi^3/32$. Hence our final result is

$$\boxed{\boldsymbol{I_2 = \int_0^1 \frac{\log^2(x)}{1+x^2}dx = \frac{\pi^3}{16}} \quad \textbf{Q.E.D.}}$$

Example 6-3. $\boldsymbol{I_3 = \int_0^{\pi/2} \cot(x)\log[\sec(x)]dx}$

To do this integral, we first need a change of variable. Let $y = \cos(x)$ so that $dy = -\sin(x)dx$ and our integration interval goes from $(0, \pi/2) \to (1, 0)$. We also have, $dx = -dy/\sqrt{1-y^2}$, $\log[\sec(x)] = \log(1/y)$, and $\cot(x) = y/\sqrt{1-y^2}$. So our integral, under this change of variable, becomes

$$I_3 = \int_1^0 \frac{y}{\sqrt{1-y^2}}\log\left(\frac{1}{y}\right)\left(-\frac{dy}{\sqrt{1-y^2}}\right) = -\int_0^1 \frac{y\log(y)}{1-y^2}dy$$

Now, writing the denominator of this last expression as a power series, we have

$$I_3 = -\int_0^1 y\log(y)(1+y^2+y^4+\cdots)dy = -\int_0^1 \log(y)\left(\sum_{k=0}^{\infty} y^{2k+1}\right)dy$$

Now comes the interchange of the summation operation and the integration operation, giving us

$$I_3 = -\sum_{k=0}^{\infty}\int_0^1 y^{2k+1}\log(y)dy$$

This last integral is easily evaluated using the integration by parts technique. Let $u = \log(y)$ so that $du = dy/y$ and let $dv = y^{2k+1}dy$ so that $v = y^{2(k+1)}/2(k+1)$.

$$I_3 = -\sum_{k=0}^{\infty}\left\{\left[\frac{y^{2(k+1)}\log(y)}{2(k+1)}\right]_0^1 - \frac{1}{2(k+1)}\int_0^1 \frac{y^{2(k+1)}}{y}dy\right\} = \sum_{k=0}^{\infty}\frac{1}{2(k+1)}\int_0^1 y^{2k+1}dy = \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}$$

This last sum converges to $\pi^2/6$ (see Table 3, entry #2, Chapter 1). The final value is therefore

$$\boldsymbol{I_3 = \int_0^{\pi/2} \cot(x)\log[\sec(x)]dx = \frac{\pi^2}{24}} \quad \textbf{Q.E.D.}$$

Example 6-4. $I_4 = \int_0^{\infty} e^{-x^2}dx$

Well, well, well—here's our old friend that dumfounded me as a college freshman and is ultimately responsible for the writing of this book. (If I hadn't gotten hooked on the puzzle aspect of non-conventional integration, I'm sure this book would never have occurred to me.) I've subsequently learned many methods of evaluating this integral, but the one I'd like to present here is the one I stumbled upon back in college. This is a very famous integral that was first evaluated in 1810 by the famous French mathematician Pierre-Simon Laplace (1749-1827). Although Laplace is given credit for first evaluating this integral, the name of Carl Friedrich Gauss (1777-1855) is usually associated with this integral because it forms the basis in probability theory of the Gaussian probability density function. As a result, I'm going to write a little bit about Gauss instead of Laplace (Laplace will be written about subsequently).

German mathematician Carl Friedrich Gauss (1777-1855)

"*Surely it is not knowledge, but learning; not owning, but earning; not being there, but getting there; that gives us the greatest pleasure.*"—Carl Friedrich Gauss

Gauss was to mathematics as Mozart was to music. History tells us that Mozart wrote a minuet at the age of four, while Gauss pointed out an arithmetical error in his father's payroll calculations at the age of three. His father, a bricklayer by trade, was adding up a long column of numbers and when writing down the sum he had calculated, his three-year old son told him he had made a mistake and that the sum was really this other value—and his three-year old son was correct. By the age of five, he was keeping his father's books. He could do this all mentally—he was a lightening calculator. His father wanted him to become a bricklayer also; fortunately for the world of mathematics, those plans never came to fruition. My favorite story about him was an incident that occurred in elementary school at the age of 7. The class had evidently been acting up and to punish the class, the teacher told every student to add up the numbers from 1 to 100. Gauss immediately put his head down on his desk and closed his eyes, as though he were sleeping. The teacher, seeing this said, "Young Gauss, have you completed the sum?" "Yes, I have" said Gauss and then proceeded to enunciate his total, namely 5050 (five thousand fifty). The teacher was stunned, as that was the correct total. What Gauss had done was recognized that within the first 100 integers, there are 50 unique pairs of numbers that total 101(namely, 1+100, 2+99, 3+98, … on up to 50+51) and therefore the correct total was simply the product of 50 and 101 which he could easily do in his head. Not much of a punishment when you are dealing with a genius.

At school, his cleverness attracted attention and eventually came to be known by the Duke of Brunswick who took an interest in young Gauss. (Brunswick is the town in which Gauss was born and currently lived with his parents.) If it weren't for this Duke, the world might very well have lost the genius of Gauss. Parental protests aside, the Duke sent young Gauss to the Collegium Carolinum and, in 1795, to Gottingen. At Gottingen, Gauss was exposed to influences that caused mathematics to become the study that he pursued the rest of his life.

I remember seeing a write-up on the life and times of Gauss wherein he was referred to as the "Prince of Mathematicians". He has had a remarkable influence in many fields of mathematics and science and is ranked as one of history's most influential mathematicians. However, being referred to as the Prince of Mathematicians sort of implies there must be a King of Mathematicians and I can't imagine who that would be (given the remarkable impact Gauss had on the field of mathematics it is difficult to imagine anybody above his level).

Throughout his career Gauss wrote voluminously but austerely; he would cut away all but the essential results. His works take a great deal of patience by the reader but with mathematics that are exceptionally stimulating. But, he probably wrote just like he thought; he possessed that inexplicable ability that allowed him to leap to the correct conclusion without, seemingly, to plod through the intermediate logic that the rest of us need in order to arrive at the same conclusion. A very good example of that is provided by his *Prime Number Theorem*. It is not known whether he proved this theorem or not, but he certainly stated it with the correct conclusion; a proof of the theorem was finally given in the 1920's. It deals with the distribution of prime numbers; it describes the asymptotic distribution of the prime numbers among the positive integers. It formalizes the intuitive idea that primes become less common as they become larger by precisely quantifying the rate at which this occurs.

Gauss kept a diary which contains notes on nearly all of his discoveries; the diary reveals work on higher trigonometry, elliptic function theory, and aspects of non-Euclidean geometry. Gauss was the last complete mathematician; since his time, mathematics has increased so extensively, that no one can hope to master the whole. One last thing about Gauss and that is Bernard Riemann, the German mathematical genius, whose integral is the subject of this book, was a student of Gauss.

Let's get back to the evaluation of the integral.

$$I_4 = \int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-y^2}dy$$

After all, the variable x is just a dummy variable and we can call it y if we want to. However, doing so allows us to write

$$(I_4)^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}dy\right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$$

Aha, a double integral! Note that changing the order of which variable to integrate with respect to in this case is useless because of the symmetry in x and y of the integrand (i.e., you can change x to y and y to x and you get the same integrand that you started with). However, a change to polar coordinates so that $x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$ can make the integration tractable. To change the intervals of integration for polar coordinates, one must look at the region of integration. In this case, x and y are both positive and increase without bound. That translates to the entire first quadrant. To cover the first quadrant in polar coordinates, r must go from 0 to ∞ and θ must go from 0 to $\pi/2$. Hence,

$$(I_4)^2 = \int_0^{\pi/2}\int_0^\infty re^{-r^2}drd\theta = \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty d\theta = \left[\frac{1}{2}\theta\right]_0^{\pi/2} = \frac{\pi}{4}$$

Obviously, if $(I_4)^2 = \pi/4$, then we have the expected result.

$$\boxed{\boldsymbol{I_4 = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}} \quad \textbf{Q.E.D.}}$$

I certainly don't mean to slight Laplace, who, as mentioned previously, was the first to solve I_4; to see how Laplace solved I_4, see Appendix A which contains Laplace's solution and other material related to I_4.

Example 6-5. $\boldsymbol{I_5 = \int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2}dx \quad a, b \in \mathbb{R}}$

The integrand of I_5 by itself can be written as an integral. Namely,

$$\frac{\cos(ax)-\cos(bx)}{x^2} = \int_a^b \frac{\sin(xu)}{x}du$$

If you are puzzled by this last equation, you just have to do integration of a definite integral backwards. That is,

$$\frac{\cos(ax)-\cos(bx)}{x^2} = \frac{1}{x}\left[-\frac{\cos(xu)}{x}\right]_a^b = \frac{1}{x}\int_a^b \sin(xu)du = \int_a^b \frac{\sin(xu)}{x}du$$

If you still don't see it, work the last equation backwards from right to left! Don't forget, the integration is with respect to the variable that I've called u and therefore x can be handled like a constant. Anyway, continuing, we can now substitute this integral for the integrand of I_5, the integral we are attempting to evaluate. When we do, we obtain the following double integral.

$$I_5 = \int_0^\infty \left[\int_a^b \frac{\sin(xu)}{x}du\right]dx$$

In keeping with the theme of this chapter, interchange the order of integration and we get

$$I_5 = \int_a^b \int_0^\infty \frac{\sin(xu)}{x} dx\, du$$

Aha! The inner integral should look familiar—it's Dirichlet's discontinuous integral from Chapter 5, example 3. So, we know the value of the inner integral; it's π/2 (we are assuming that *u* is positive). Thus,

$$I_5 = \int_a^b \frac{\pi}{2} du = \frac{\pi}{2}(b-a)$$

Or, our final result is

$$\boxed{\boldsymbol{I_5 = \int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2} dx = \frac{\pi}{2}(b-a)} \quad \text{Q.E.D.}}$$

Example 6-6. $\boldsymbol{I_6 = \int_0^\infty \frac{e^{-ax^2} - e^{-bx^2}}{x^2} dx \;\; a, b \in \mathbb{R}}$

Chapter 5 was devoted to the technique of differentiating an integral's integrand with respect to some parameter to eventually evaluate the integral. If one can differentiate an integral, it doesn't seem unreasonable to surmise that one can probably integrate an integral also. For this example, let's start with our old friend $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$. Let's do a change of variable on our old friend. Let $x = y\sqrt{p}$ where *p* is just some parameter. So, $dx = \sqrt{p}dy$ and (0, ∞) → (0, ∞).

So we have $\int_0^\infty e^{-py^2}(\sqrt{p})dy = \frac{\sqrt{\pi}}{2}$ or $\int_0^\infty e^{-py^2} dy = \frac{\sqrt{\pi}}{2\sqrt{p}}$

Now, integrate with respect to *p* both sides of this last expression from *a* to *b*.

$$\int_a^b \left[\int_0^\infty e^{-py^2} dy\right] dp = \frac{\sqrt{\pi}}{2} \int_a^b \frac{dp}{\sqrt{p}} = \left[\frac{\sqrt{\pi}}{2}(2)p^{1/2}\right]_a^b = \sqrt{\pi}\left(\sqrt{b} - \sqrt{a}\right)$$

Let's reverse the order of integration in the double integral. Doing that yields

$$\int_0^\infty \int_a^b e^{-py^2} dp dy = \sqrt{\pi}\left(\sqrt{b} - \sqrt{a}\right).$$

But, now the inner integral gives us the following

$$\int_0^\infty \left[-\frac{e^{-py^2}}{y^2}\right]_a^b dy = -\int_0^\infty \left[\frac{e^{-by^2}}{y^2} - \frac{e^{-ay^2}}{y^2}\right] dy = \int_0^\infty \frac{e^{-ay^2} - e^{-by^2}}{y^2} dy$$

WOW—that's I_6, the integral that we set out to solve!

$$\boxed{\boldsymbol{I_6 = \int_0^\infty \frac{e^{-ax^2} - e^{-bx^2}}{x^2} dx = \sqrt{\pi}\left(\sqrt{b} - \sqrt{a}\right)} \quad \text{Q.E.D.}}$$

This last beautiful solution certainly (in my mind) epitomizes unconventional integration!!!!

Quite ingenious!!!

Example 6-7. $\boldsymbol{I_7 = \int_1^\infty \frac{\log(x)}{x^2+1} dx}$

A change of variable of $x = e^y$, $dx = e^y dy$, and $(1, \infty) \to (0, \infty)$ gives us

$$I_7 = \int_0^\infty \frac{\log(e^y)e^y}{e^{2y}+1} dy = \int_0^\infty \frac{ye^y}{e^{2y}+1} dy$$

Now, expanding the denominator into a power series, we obtain

$$I_7 = \int_0^\infty ye^y \left(\sum_{k=0}^\infty (-1)^k e^{-2ky} \right) dy$$

As you probably suspect, we now change the order of the sum and integral. Thus, we have

$$I_7 = \sum_{k=0}^\infty (-1)^k \left[\int_0^\infty ye^{-(2k+1)y} dy \right]$$

This integral yields quite readily to integration by parts, with $u = y$ and $dv = e^{-(2k+1)y}dy$, making $du = dy$ and $v = -e^{-(2k+1)y}/(2k+1)$. Hence

$$I_7 = \sum_{k=0}^\infty (-1)^k \left\{ \left[\frac{-ye^{-(2k+1)y}}{2k+1} \right]_0^\infty + \frac{1}{2k+1} \int_0^\infty e^{-(2k+1)y} dy \right\}$$

The square-bracketed term vanishes at both limits leaving us with the integral that integrates directly. We therefore have (see series #10 in Table 3 of Chapter 1).

$$\boldsymbol{I_7 = \int_1^\infty \frac{\log(x)}{x^2+1} dx = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^2} = G} \quad \text{Q.E.D.}$$

Catalan's constant is a constant that commonly appears in estimates of combinatorial functions, certain classes of sums, and in many definite integrals (hence our interest in it). It is usually denoted by G and may be defined simply as the summation above. Interestingly, it has not been proven that G is irrational although I don't believe that there is a mathematician on the planet who believes it is rational. It has been computed to 31,026,000,000 decimal places without any repetitive pattern. It is named for Eugene Catalan (1814-1894), a French and Belgian mathematician who first gave an equivalent series and expressions in terms of integrals.

Example 6-8. $\boldsymbol{I_8 = \int_0^1 \frac{\tan^{-1}(x)}{x} dx}$

First, note that the numerator of I_8's integrand can be written as:

$$\tan^{-1}(x) = \int_0^x \frac{du}{1+u^2}$$

But,

$$\frac{1}{1+u^2} = 1 - u^2 + u^4 - u^6 + \cdots = \sum_{k=0}^{\infty} (-1)^k u^{2k}$$

Therefore, we can write

$$\tan^{-1}(x) = \int_0^x \left[\sum_{k=0}^{\infty} (-1)^k u^{2k}\right] du$$

Now using the IO property, that is, interchange the integration and summation operations, giving us

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \left[\int_0^x u^{2k} du\right]$$

Upon evaluating the integral we obtain

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \left[\frac{1}{2k+1} u^{2k+1}\right]_0^x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

I_8, the integral we are trying to crack, can now be obtained by simply dividing both sides of this last equation by x, integrating from 0 to 1 by using the IO property once again as follows:

$$I_8 = \int_0^1 \frac{\tan^{-1}(x)}{x} dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k+1} dx = \sum_{k=0}^{\infty} \left[\frac{(-1)^k x^{2k+1}}{(2k+1)^2}\right]_0^1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

And, remarkably, we have the final value

$$\boxed{\boldsymbol{I_8 = \int_0^1 \frac{\tan^{-1}(x)}{x} dx = G \quad \text{Q.E.D.}}}$$

Formulae for Catalan's constant abound. Not all involve properly improper integrals, but there are a plethora of such integrals. We could go on and derive many of them, but it would not contribute much more about methods of doing properly improper integrals and that, don't forget, is the purpose of this book. Instead, we've built a table that shows merely a representative 24 such integrals. We will examine one more, simply to show that once you have some results, others may follow; in-other-words, a power series of a portion of the integrand is not necessarily required nor is use of the IO property, as has been the case up until now.

Example 6-9. $I_9 = \int_0^{\infty} \frac{\log(x+1)}{x^2+1} dx$

I_9 is entry #6 in the table of Catalan integrals that appears on the next page. You will see as we go through the derivation that it doesn't require use of the IO property even though that is the subject of this chapter. Is it out of context? I don't think so. Some of the intermediate results that it uses could not have been achieved without the IO property; further, it is such a beautiful little derivation, I couldn't bear to leave it out.

Table 5: Catalan Integrals	
1. $\int_0^1 \frac{\pi/4 - \tan^{-1}(x)}{1-x^2}\,dx = G$	13. $\int_1^{\sqrt{2}} \frac{\log(x)}{x\sqrt{x^2-1}}\,dx = \frac{\pi\log(2)}{4} - \frac{G}{2}$
2. $\int_0^1 \left(x - \frac{1}{2}\right)\sec(\pi x)\,dx = -\frac{4G}{\pi^2}$	14. $\int_0^1 \frac{\log\left[\frac{1}{\sqrt{2}}(1-x)\right]}{1+x^2}\,dx = -G$
3. $\int_0^{\pi/4} \log[\cot(x)]dx = G$	15. $\int_1^{\frac{\sqrt{2}+1}{\sqrt{2}-1}} \frac{(x+1)\log(x)}{4x\sqrt{6x-x^2-1}}\,dx = G$
4. $\int_0^1 \frac{\log(x)}{(x+1)\sqrt{x}}\,dx = -4G$	16. $\int_0^{\pi/2} \left[\frac{1}{1+\cos^2(x)}\right]\log\left[\frac{\sqrt{2}+\sin(x)}{\sqrt{2}-\sin(x)}\right]dx = \sqrt{2}G$
5. $\int_0^{\pi/4} \frac{\log\left[\frac{1+\sin(x)}{1-\sin(x)}\right]}{\cos(x)\sqrt{\cos(2x)}}\,dx = 2G$	17. $\int_1^{\infty} \frac{\log(x)}{(1+x)\sqrt{x}}\,dx = 4G$
6. $\int_0^{\infty} \frac{\log(1+x)}{1+x^2}\,dx = G + \frac{\pi\log(2)}{4}$	18. $\int_0^{\pi/4} \log[\tan(x)]dx = -G$
7. $\int_0^{\pi/2} \log[\sin(x)+\cos(x)]dx = G - \frac{\pi\log(2)}{4}$	19. $\int_0^{\pi/2} \log\left[\frac{1+\cos(x)}{1-\cos(x)}\right]dx = 4G$
8. $\int_0^1 [\tan^{-1}(x)]^2 dx = \frac{\pi\log(2)}{4} + \frac{\pi^2}{16} - G$	20. $\int_0^{\pi/2} \log\left[\frac{1+\sin(x)}{1-\sin(x)}\right]dx = 4G$
9. $\int_0^{\pi/6} \frac{x}{\sin(x)}\,dx = \frac{4}{3}\left[G - \frac{\pi\log(2+\sqrt{3})}{8}\right]$	21. $\int_0^{\pi/4} \log[2\sin(x)]\,dx = -\frac{G}{2}$
10. $\int_0^{\pi/2} \frac{x\csc(x)}{\sin(x)+\cos(x)}\,dx = G + \frac{\pi\log(2)}{4}$	22. $\int_0^{\pi/4} \log[2\cos(x)]\,dx = \frac{G}{2}$
11. $\int_0^{2-\sqrt{3}} \frac{\log(x)}{1+x^2}\,dx = -\frac{2}{3}G$	23. $\int_0^{\infty} \frac{x}{\cosh(x)}\,dx = 2G$
12. $\int_0^{\pi/2} \sinh^{-1}[\sin(x)]dx = G$	24. $\int_0^{\pi/2} \frac{x}{\sin(x)}\,dx = 2G$

The first thing you should notice about this example is that the integrand is exactly the same as an example we have done previously; the difference is the integration interval. Examine example 3-5 of Chapter 3 and you will see that except for the interval, this is Serret's integral. That, of course, gives us a hint on how to proceed, i.e., break I_9 into two integrals, the first with Serret's interval and the second with the rest, ala

$$I_9 = \int_0^1 \frac{\log(x+1)}{x^2+1}\,dx + \int_1^{\infty} \frac{\log(x+1)}{x^2+1}\,dx = \frac{\pi}{8}\log(2) + \int_1^{\infty} \frac{\log(x+1)}{x^2+1}\,dx$$

Now factor an x from the logarithm argument and by the property of logarithms, we have

$$I_9 = \frac{\pi}{8}\log(2) + \int_1^{\infty} \frac{\log[x(1+\frac{1}{x})]}{x^2+1}\,dx = \frac{\pi}{8}\log(2) + \int_1^{\infty} \frac{\log(x)}{x^2+1}\,dx + \int_1^{\infty} \frac{\log(1+\frac{1}{x})}{x^2+1}\,dx$$

By example 6-7, the first integral to the right of the second equal sign is simply G. In the second integral to the right of the second equal sign make a change of variable of $x = 1/u \Rightarrow dx = -1/u^2$ and $(1, \infty) \rightarrow (1, 0)$. We therefore have

$$I_9 = \frac{\pi}{8}\log(2) + G + \int_0^1 \frac{\log(1+u)}{u^2+1}\,du$$

The remaining integral is Serret's integral again and our final result is

$$\boldsymbol{I_9 = \int_0^{\infty} \frac{\log(x+1)}{x^2+1}\,dx = \frac{\pi}{4}\log(2) + G} \quad \textbf{Q.E.D.}$$

Example 6-10. $\boldsymbol{I_{10} = \int_0^1 \frac{\log(1-x)}{x} dx}$

As we developed a power series for log(1+*x*) in example 6-1 above, we can do a similar development for log(1–*x*) here, that is, $\log(1-x) = -\int_0^x \frac{du}{1-u}$. However,

$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots = \sum_{k=0}^{\infty} u^k$. Hence,

$$\log(1-x) = -\int_0^x \left(\sum_{k=0}^{\infty} u^k\right) du = -\sum_{k=0}^{\infty} \int_0^x u^k du = -\sum_{k=0}^{\infty} \left[\frac{u^{k+1}}{k+1}\right]_0^x = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}.$$

Now dividing by *x* and integrating from 0 to 1, we get,

$$I_{10} = \int_0^1 \frac{\log(1-x)}{x} dx = -\int_0^1 \left(\sum_{k=0}^{\infty} \frac{x^k}{k+1}\right) dx = -\sum_{k=0}^{\infty} \frac{1}{(k+1)^2}$$

And we finally obtain (see infinite series #3 from chapter 1),

$$\boldsymbol{I_{10} = \int_0^1 \frac{\log(1-x)}{x} dx = -\frac{\pi^2}{6}} \quad \textbf{Q.E.D.}$$

Example 6-11. $\boldsymbol{I_{11} = \int_0^1 \frac{1}{x} \log\left(\frac{1+x}{1-x}\right)^n dx,\ n \in \mathbb{N}^+}$

This is an easy one given the results of previous examples in this chapter, i.e., examples 6-1 and 6-10. By the property of logarithms, we have

$$I_{11} = n\int_0^1 \frac{\log(1+x)}{x} dx - n\int_0^1 \frac{\log(1-x)}{x} dx = nI_1 - nI_{10}$$

Therefore, our final value is

$$\boldsymbol{I_{11} = \int_0^1 \frac{1}{x} \log\left(\frac{1+x}{1-x}\right)^n dx = \frac{n\pi^2}{4}} \quad \textbf{Q.E.D.}$$

Example 6-12. $\boldsymbol{I_{12} = \int_0^\infty \log\left(\frac{1+e^{-x}}{1-e^{-x}}\right) dx}$

As a final integral to this chapter, let us attack this formidable looking I_{12}. However, if we make the change of variable $u = e^{-x}$ so that $du = -e^{-x}dx$, $(0, \infty) \to (1, 0)$ and, of course, $dx = -du/u$, it's not quite so formidable looking anymore, e.g.,

$$I_{12} = \int_1^0 \log\left(\frac{1+u}{1-u}\right)\left(-\frac{du}{u}\right) = \int_0^1 \frac{1}{u} \log\left(\frac{1+u}{1-u}\right) du$$

because it is now apparent that this is merely I_{11}, above, with $n = 1$. Therefore, its value is $\pi^2/4$. Nevertheless, it is a useful and interesting exercise to attempt to crack this integral pretending that we didn't recognize the relationship to I_{11}. Let's do that as it really illustrates the IO technique that we are trying to illustrate in this chapter. First of all, let us use the property of logarithms to write this integral as follows:

$$I_{12} = \int_0^1 \frac{1}{u}[\log(1+u) - \log(1-u)]du$$

Now, we can do a power series for the two log functions in the integrand. In point-of-fact, we have already developed such power series. For log(1 + *u*) see example 6-1 and for log(1 – *u*) see example 6-10. Therefore, our integral becomes

$$I_{12} = \int_0^1 \frac{1}{u}\left[\sum_{k=0}^{\infty} \frac{(-1)^k u^{k+1}}{k+1} - \left(-\sum_{k=0}^{\infty} \frac{u^{k+1}}{k+1}\right)\right] du$$

Instead of using the compact summation notation (i.e., the upper case sigma's) let's do away with them and simply write out the first few terms in each series; it will be a bit easier to see what is going on. Our equation becomes

$$I_{12} = \int_0^1 \frac{1}{u}\left[\left(u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots\right) - \left(-u - \frac{u^2}{2} - \frac{u^3}{3} - \frac{u^4}{4} - \cdots\right)\right] du$$

Collecting like terms, we have

$$I_{12} = \int_0^1 \frac{1}{u}\left(2u + \frac{2u^3}{3} + \frac{2u^5}{5} + \frac{2u^7}{7} + \cdots\right) du$$

As you can see, all even powers of *u* vanish (sum to zero) and all odd powers of *u* double. And then when we multiply through by the $1/u$ term immediately following the integral sign we have

$$I_{12} = 2\int_0^1 \left(1 + \frac{u^2}{3} + \frac{u^4}{5} + \frac{u^6}{7} + \cdots\right) du = 2\int_0^1 \left(\sum_{k=0}^{\infty} \frac{u^{2k}}{2k+1}\right) du$$

At last we get to the point of this entire chapter—using the IO property—interchanging the operations of integration and summation we obtain

$$I_{12} = 2\sum_{k=0}^{\infty} \left[\frac{u^{2k+1}}{(2k+1)^2}\right]_0^1 = 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

This last sum is useful infinite series #4 from chapter 1 and as such it sums to $\pi^2/8$ and our final value is as we expected

$$\boldsymbol{I_{12} = \int_0^{\infty} \log\left(\frac{1+e^{-x}}{1-e^{-x}}\right) dx = \frac{\pi^2}{4}} \quad \textbf{Q.E.D.}$$

Descartes commanded the future from his study more than Napoleon from the throne.

—Oliver Wendell Holmes

Chapter 7. Interval Normalization (IN)

Interval normalization (see Table 4, Chapter 1) is the name that I've given to a technique that changes the specific integration interval of (0, ∞) to that of (0, 1). In Quantum Mechanics normalizing the wave function means a scaling so that all of the probabilities add up to 1. Further, normalization in String Theory means getting rid of the infinities when attempting to incorporate gravitation into a unified field theory. Since we will use this technique to get rid of the upper infinity limit, it seems like a fairly good match when we change the integration interval to go from zero to one. Anyway, that's why I'm calling this technique Interval Normalization. Table 4 of Chapter 1 shows the following entry for IN:

$$\int_0^\infty f(x)dx = \int_0^1 \frac{x^2 f(x) + f\left(\frac{1}{x}\right)}{x^2} dx$$

The above equality is obtained by combining three of the other techniques that are also entries in Table 4, Chapter 1, namely, Interval Subdivision, Change of Variable, and Dummy Variable. We have, by the Interval Subdivision technique

$$\int_0^\infty f(x)dx = \int_0^1 f(x)dx + \int_1^\infty f(x)dx$$

Now, in the second interval make a change of variable. Let $x = 1/u$ so that $dx = -du/u^2$, and so that $(1, \infty) \to (1, 0)$. We then have

$$\int_0^\infty f(x)dx = \int_0^1 f(x)dx - \int_1^0 f\left(\frac{1}{u}\right)\frac{du}{u^2} = \int_0^1 f(x)dx + \int_0^1 \frac{f\left(\frac{1}{x}\right)}{x^2} dx$$

The last step, of course, is making use of the fact that u is just a dummy variable and we can call it x (or any other symbol we choose to use). However, by calling it x and since the integration intervals are now the same, we can combine the two integrals into one and we have the Table 4, Chapter 1 entry for IN. namely

$$\int_0^\infty f(x)dx = \int_0^1 \frac{x^2 f(x) + f\left(\frac{1}{x}\right)}{x^2} dx$$

This is what is meant by normalizing the integral, that is, taking an integral whose integration interval is given as (0, ∞), subdivide it into two integrals, the 1st whose interval is (0, 1) and the 2nd whose interval is (1, ∞)—step 1, and then performing a CV on the 2nd integral such that its interval also becomes (0, 1)—step 2, and then combining the two integrals into one—step 3. So where does that get us? What we've derived is possibly more complicated than what we started with, depending on the specific form of $f(x)$. Well, what we have is a general principle that allows us to change an integration interval of (0, ∞) to one of (0, 1). That can be a big benefit for a very large class of functions. For example, suppose that the function, $f(x)$, has the property that $f(x) = f(1/x)$, in-other-words, $f(x)$ is a function that is a symmetric function of x and $1/x$. In that case, all integrals of the form $I = \int_0^\infty \frac{f(x)}{x} dx$ can be set equal to $2\int_0^1 \frac{f(x)}{x} dx$, provided that $f(x)/x$ remains finite for all $x \geq 0$. The proof of this is simple. Normalize this integral and look what one gets:

$$I = \int_0^\infty \frac{f(x)}{x} dx = \int_0^1 \frac{f(x)}{x} dx + \int_1^\infty \frac{f(x)}{x} dx \quad \text{(the subdivide step} - \text{step 1)}$$

$$I = \int_0^\infty \frac{f(x)}{x} = \int_0^1 \frac{f(x)}{x} dx + \int_1^0 \frac{f\left(\frac{1}{u}\right)\left(-\frac{du}{u^2}\right)}{\frac{1}{u}} \quad \text{(the change of variable step – step 2)}$$

$$I = \int_0^\infty \frac{f(x)}{x} dx = \int_0^1 \frac{f(x)}{x} dx + \int_0^1 \frac{f(u)}{u} du = 2\int_0^1 \frac{f(x)}{x} dx \quad \text{(the combining step – step 3)}$$

As a simple corollary to this idea, if $f(x) = -f(1/x) \Rightarrow \int_0^\infty \frac{f(x)}{x} dx = 0$. As another corollary, if $I_1 = \int_0^\infty \frac{f(x)}{x} dx$ and $f(x) = f\left(\frac{1}{x}\right)$ and $n \in \mathbb{N}^+$ then $I_2 = \int_0^\infty \frac{f(x)}{x(1+x^n)} dx = \frac{1}{2} I_1$. This proof is also simple. In I_2, make the change of variable $x = 1/u$ so that $dx = -1/u^2$ and $(0, \infty) \to (\infty, 0)$. I_2 then becomes

$$I_2 = \int_\infty^0 \frac{f\left(\frac{1}{u}\right)\left(-\frac{du}{u^2}\right)}{\left(1+\frac{1}{u^n}\right)\left(\frac{1}{u}\right)} = \int_0^\infty \frac{u^n f\left(\frac{1}{u}\right)}{u^n+1}\left(\frac{du}{u}\right) = \int_0^\infty \frac{x^n f(x)}{1+x^n}\left(\frac{dx}{x}\right)$$

Therefore,

$$2I_2 = \int_0^\infty \frac{f(x)}{1+x^n} \cdot \frac{dx}{x} + \int_0^\infty \frac{x^n f(x)}{1+x^n}\left(\frac{dx}{x}\right) = \int_0^\infty \frac{f(x)}{x} dx = I_1.$$

Hence

$$I_2 = \frac{1}{2} I_1$$

What does all this mean? Well, there are a plethora of properly improper integrals that can be evaluated using the ideas outlined above. Hopefully, the examples that follow will make it clear what can be done with this idea of interval normalization.

Example 7-1 $\boldsymbol{I_1 = \int_0^\infty \frac{dx}{1-x^2}}$

We are starting out with what looks like an extremely simple integral that appears quite innocent. But watch out! At first glance, I would say, "Gee, the denominator is the difference of two squares. Factor it and use the partial fraction technique to split it into two integrals, each of which will be the recognizable form of the natural logarithm function". Oh yeah?

$$I_1 = \int_0^\infty \frac{dx}{(1+x)(1-x)} = \int_0^\infty \left(\frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}\right) dx = \frac{1}{2}\left[\log\left(\frac{1+x}{1-x}\right)\right]_0^\infty = \frac{1}{2}\lim_{x\to\infty} \log\left(\frac{\frac{1}{x}+1}{\frac{1}{x}-1}\right) - \frac{1}{2}\lim_{x\to 0} \log\left(\frac{1+x}{1-x}\right)$$

The second limit above easily evaluates to zero, but what does the first limit evaluate to? It looks like log(–1). Can't be! Time to get out L'Hopital's rule; however, you must use it on the 1st limit above in the form in which I've written it. So that limit becomes

$$\frac{1}{2}\lim_{x\to\infty} \log\left(\frac{-\frac{1}{x^2}}{-\frac{1}{x^2}}\right) = \frac{1}{2}\log(1) = 0$$

And, that is correct, both limits are zero and therefore $I_1 = 0$. Not so easy though. It took me a long time to figure out how to do that limit. However, look how easy and free of complications the original integral is if we normalize it.

$$I_1 = \int_0^\infty \frac{dx}{1-x^2} = \int_0^1 \frac{dx}{1-x^2} + \int_1^\infty \frac{dx}{1-x^2}$$

In the 2nd integral, let $x = 1/u$ so that $dx = -1/u^2$ and $(1, \infty) \rightarrow (1, 0)$ giving

$$I_1 = \int_0^1 \frac{dx}{1-x^2} + \int_1^0 \frac{-\frac{du}{u^2}}{1-\frac{1}{u^2}} = \int_0^1 \frac{dx}{1-x^2} + \int_0^1 \frac{dx}{u^2-1} = \int_0^1 \frac{dx}{1-x^2} - \int_0^1 \frac{dx}{1-x^2} = 0$$

Our final result is therefore

$$\boxed{\boldsymbol{I_1 = \int_0^\infty \frac{1}{1-x^2} dx = 0 \quad \text{Q.E.D.}}}$$

Example 7-2 $\boldsymbol{I_2 = \int_0^\infty \frac{\log^2(x)}{1+x^2} dx}$

In Chapter 6, example 6.3 we evaluated $\int_0^1 \frac{\log^2(x)}{1+x^2} dx$ and we found its value to be $\pi^3/16$. Note that the integrands of I_2 and that of example 6.3 are exactly the same; the difference in the two integrals is solely the interval of integration; in one case (0, 1) and in the other $(0, \infty)$. Well, we should certainly know what to do to attack I_2.

$$I_2 = \int_0^1 \frac{\log^2(x)}{1+x^2} dx + \int_1^\infty \frac{\log^2(x)}{1+x^2} dx$$

In the second integral above, let $x = 1/u$ so that $dx = -du/u^2$ and $(1, \infty) \rightarrow (1, 0)$. We then have

$$I_2 = \int_0^1 \frac{\log^2(x)}{1+x^2} dx + \int_1^0 \frac{\log^2\left(\frac{1}{u}\right)}{1+\left(\frac{1}{u}\right)^2}\left(-\frac{du}{u^2}\right) = \int_0^1 \frac{\log^2(x)}{1+x^2} dx + \int_0^1 \frac{\log^2(x)}{1+x^2} dx = 2\int_0^1 \frac{\log^2(x)}{1+x^2} dx$$

So, we have our final value, namely

$$\boxed{\boldsymbol{I_2 = \int_0^\infty \frac{\log^2(x)}{1+x^2} dx = 2\left(\frac{\pi^3}{16}\right) = \frac{\pi^3}{8} \quad \text{Q.E.D.}}}$$

Example 7-3 $\boldsymbol{I_3 = \int_0^\infty \frac{\log(x)}{x^2-1} dx}$

Again, using the interval normalization technique, we have

$$I_3 = \int_0^1 \frac{\log(x)}{x^2-1} dx + \int_1^\infty \frac{\log(x)}{x^2-1} dx$$

In the second integral make a change of variable. Let $x = 1/z$ so that $dx = -dz/z^2$ and $(1, \infty) \rightarrow (1, 0)$.

$$I_3 = \int_0^1 \frac{\log(x)}{x^2-1} dx + \int_1^0 \frac{\log\left(\frac{1}{z}\right)}{\frac{1}{z^2}-1}\left(-\frac{dz}{z^2}\right) = 2\int_0^1 \frac{\log(x)}{x^2-1} dx$$

Now, representing the denominator of the integrand by a power series, we have

$$I_3 = 2\int_0^1 \left[\log(x)\left(-\sum_{k=0}^{\infty} x^{2k}\right)\right] dx$$

Interchanging the operations of summation and integration, we obtain

$$I_3 = -2\sum_{k=0}^{\infty}\int_0^1 x^{2k}\log(x)\,dx$$

The integral can now be done by parts with $u = \log(x)$ and $dv = x^{2k}dx$, giving us

$$I_3 = -2\sum_{k=0}^{\infty}\left\{\left[\frac{x^{2k+1}\log(x)}{2k+1}\right]_0^1 - \int_0^1 \frac{x^{2k+1}dx}{(2k+1)x}\right\} = 2\sum_{k=0}^{\infty}\frac{1}{2k+1}\int_0^1 x^{2k}dx = 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$$

Entry #4 in the Table of Useful infinite series of Chapter 1 gives us this sum as $\pi^2/8$. We therefore have our final result

$$\boxed{\boldsymbol{I_3 = \int_0^{\infty}\frac{\log(x)}{x^2-1}dx = \frac{\pi^2}{4}} \quad \textbf{Q.E.D.}}$$

Example 7-4 $\boldsymbol{I_4 = \int_0^{\infty}\frac{\log\left(\frac{1}{x}\right)}{1+x^2}dx}$

$$I_4 = \int_0^1 \frac{\log\left(\frac{1}{x}\right)}{1+x^2}dx + \int_1^{\infty}\frac{\log\left(\frac{1}{x}\right)}{1+x^2}dx$$

Now make a CV in the 2nd integral above of $x = 1/u$ so that $dx = -du/u^2$, and $(1, \infty) \to (1, 0)$. Upon doing that, we have

$$I_4 = \int_0^1 \frac{\log\left(\frac{1}{x}\right)}{1+x^2}dx + \int_1^{\infty}\frac{\log(u)}{1+\frac{1}{u^2}}\left(-\frac{du}{u^2}\right) = \int_0^1 \frac{\log\left(\frac{1}{x}\right)}{1+x^2}dx + \int_0^1 \frac{\log(u)}{u^2+1}du$$

However, since u is just a dummy variable, we can call it x and also by the property of logarithms, $\log(u) = -\log(1/u)$, the above equation becomes

$$I_4 = \int_0^1 \frac{\log\left(\frac{1}{x}\right)}{1+x^2}dx - \int_0^1 \frac{\log\left(\frac{1}{x}\right)}{1+x^2}dx = 0$$

Lo and behold, we have our final answer, that is,

$$\boxed{\boldsymbol{I_4 = \int_0^{\infty}\frac{\log\left(\frac{1}{x}\right)}{1+x^2}dx = 0} \quad \textbf{Q.E.D.}}$$

Example 7-5 $\boldsymbol{I_5 = \int_0^{\infty}\left[\frac{\log(x)}{x-1}\right]^2 dx}$

Here is a relatively simple looking properly improper integral. It is certainly not an elementary integral, but not an overly complex integrand. Never-the-less, before we arrive at a final value, we will have used the following techniques: IN, CV, IO, and IBP twice, not to mention a power series expansion. Hang onto your hats, because here we go! Using the IN technique, we have

$$I_5 = \int_0^1 \frac{\log^2(x)}{(x-1)^2} dx + \int_1^\infty \frac{\log^2(x)}{(x-1)^2} dx = 2\int_0^1 \frac{\log^2(x)}{(x-1)^2} dx$$

If you are beginning to understand this IN technique, you realize that its purpose is to change the integration interval from $(0, \infty)$ to $(0, 1)$. Well, now we are going to change it right back with a CV of $x = e^{-y}$. Under this CV we get $dx = -e^{-y}dy$ and $(0, 1) \rightarrow (\infty, 0)$. Thus,

$$I_5 = 2\int_\infty^0 \frac{y^2(-e^{-y}dy)}{(e^{-y}-1)^2} = 2\int_0^\infty y^2 e^{-y}(1 + 2e^{-y} + 3e^{2y} + 4e^{-3y} + \cdots)dy$$

Let's use the more compact sigma notation for the power series, i.e., we then have

$$I_5 = 2\int_0^\infty y^2 e^{-y} \sum_{k=0}^{\infty}[(k+1)e^{-ky}]dy$$

Now, interchanging the order of summation and integration we have,

$$I_5 = 2\sum_{k=0}^{\infty}(k+1)\int_0^\infty y^2 e^{-(k+1)y}dy$$

The integral can now be evaluated by using integration by parts twice. The first time let $u = y^2$ and $dv = e^{-(k+1)y}dy$. The second time, let $u = y$ and $dv = e^{-(k+1)y}dy$. Doing that, we obtain

$$I_5 = 4\sum_{k=0}^{\infty}\left[-\frac{1}{(k+1)^2}e^{-(k+1)y}\right]_0^\infty = 4\sum_{k=0}^{\infty}\frac{1}{(k+1)^2} = \frac{2\pi^2}{3}$$

This last step, of course, is from entry 2 of table 3 (Useful Infinite Series) in Chapter 1. So our final result is

$$\boxed{\boldsymbol{I_5 = \int_0^\infty \left[\frac{\log(x)}{x-1}\right]^2 dx = \frac{2\pi^2}{3}} \quad \textbf{Q.E.D.}}$$

Alternately, one can eliminate the IN step at the very beginning and make the $x = e^{-y}$ change of variable directly to the original integral. This gives an integral with the same integrand as we got above, however its interval runs from $(-\infty, \infty)$ instead of $(0, \infty)$. Then, due to a symmetry argument, we can multiply the integral by 2 and make the interval $(0, \infty)$, and we are right back on course with the above derivation.

Example 7-6 $\boldsymbol{I_6 = \int_0^\infty \left[\frac{\log(x)}{x-1}\right]^3 dx}$

This is a bear, so bear with me! Normalizing the interval we obtain

$$I_6 = \int_0^1 \left[\frac{\log(x)}{x-1}\right]^3 dx + \int_1^\infty \left[\frac{\log(x)}{x-1}\right]^3 dx$$

In the 2nd integral, let $x = 1/u$ so that $dx = -du/u^2$ and $(1, \infty) \rightarrow (1, 0)$. We then get

$$I_6 = \int_0^1 \left[\frac{\log(x)}{x-1}\right]^3 dx + \int_1^0 \left[\frac{-\log(u)}{\frac{1-u}{u}}\right]^3 \left(-\frac{du}{u^2}\right) = \int_0^1 \left[\frac{\log(x)}{x-1}\right]^3 dx - \int_0^1 \frac{x[\log(x)]^3}{(x-1)^3} dx$$

In the last integral, the dummy variable u has been changed to an x. These last two integrals can now be combined under one integral sign and we have

$$I_6 = \int_0^1 \frac{(1+x)}{(1-x)^3}\left[\log\left(\frac{1}{x}\right)\right]^3 dx$$

We will now make a change of variable, namely, let $x = e^{-z}$ so that $dx = -e^{-z}dz$ and $(0, 1) \to (\infty, 0)$. As you can see, the integration interval is going right back to 0 to ∞. Could we have done this at the beginning instead of normalizing the interval? In this case, no, the integrand that we would obtain is not symmetric about the vertical axis. Continuing with this change of variable, we have

$$I_6 = \int_0^\infty \frac{z^3(e^{-z}+e^{-2z})}{(1-e^{-z})^3}dz = \int_0^\infty z^3(e^{-z}+e^{-2z})(1+3e^{-z}+6e^{-2z}+10e^{-3z}+15e^{-4z}+\cdots)dz$$

Now, multiplying the power series expansion by $e^{-z} + e^{-2z}$ we obtain

$$I_6 = \int_0^\infty z^3(e^{-z}+4e^{-2z}+9e^{-3z}+16e^{-4z}+25e^{-5z}+\cdots)dz = \int_0^\infty z^3\sum_{k=0}^{\infty}(k+1)^2e^{-(k+1)z}\,dz$$

By interchanging the order of summation and integration we can write this last expression as

$$I_6 = \sum_{k=0}^{\infty}(k+1)^2\int_0^\infty z^3e^{-(k+1)z}dz$$

Now we will integrate by parts a few times—namely three. Let $u = z^3$, $du = 3z^2dz$ while letting $dv = e^{-(k+1)z}dz$, $v = -e^{-(k+1)z}/(k+1)$. Thus, for the 1st integration by parts we obtain

$$I_6 = \sum_{k=0}^{\infty}(k+1)^2\left\{\left[\frac{-z^3}{k+1}e^{-(k+1)z}\right]_0^\infty + \frac{3}{k+1}\int_0^\infty z^2e^{-(k+1)z}dz\right\} = 3\sum_{k=0}^{\infty}(k+1)\int_0^\infty z^2e^{-(k+1)z}dz$$

Again, by parts with $u = z^2$, $du = 2zdz$ and $dv = e^{-(k+1)z}dz$, $v = -e^{-(k+1)z}/((k+1)$.

$$I_6 = 3\sum_{k=0}^{\infty}(k+1)\left\{\left[\frac{-z^2}{k+1}e^{-(k+1)z}\right]_0^\infty + \frac{2}{k+1}\int_0^\infty ze^{-(k+1)z}dz\right\} = 6\sum_{k=0}^{\infty}\int_0^\infty ze^{-(k+1)z}dz$$

This time with $u = z$, $du = dz$ and $dv = e^{-(k+1)z}dz$, $v = -e^{-(k+1)z}/(k+1)$. So, the final integration by parts gives us

$$I_6 = 6\sum_{k=0}^{\infty}\left\{\left[\frac{-z}{k+1}e^{-(k+1)z}\right]_0^\infty + \frac{1}{k+1}\int_0^\infty e^{-(k+1)z}dz\right\} = 6\sum_{k=0}^{\infty}\frac{1}{k+1}\int_0^\infty e^{-(k+1)z}dz$$

Of course, the last integral is the recognizable form of the exponential function and so we have

$$I_6 = \sum_{k=0}^{\infty}\left[\frac{-1}{(k+1)^2}e^{-(k+1)z}\right]_0^\infty = 6\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}$$

Refer to Chapter 1, Table 3, entry #2 and we see that this sum is equal to $\pi^2/6$. As a result, we have our final value, namely

$$\boxed{I_6 = \int_0^\infty \left[\frac{\log(x)}{x-1}\right]^3 dx = \pi^2 \quad \text{Q.E.D.}}$$

Example 7-7 $I_7 = \int_0^\infty \frac{\log(x)}{x^2+a^2} dx \;\; a \in \mathbb{R}$

This is a very simple looking example and I will use it to close out this chapter on normalization. I am going to show how to evaluate this integral in two different ways, one of which embodies the normalization process and one that does not. I think it's quite interesting!

A good thing to try when you see a log(x) in the numerator of the integrand is to do a CV where $x = 1/u$. Why is that? Because by the property of logarithms, the log(1/u) becomes –log(u) and that can be convenient depending on the rest of the integrand's transformation. Anyway, let's try it. Let $x = 1/u$ so that $dx = -du/u^2$ and $(0, \infty) \rightarrow (\infty, 0)$. Hence,

$$I_7 = -\int_\infty^0 \frac{\log\left(\frac{1}{u}\right)}{\frac{1}{u^2}+a^2} \cdot \frac{1}{u^2} du = \int_0^\infty \frac{\log\left(\frac{1}{u}\right)}{1+a^2u^2} du = -\int_0^\infty \frac{\log(u)}{1+a^2u^2} du.$$

Now, let's do another CV, namely, let $y = au$ so that $du = dy/a$ and $(0, \infty) \rightarrow (0, \infty)$. We now obtain

$$I_7 = -\frac{1}{a}\int_0^\infty \frac{\log\left(\frac{y}{a}\right)}{1+y^2} dy = -\frac{1}{a}\int_0^\infty \frac{\log(y)-\log(a)}{1+y^2} dy = \frac{\log(a)}{a}\int_0^\infty \frac{dy}{1+y^2} - \frac{1}{a}\int_0^\infty \frac{\log(y)}{1+y^2} dy.$$

The first integral on the right of the second equal sign is easy—it's the recognizable form of the inverse tangent. So for that integral we have

$$\left[\frac{\log(a)}{a}\tan^{-1}(y)\right]_0^\infty = \frac{\pi\log(a)}{2a}.$$

For the second integral, use the normalization process property so that we get (and to me, this was the aha moment which is further explained below) the following

$$-\frac{1}{a}\int_0^\infty \frac{\log(y)}{1+y^2} dy = -\frac{1}{a}\int_0^1 \frac{\log(y)}{1+y^2} dy - \frac{1}{a}\int_1^\infty \frac{\log(y)}{1+y^2} dy.$$

Now, complete the normalization process by making the expected CV to the integral above on the far right. Let $y = 1/z$ so that $dy = -dz/z^2$ and $(1, \infty) \rightarrow (1, 0)$. You will find that after making the CV that these two integrals are equal but opposite in sign, thereby canceling one another. So, we already had our final answer, we just didn't know it. That is,

$$\boxed{I_7 = \int_0^\infty \frac{\log(x)}{x^2+a^2} dx = \frac{\pi\log(a)}{2a} \quad \text{Q.E.D}}$$

Note that if we applied this last CV to the other integral on the right of the equal sign (i.e., the integral that goes from 0 to 1), we would see that it is exactly the negative of the integral that goes from 1 to ∞ with the exception of the sign; it is opposite in sign. We get the correct result no matter which way we do it.

Let us now see how one might go about evaluating this integral without use of the normalization property. It was stated in a previous chapter that whenever an integrand exhibits a constant plus the variable of integration squared (as I_7 does in the denominator), it is a good idea to consider a CV in which the variable of integration is set equal to the square root of the constant times the tangent of the new variable. Let us try that and see what happens. Let $x = a\tan(\theta)$ so that $dx = a\sec^2(\theta)d\theta$ and $(0, \infty) \rightarrow (0, \pi/2)$. Under this CV, our integral becomes

$$I_7 = \int_0^{\pi/2} \frac{\log(a\tan\theta)a\sec^2(\theta)d\theta}{a^2\tan^2(\theta)+a^2} = \int_0^{\pi/2} \frac{\log(a\tan\theta)a\sec^2(\theta)d\theta}{a^2\sec^2(\theta)} = \frac{1}{a}\int_0^{\pi/2} \log(a\tan\theta)\, d\theta$$

Now, by the property of logarithms, this last integral becomes

$$I_7 = \frac{1}{a}\int_0^{\pi/2}[\log(a) + \log(\tan\theta)]\, d\theta = \frac{\log(a)}{a}\int_0^{\pi/2} d\theta + \frac{1}{a}\int_0^{\pi/2} \log(\tan\theta) d\theta$$

Of course, the 1st integral to the right of the equal sign above is easily integrated while the 2nd integral can be written as

$$I_7 = \left[\frac{\log(a)}{a}\theta\right]_0^{\pi/2} + \frac{1}{a}\int_0^{\pi/2} \log\left[\frac{\sin(\theta)}{\cos(\theta)}\right] d\theta = \frac{\pi\log(a)}{2a} + \frac{1}{a}\int_0^{\pi/2}[\log(\sin\theta) - \log(\cos\theta)]d\theta$$

This last integral can be broken into two giving us

$$I_7 = \frac{\pi\log(a)}{2a} + \frac{1}{a}\int_0^{\pi/2} \log(\sin\theta)\, d\theta - \frac{1}{a}\int_0^{\pi/2} \log(\cos\theta) d\theta$$

It is easy to show that $\int_0^{\pi/2} \log(\sin\theta) d\theta = \int_0^{\pi/2} \log(\cos\theta)\, d\theta$ (as you will see in the next chapter) by means of an IP CV thereby giving us the correct value of $\pi\log(a)/2a$, e.g., let $\theta = \pi/2 - \phi$ in either integral.

Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.

—David Hilbert

Chapter 8. Crème de la Crème

This chapter, as explained in the preface, is devoted to those integral solutions that I have deemed to be the best, most exciting, and cleverest of all those that I have studied, that is, the so-called crème de la crème of properly improper integrals. Of course, this is purely my opinion and others might very well have included an entirely different set of integrals in such a chapter. Never-the-less, as a result, this chapter is a potpourri of all the techniques and methodologies that I have tried to delineate and explain in the preceding chapters. With that short introduction, have at it!

8.1 The Frullani/Cauchy Theorem

We will start this chapter off by establishing a general result that then can be used to evaluate an entire host of properly improper integrals. It is known as Frullani's Theorem, named after an Italian mathematician Giuliano Frullani (1795-1834). The theorem can be stated in the following manner. If we have a function, $f(x)$, such that both $f(0)$ and $f(\infty)$ exist, then

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}dx = [f(\infty)-f(0)]\log\left(\frac{a}{b}\right), \quad \text{where } a,b \in \mathbb{R}^+.$$

There is some evidence that the theorem was first published by Cauchy in 1823. Frullani did not publish the result until 1829; however, he claimed to have communicated the result in a letter to the Italian astronomer and mathematician Plana (1781-1864). As to who should get credit for the theorem is irrelevant – it's very clever (a good candidate for the crème de la crème chapter) and we will subsequently prove the theorem. However, let us first discuss a bit about Cauchy as he is much more interesting than Frullani. But, instead of the usual bio that includes Cauchy's academic education and mathematical credentials, let's sum that up quickly and talk instead about Cauchy's personality—a much more interesting subject.

Figure 8-1. French Mathematician Augustin Louis Cauchy (1789-1857)

"Men pass away, but their deeds abide."—Augustin-Louis Cauchy

Augustin-Louis Cauchy was a French mathematician and one of the greatest modern mathematicians of the 19th century; he was an extreme mathematical genius. His forte was mathematical rigor which he brought to the subject of Newton's and Leibniz's Calculus. He almost singlehandedly created Complex Analysis. Numerous terms in mathematics bear Cauchy's name; no less than 45 (at my count) theorems, constants, formulas, and/or mathematical structures are named for him. To name just a few, how about the Cauchy integral theorem, in the theory of complex functions, the Cauchy-Kovalevskaya existence theorem for the solution of partial differential equations, and the Cauchy-Riemann equations that govern the conditions for a function of a complex variable to be analytic. Cauchy produced almost 800 mathematics papers, a truly overwhelming achievement. Cauchy's genius found expression not only in his work on the foundations of real and complex analysis, areas to which his name is inextricably linked, but also in many other fields such as his major contributions to the development of mathematical physics and to theoretical mechanics. His two theories of elasticity and his investigations on the theory of light, research which required that he develop whole new mathematical techniques such as Fourier transforms, diagonalization of matrices, and the calculus of residues.

As I write the phrase "Cauchy-Riemann equations that govern . . .," above, an image comes to mind that makes me laugh out loud and I just have to share it with you. I see the complex-plane in front of me and it is pictured as a flat horizontal plane going off into the distance to infinity; a surreal image as often seen in Salvador Dali paintings. There is mist or wispy fog rising slowly from various parts of the plane and there are a few large, dark holes in the plane that seem to be bottomless. And, way off in the distance, are two tiny figures, hand-in-hand, trudging around one of the holes. This is Cauchy leading Riemann around a singularity. (For further elucidation, see the subject of contour integration and the Cauchy integral equations.) Enough nonsense, back to Cauchy's personality.

Obviously, Cauchy was an outstanding student, researcher, and mathematical genius, but had a reputation of arrogance and self-infatuation. Take a look at the portrait of Cauchy above. Whoever the artist was that did that portrait certainly captured the arrogance and self-infatuation. Cauchy did not have particularly good relations with other scientists. He was a staunch Catholic and his religious views often caused trouble; he would bring religion into his scientific work as he did on giving a report on the theory of light in 1824 when he attacked the author (the deceased Isaac Newton) for his belief that people did not have souls. In short, he was a genius but an extreme religious nut! Cauchy was once described by a journalist who said:- *... it is certainly a curious thing to see an academician who seemed to fulfill the respectable functions of a missionary preaching to the heathens.* An example of how Cauchy treated colleagues is given by Poncelet whose work on projective geometry had, in 1820, been criticized by Cauchy:- *... I managed to approach my too rigid judge at his residence ... just as he was leaving ... During this very short and very rapid walk, I quickly perceived that I had in no way earned his regards or his respect as a scientist ... without allowing me to say anything else, he abruptly walked off, referring me to a forthcoming publication where, according to him, 'the question would be very properly explored'.* Again his treatment of Norwegian mathematician Niels Henrik Abel during this period was unfortunate. Abel, wrote of him:- *Cauchy is mad and there is nothing that can be done about him, although, right now, he is the only one who knows how mathematics should be done.*

One last thing about Cauchy. In a May 2001 article in "The American Mathematical Monthly" (pages 432-436) by Erik Talvila entitled "Some Divergent Trigonometric Integrals", the author writes that while examining a table of definite integrals, he came across four divergent trigonometric entries with incorrect finite values. Guess what–the well-known mathematician who made the original error was none other than the famous French genius Augustin-Louis Cauchy. Even geniuses are sometimes wrong; his error had persisted for over 140 years. Enough about Cauchy—let's prove the theorem.

Let $I(a,b)=\int_0^\infty \frac{f(ax)-f(bx)}{x}dx$. Now, make a CV of $x = u/b$, so that $dx = du/b$, and $(0, \infty) \rightarrow (0, \infty)$. Hence,

$$I(a,b)=\int_0^\infty \frac{f(ax)-f(bx)}{x}dx=\int_0^\infty \frac{f\left(\frac{a}{b}u\right)-f(u)}{\frac{u}{b}}\frac{du}{b}=\int_0^\infty \frac{f\left(\frac{a}{b}u\right)-f(u)}{u}du$$

The integral now depends upon the single parameter – the ratio of a to b, so let us call this ratio q, i.e., $q = a/b$. We then have,

$$I(q)=\int_0^\infty \frac{f(qu)-f(u)}{u}du$$

Let us now differentiate with respect to q, that is,

$$\frac{dI}{dq}=\frac{d\left(\int_0^\infty \frac{f(qu)-f(u)}{u}du\right)}{dq}=\int_0^\infty \frac{d[f(qu)]}{dq}\frac{du}{u}$$

Now, using the chain rule, we have

$$\frac{d[f(qu)]}{dq}=\frac{d[f(qu)]}{d(qu)}\cdot\frac{d(qu)}{dq}=\frac{d[f(qu)]}{d(qu)}\cdot u$$

Therefore,

$$\frac{dI}{dq}=\int_0^\infty \frac{df(qu)}{d(qu)}du=\frac{1}{q}\int_0^\infty \frac{df(qu)}{d(qu)}d(qu)=\frac{1}{q}[f(qu)]_0^\infty$$

And, evaluation of this last expression gives us

$$\frac{dI}{dq}=\frac{f(\infty)-f(0)}{q}$$

Solving this differential equation yields $I=[f(\infty)-f(0)]\log(q)+C$ where C is an arbitrary constant of integration. Now, obviously, when $a = b$, I will be zero, and this condition allows us to evaluate C, namely, $C = 0$. Thus, we have proved the proposition, namely

$$\boxed{\int_0^\infty \frac{f(ax)-f(bx)}{x}dx=[f(\infty)-f(0)]\log\left(\frac{a}{b}\right)\quad \textbf{Q.E.D.}}$$

By using the Frullani/(Cauchy) Theorem, some very, very, extremely complex integrals can be evaluated without doing much work at all. The following table gives a representative list of Frullani integrals and their values. The 1st column of the table merely lists the function used in the integrand, namely, $f(x)$. The 2nd column shows the entire integrand, while the 3rd column lists any restrictions or limitations on any parameters in the integrand (sans parameters a and b).

Finally, the 4th and last column specifies the value of the Frullani integral. It is understood that the integration interval is always $(0, \infty)$ and that the parameters a and b belong to $\mathbb{R}^+$.

$f(x)$	Integrand	Parameters	Integral Value
e^{-x}	$\frac{e^{-ax}-e^{-bx}}{x}$	------	$\log\left(\frac{b}{a}\right)$
e^{-x^p}	$\frac{e^{-ax^p}-e^{-bx^p}}{x}$	$p \in \mathbb{R}^+$	$\frac{1}{p}\log\left(\frac{b}{a}\right)$
$\frac{e^{-px}-e^{-qx}}{x}$	$\frac{e^{-apx}-e^{-aqx}}{ax^2} - \frac{e^{-bpx}-e^{-bqx}}{bx^2}$	$p, q \in \mathbb{R}^+$	$(q-p)\log\left(\frac{b}{a}\right)$
$\frac{xe^{-cx}}{(1-e^{-x})}$	$\frac{ae^{-ce^{ax}}}{1-e^{-ax}} - \frac{be^{-ce^{bx}}}{1-e^{-bx}}$	$c \in \mathbb{R}^+$	$e^{-c}\log\left(\frac{b}{a}\right)$
$(x+c)^{-\mu}$	$\frac{(ax+c)^{-\mu}-(bx+c)^{-\mu}}{x}$	$c, \mu \in \mathbb{R}^+$	$c^{-\mu}\log\left(\frac{b}{a}\right)$
$\tan^{-1}(x)$	$\frac{\tan^{-1}(ax)-\tan^{-1}(bx)}{x}$	-------	$\frac{\pi}{2}\log\left(\frac{a}{b}\right)$
$\log(p+qe^{-x})$	$\frac{\log(p+qe^{-ax})-\log(p+qe^{-bx})}{x}$	$p, q \in \mathbb{R}^+$	$\log\left(\frac{p}{p+q}\right)\log\left(\frac{a}{b}\right)$
$(ab/x)\log(1+x)$	$\frac{b\log(1+ax)-a\log(1+bx)}{x^2}$	------	$ab\log\left(\frac{b}{a}\right)$
$\left(1+\frac{p}{x}\right)^x$	$\frac{\left(1+\frac{p}{ax}\right)^{ax}-\left(1+\frac{p}{bx}\right)^{bx}}{x}$	$p \in \mathbb{R}^+$	$(e^p-1)\log\left(\frac{a}{b}\right)$
$\frac{p+qe^{-x}}{ce^x+g+qe^{-x}}$	$\left(\frac{p+qe^{-ax}}{ce^{ax}+g+he^{-ax}} - \frac{p+qe^{-bx}}{ce^{bx}+g+he^{-bx}}\right)\frac{1}{x}$	$p, q, c, g, h \in \mathbb{R}^+$	$\frac{p+q}{c+g+h}\log\left(\frac{b}{a}\right)$
$\log(p+qe^{-x})$	$\frac{1}{x}\log\left(\frac{p+qe^{-ax}}{p+qe^{-bx}}\right)$	$p, q \in \mathbb{R}^+$	$\log\left(1+\frac{q}{p}\right)\log\left(\frac{b}{a}\right)$
$\left(\frac{x+p}{x+q}\right)^n$	$\left[\left(\frac{ax+p}{ax+q}\right)^n - \left(\frac{bx+p}{bx+q}\right)^n\right]\frac{1}{x}$	$p, q \in \mathbb{R}^+ \quad n \in \mathbb{N}^+$	$\left(1-\frac{p^n}{q^n}\right)\log\left(\frac{a}{b}\right)$

These last two entries in the table were found in volume 1 of Ramanujan's Notebook (Hardy would have been impressed). The integrand for the next-to-the-last entry does not look like a Frullani type of integral until you remember that the logarithm of a fraction is the same as the difference of the logarithms of the numerator and denominator. There are, of course, many more entries (an unending list) that could be put into the table, but I think that the point we are trying to emphasize has been made. That is, some extremely complex integrals can be easily evaluated using this theorem. Enough said!

8.2 Euler's Log Sine Integral

Figure 8-2. Swiss mathematician Leonhard Euler (1707 – 1783)

"Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate."—Leonhard Euler

In 1769, Swiss mathematician Leonhard Euler (1707-1783) solved the following integral and that accounts for the fact that we sometimes refer to it as the Euler Log Sine integral:

$$\int_0^{\pi/2} \log(\sin\theta)d\theta$$

For many years it was thought that this integral was best tackled using the techniques of contour integration. As you will shortly see, that is incorrect. A very creative and ingenious manipulation of this integral can easily arrive at the value of this properly improper integral. Before we do that, however, let's address a little bit about Euler's mathematical life. A "little bit" is the operative phrase here because Euler's work in mathematics is so vast that one can only give a very superficial account of it. He was the most prolific writer of mathematics of all time (60 to 80 volumes—80% of which are in Latin). A 20th century science historian has estimated that of all the mathematical and scientific work published during the whole of the 18th century, a full 25% was written by Euler. Incredible!!!!!!

As I write these words, I recall an episode that occurred when I was a graduate student studying for my Master's oral exam. My advisor had called me into his office a few days before the exam was to take place to ensure that I was ready, ask me a few questions, and see if I had any questions for him. At one point, we were going over a power series expansion of an exponential function (if I remember correctly) and he asked me if I knew who was responsible for this particular series. I didn't and then I asked him if I would encounter questions like that during the exam. He answered by saying it was unlikely; however, if you do and don't know, simply look the professor who asked the question directly in the eye and say that Euler did it. He then said, chances are that Euler did do it, but even if he didn't, Euler's work is so voluminous and mostly written in Latin, the professor who asked will have no earthly idea whether you are right or wrong.

Euler was not only a magnificent mathematician, but he made contributions to mathematics that even most mathematicians don't know about. We have Euler to thank for the notation $f(x)$ for a function, e for the base of the natural logarithms, i for the square root of -1, π for the ratio of a circle's circumference to its diameter, Σ for summation, Δ for finite differences, and many, many more. Euler was the first who defined the trigonometric functions as proportions of lines. He was also the one who introduced the notation for these functions. Incredible!!!!

Euler is also responsible for the following equation, now known as Euler's identity:

$$e^{i\pi} + 1 = 0$$

Euler's identity has been called the most remarkable formula in mathematics for its single use of addition, multiplication, exponentiation, and equality and its single use of the five constants, e, i, π, 1, and 0. In 1988, readers of the *Mathematical Intelligencer* voted it the most beautiful mathematical formula ever. In fact, Euler was responsible for 3 of the top 5 formulae in that poll. Incredible!!!!!

However, enough about Euler, let's see how he went about evaluating this integral. Let

$$I = \int_0^{\pi/2} \log(\sin x)\, dx$$

The first thing we need to do is convince ourselves that *I* also equals $\int_0^{\pi/2} \log(\cos x) dx$. One can do this in many different ways. One way, is to examine the graph of the two integrands that are shown in figure 8-3. You will notice that between 0 and π/2, the integration interval, the two

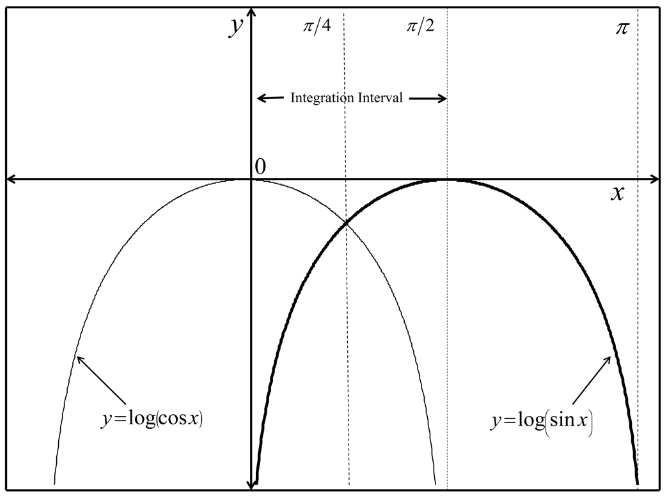


Figure 8-3. Graph of log(sin*x*) and log(cos*x*)

functions take on exactly the same values–just not in the same order; log(cos*x*) goes from 0 to – ∞ while log(sin*x*) goes from – ∞ to 0. So obviously, the integrals of these two functions over that interval have to be the same. If you don't find this argument convincing, then make the following change of variable in the original integral, i.e., $x = \pi/2 - u$ (note that this is the IP CV addressed in Chapter 3). With this change of variable, we have

$$I = \int_0^{\pi/2} \log(\sin x)\, dx = -\int_{\pi/2}^{0} \log\left[\sin\left(\tfrac{\pi}{2} - u\right)\right] du = \int_0^{\pi/2} \log(\cos x).$$

Since the sine of an angle is the cosine of its compliment and the compliment of $\pi/2 - u$ is u (or x—since u is a dummy variable). The point is, we can now write

$$2I = \int_0^{\pi/2} \log(\sin x) dx + \int_0^{\pi/2} \log(\cos x) dx = \int_0^{\pi/2} [\log(\sin x) + \log(\cos x)] dx.$$

However, by the property of logarithms (the sum of two logarithms is the same as the logarithm of the product of their arguments), this can be rewritten as

$$2I = \int_0^{\pi/2} \log(\sin x \cos x)\, dx.$$

Now using the trigonometric identity $\sin(2x) = 2\sin(x)\cos(x)$, this last equality becomes

$$2I = \int_0^{\pi/2} \log\left[\tfrac{1}{2}\sin(2x)\right] dx.$$

Again, we use the property of logarithms, and we have

$$2I = \int_0^{\pi/2} \log\left(\frac{1}{2}\right) dx + \int_0^{\pi/2} \log[\sin(2x)]\, dx.$$

Note that the first integral can now be evaluated. In the second integral, again a change of variable is required. Let $2x = u$ so that $dx = ½du$ and $(0, \pi/2) \to (0, \pi)$. We then have

$$2I = \log\left(\frac{1}{2}\right) \int_0^{\pi/2} dx + \frac{1}{2}\int_0^{\pi} \log(\sin u)\, du = \frac{\pi}{2}\log\left(\frac{1}{2}\right) + \frac{1}{2}\int_0^{\pi} \log(\sin x) dx.$$

In this last expression on the right of the second equal sign, we have evaluated the first integral and rewritten the second integral in terms of x instead of the dummy variable u. Now, re-examine the graph of $y = \log(\sin x)$ in the previous figure. You will note that it is symmetric about the vertical line $x = \pi/2$ which halves the integration interval $(0, \pi)$. Therefore, we can double the integral and halve the integration interval, i.e.,

$$2I = \frac{\pi}{2}\log\left(\frac{1}{2}\right) + \int_0^{\pi/2} \log(\sin x) dx = \frac{\pi}{2}\log\left(\frac{1}{2}\right) + I\,.$$

Eureka! This last integral is I, our original integral. Solving for I, we get Euler's final result. Namely

$$\boldsymbol{I = \int_0^{\frac{\pi}{2}} \log(\sin x) dx = \frac{\pi}{2}\log\left(\frac{1}{2}\right)} \quad \textbf{Q.E.D.}$$

What a brilliant piece of work! Also, note the simplicity of the mathematics involved in Euler's derivation—it's the total idea that required Euler's genius. In my opinion, the "aha moment" is when Euler realized that $2I$ was the sum of the two integrals of $\log(\sin x)$ and $\log(\cos x)$.

By "piggy-backing" off of Euler's dazzling derivation, many related integrals can also be evaluated. The following table contains a few of the more obvious examples.

Integral and Value
$\int_0^{\pi/2} \log(\tan x) dx = \int_0^{\pi/2} \log(\cot x) dx = 0$
$\int_0^{\pi/2} \log(\sec x)\, dx = \int_0^{\pi/2} \log(\csc x) dx = \frac{\pi}{2}\log(2)$
$\int_0^{\pi/2} \log[a\sin^n(x)]\, dx = \frac{\pi}{2}\log\left(\frac{a}{2^n}\right) \quad a \in \mathbb{R}^+ \quad n \in \mathbb{N}^+$
$\int_0^{\pi/2} \log[a\sin(x)]^n\, dx = \frac{\pi}{2}\log\left(\frac{a}{2}\right)^n \quad a \in \mathbb{R}^+ \quad n \in \mathbb{N}^+$
$\int_0^{\pi/2} \log[a\tan(x)]^n dx = \frac{\pi}{2}\log(a)^n \quad a \in \mathbb{R}^+ \quad n \in \mathbb{N}^+$
$\int_0^{\pi/2} \log[a\sec(x)]^n dx = \frac{\pi}{2}\log(2a)^n \quad a \in \mathbb{R}^+ \quad n \in \mathbb{N}^+$
$\int_0^{\pi/2} \log\left[\frac{a\sin(x)}{x}\right]^n dx = \frac{n\pi}{2}\left[\log\left(\frac{a}{\pi}\right) + 1\right] \quad a \in \mathbb{R}^+ \quad n \in \mathbb{N}^+$
$\int_0^{\infty} \frac{\log(1+x^2)^n}{1+x^2} dx = \pi\log 2^n \quad n \in \mathbb{N}^+$
$\int_0^{1} \frac{\log\left(x+\frac{1}{x}\right)}{1+x^2} dx = \frac{\pi}{2}\log(2)$
$\int_0^{1} \frac{\log(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{2}\log\left(\frac{1}{2}\right)$
$\int_0^{\infty} \frac{xe^{-x}}{\sqrt{1-e^{-2x}}} dx = \frac{\pi}{2}\log(2)$

The last four integrals in the above table do not, at first glance, appear to be related to Euler's log sine integral. However, using the appropriate integral properties (such as changes of variable, breaking the integral up into different integration intervals, etc.) these integrals can be manipulated so that Euler's log sine integral can be used to obtain their value. For example, in the next-to-the-last integral in the table, make the change of variable $x = \sin(u)$.

I would now like to present an alternate approach to Euler's log sine integral that, I think, deserves to be in the crème de la crème chapter. It's extremely clever. Back in Chapter 5 (the DUI Chapter), the following result was shown in example #6.

$$\int_0^\infty \frac{\tan^{-1}\left(\frac{x}{a}\right)}{x(x^2+b^2)}dx = \frac{\pi}{2b^2}\log\left(\frac{a+b}{a}\right).$$

Now make a change of variable. Let $x = b\tan(\theta)$ so that $dx = b\sec^2(\theta)d\theta$ and $(0, \infty) \to (0, \pi/2)$. Under this change of variable, we get

$$\int_0^{\pi/2} \frac{\tan^{-1}\left[\frac{b}{a}\tan(\theta)\right]b\sec^2(\theta)}{b\tan(\theta)[b^2\tan^2(\theta)+b^2]}d\theta = \frac{1}{b^2}\int_0^{\pi/2} \frac{\tan^{-1}\left[\frac{b}{a}\tan(\theta)\right]}{\tan(\theta)}d\theta = \frac{\pi}{2b^2}\log\left(\frac{a+b}{a}\right)$$

Clearing the b^2 term on both sides leaves us with the following equality

$$\int_0^{\pi/2} \frac{\tan^{-1}\left[\frac{b}{a}\tan(\theta)\right]}{\tan(\theta)}d\theta = \frac{\pi}{2}\log\left(\frac{a+b}{a}\right)$$

Now let's consider the particular case of $b/a = 1$. The above equality simplifies to

$$\int_0^{\pi/2} \frac{\tan^{-1}[\tan(\theta)]}{\tan(\theta)}d\theta = \int_0^{\pi/2} \theta\cot(\theta) = \frac{\pi}{2}\log(2)$$

Integrate the last integral by parts with $u = \theta$ so that $du = d\theta$ and $dv = \cot(\theta)d\theta$ so that $v = \log[\sin(\theta)]$. We therefore have

$$[\theta\log[\sin(\theta)]]_0^{\pi/2} - \int_0^{\pi/2}\log[\sin(\theta)]\,d\theta = \frac{\pi}{2}\log(2)$$

The first term above vanishes and we are left with the desired result—WOW!

$$\boxed{\int_0^{\pi/2}\log[\sin(\theta)]\,d\theta = \frac{\pi}{2}\log\left(\frac{1}{2}\right)}$$

8.3 Wolstenholme's Integrals

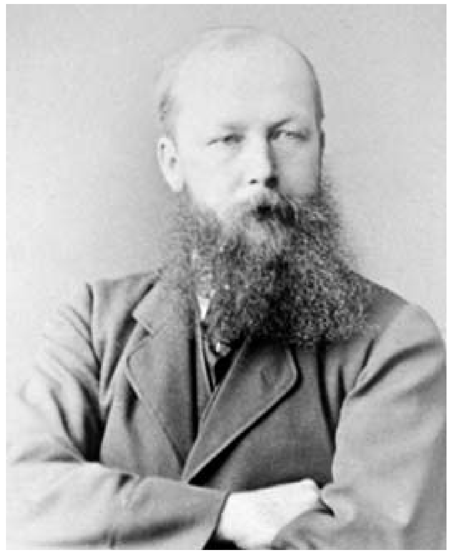


Figure 8-4. English Mathematician Joseph Wolstenholme (1829-1891)

As our next crème de la crème integral we are going to take up two integrals that look very much like Euler's log sine integral. Joseph Wolstenholme is the mathematician responsible for the solution of these integrals which we will designate as I_1 and I_2, where

$$I_1 = \int_0^{\pi/2} \log^2(a \sin x) dx \text{ and } I_2 = \int_0^{\pi/2} [\log(a \sin x)]\,[\log(a \cos x)] dx.$$

These two integrals look like they are related to Euler's log sine integral and, indeed, they are in the sense that Wolstenholme's solution cannot be completed without Euler's result. However, Wolstenholme's solution is extremely clever—so clever, that I consider it to be a very good entrant for this crème de la crème chapter. Before we get into the mathematics of the solution, however, let's address a bit about this lesser known English mathematician.

Joseph Wolstenholme was born near Manchester, England to the wife of a Methodist minister. He studied mathematics at Wesley College in Sheffield, and then entered St John's College at Cambridge on 1 July 1846 and, four years later, he graduated third in his class. While at Cambridge, he became good friends with Leslie Stephen who also studied mathematics and would eventually become the father of the author Virginia Woolf. Virginia (Stephen) Woolf was a young girl when Wolstenholme shared the family holidays in St. Ives so she got to know Wolstenholme pretty well. She later incorporated Wolstenholme into one of her most famous books, "*To the Lighthouse*" as the character Mr. Augustus Carmichael although the character Carmichael is not portrayed as a mathematician in the novel.

Wolstenholme was the author of a number of mathematical papers. They were usually concerned with questions of analytical geometry, and they were earmarked by an unusual analytical skill and ingenuity. This ingenuity becomes apparent in the derivation of the aforementioned integrals, I_1 and I_2. His greatest contribution towards mathematics was his volume of mathematical problems. Wolstenholme's problems have proved a help and a stimulus to many students. A collection of some three thousand problems naturally varies widely in value, but many of them contain important results, which in other places or at other times would not infrequently have been embodied in original papers. As they stand, they form a curious and almost unique monument of ability and industry.

Now let's see how Wolstenholme went about obtaining the values of I_1 and I_2. To do so however, we need, as did Wolstenholme, some preliminary results, namely the values of three other integrals which we will call I_3, I_4, and I_5. They are,

$$I_3 = \int_0^1 \frac{[\log(x)]^2}{1+x^2} dx \quad I_4 = \int_0^\infty \frac{[\log(x)]^2}{1+x^2} dx \quad \text{and} \quad I_5 = \int_0^{\pi/2} \log^2(\tan x) dx$$

To be more accurate, only the value of I_5 is needed to solve Wolstenholme's integrals, however, in order to obtain the value of I_5, one must first obtain the value of I_4 and similarly, to obtain the value of I_4, one needs the value of I_3. Fortunately, both I_3 and I_4 have already been evaluated. I_3 was done in Chapter 6 as an example of the technique of swapping the order of integration and summation; its value was found to be $\pi^3/16$ (see example #2 of Chapter 6). Similarly, I_4 was done in Chapter 7 as an example of the technique of interval normalization; its value was found to be $\pi^3/8$ (see example #2 of Chapter 7). The third integral we need, I_5, is "duck soup". Make a change of variable in I_4. Let $x = \tan(u)$ and you will see that I_4 and I_5 are equal. Therefore, we are all set to begin Wolstenholme's brilliant derivation.

First, note that by the simple IP transformation of $y = \pi/2 - x$, $I_1 = \int_0^{\pi/2} \log^2(a \cos x) dx$ (just as Euler noted in his log sine integral). We will start this derivation with, what I'm guessing was Wolstenholme's "aha" moment, namely, the following expression.

$$\int_0^{\pi/2} [\log(a \sin x) - \log(a \cos x)]^2 dx$$

To me, this expression is key to the following thoughts. The integrand of this expression is merely the square of a simple binomial; granted, each of the two terms of the binomial are somewhat complex, but that's not relevant. What is relevant is that when we complete the squaring process, we are going to have an expression that involves both I_1 and I_2. Then, by using the property of logarithms (ala Euler's log sine integral), we may be able to eventually arrive at an expression that is a function of both I_1 and I_2 but that involves no integral, in other words, a linear equation in which the variables are I_1 and I_2. Then, if we can follow that same idea with a different starting binomial expression, we may be able to arrive at another (but different) linear equation, also involving I_1 and I_2. If so, we could then solve the two linear equations (like young Jr. High School students are taught to do) as a pair of simultaneous equations. Aha!!!!! And oh, the different binomial expression might simply be a change of sign (from – to +) in the above starting expression.

Back to the starting expression—when the starting expression is squared, we obtain

$$\int_0^{\pi/2} \log^2(a \sin x)\, dx - 2\int_0^{\pi/2} \log(a \sin x) \log(a \cos x) dx + \int_0^{\pi/2} \log^2(a \cos x) dx = 2I_1 - 2I_2 \,.$$

However, the starting expression, using the property of logarithms, can also be written as

$$\int_0^{\pi/2} \log^2\left(\frac{a \sin x}{a \cos x}\right) dx = \int_0^{\pi/2} \log^2(\tan x)\, dx = I_5 = \frac{\pi^3}{8} = 2I_1 - 2I_2 \,.$$

Yes, indeed! Sure enough, we have a linear equation in two variables (the two variables being I_1 and I_2). Now, as mentioned above, we follow the same methodology as before with the different binomial expression, that is, squaring and we have

$$\int_0^{\pi/2}[\log(a\sin x)+\log(a\cos x)]^2 dx = 2I_1 + 2I_2\ .$$

As before, using the property of logarithms, we can also write

$$2I_1 + 2I_2 = \int_0^{\pi/2}\log^2(a^2\sin x\cos x)\,dx = \int_0^{\pi/2}\log^2\left(\frac{a^2}{2}\sin 2x\right)dx = \int_0^{\pi/2}\left[\log\left(\frac{a}{2}\right)+\log(a\sin 2x)\right]^2 dx$$

Now, squaring this last integrand, we obtain

$$2I_1 + 2I_2 = \log^2\left(\frac{a}{2}\right)\int_0^{\pi/2}dx + 2\log\left(\frac{a}{2}\right)\int_0^{\pi/2}\log(a\sin 2x)dx + \int_0^{\pi/2}\log^2(a\sin 2x)dx\ .$$

In the 2nd and 3rd integrals, make a change of variable so that $x = u/2$ so that $dx = du/2$ and $(0,\pi/2) \to (0,\pi)$. Hence

$$2I_1 + 2I_2 = \log^2\left(\frac{a}{2}\right)\int_0^{\pi/2}dx + 2\log\left(\frac{a}{2}\right)\int_0^{\pi}\log(a\sin u)\left(\frac{1}{2}\right)du + \frac{1}{2}\int_0^{\pi}\log^2(a\sin u)du.$$

Note that now the integration intervals for the 2nd and 3rd integrals in the above expression now go from 0 to π. However, both functions in the integrands of those two integrals are symmetric about the vertical line $x = \pi/2$ (see figure 8-3). Therefore, we can double the integral's value and integrate merely from 0 to $\pi/2$. Thus,

$$2I_1 + 2I_2 = \log^2\left(\frac{a}{2}\right)\int_0^{\pi/2}dx + 2\log\left(\frac{a}{2}\right)\int_0^{\pi/2}\log(a\sin x)dx + \int_0^{\pi/2}\log^2(a\sin x)dx.$$

Now note that the 1st integral above can be evaluated, the 2nd integral is Euler's log sine integral, and the 3rd integral is I_1. So we now have the second linear equation that we needed, namely,

$$2I_1 + 2I_2 = \log^2\left(\frac{a}{2}\right)[x]_0^{\pi/2} + 2\log\left(\frac{a}{2}\right)\left[\frac{\pi}{2}\log\left(\frac{a}{2}\right)\right] + I_1 \quad \text{or} \quad I_1 + 2I_2 = \frac{3\pi}{2}\log^2\left(\frac{a}{2}\right).$$

Lest you've forgotten, the first linear equation was $2I_1 - 2I_2 = \frac{\pi^3}{8}$. Solving these two equations simultaneously, we have the final value of the two Wolstenholme integrals, namely,

$$\boxed{\boldsymbol{I_1 = \int_0^{\frac{\pi}{2}}\log^2(a\sin x)dx = \frac{\pi}{2}\log^2\left(\frac{a}{2}\right) + \frac{\pi^3}{24}}}$$

$$\boxed{\boldsymbol{I_2 = \int_0^{\pi/2}\log(a\sin x)\log(a\cos x)dx = \frac{\pi}{2}\log^2\left(\frac{a}{2}\right) - \frac{\pi^3}{48}}.\quad \textbf{Q.E.D.}}$$

Thank you Wolstenholme, that was truly very, very ingenious.

8.4 The Methodology of Leibniz

Figure 8-5. German Mathematician Gottfried Wilhelm Leibniz (1646-1716)

"Music is the pleasure the human mind experiences from counting without being aware it is counting."—Gottfried Leibniz

The technique of evaluating an integral by finding a differential equation for which the integral is the solution can be used to evaluate the following integral:

$$I(a) = \int_0^\infty \frac{\cos(ax)}{x^2+b^2}\,dx \quad a, b, \ \epsilon \ \mathbb{R}^+,$$

where *a* is the parameter and *b* is a constant. The cleverness of this specific evaluation makes it a good candidate for the crème de la crème chapter. Leibniz is not the person responsible for coming up with this evaluation, however; it is not known who did. We shall therefore bio Leibniz himself in place of the unknown genius since it is Leibniz's technique that allows for the evaluation of this integral.

Today, Leibniz is credited, along with Sir Isaac Newton, with the discovery of the differential and integral calculus. However, back in their day, the so-called calculus controversy (calculus was not the name used back in the 17^{th} century) was an argument between Newton and Leibniz over who had first invented the mathematical study of changing variables (i.e., "calculus"). The controversy was often referred to with the German term *Prioritatsstreit*—meaning priority dispute. Instead of a brief description of Leibniz's early life and credentials (as has been done with the other mathematicians I've included in this book), I would like to address this controversy, since it's quite interesting.

As a student, I had always heard that there was great contention between Leibniz and Newton. And, indeed, there surely was. From what I've read about them, I would say that Leibniz was not only a very brilliant man but also a very ambitious man. Meanwhile, Newton is also this very talented genius who is so absorbed in his work that he doesn't bother to publish anything he discovers. This difference in the two personalities is at the core of the problem between them. Leibniz is busy publishing and Newton has already discovered everything that Leibniz thinks he should get credit for; but "publish or perish" has evidently always been the protocol so Leibniz has every right to feel put upon.

From about 1710 until his death 6 years later, Leibniz was engaged in a long, bitter dispute with Newton and others, over whether he (Leibniz) had invented calculus independently of Newton, or whether he had merely invented another notation for ideas that were fundamentally Newton's. Leibniz began working on his variant of calculus in 1674, and in 1684 published his first paper employing it, "Nova Methodus pro Maximis et Minimis". Newton claimed to have begun working on a form of calculus (which he called "the method of fluxions and fluents") in 1666, at the age of 23, but did not publish it except as a minor annotation in the back of one of his publications decades later. Yet there was seemingly no proof beyond Newton's word. He had published a calculation of a tangent with the note: "This is only a special case of a general method whereby I can calculate curves and determine maxima, minima, and centers of gravity." However, the most remarkable aspect of this dispute was that no participant involved (and there were many) doubted for a moment that Newton had already developed his method of fluxions when Leibniz began working on the differential calculus.

Today, we make use of the calculus and consider it to have been "invented" by both Leibniz and Newton, but it is Leibniz's notation that we use (and what great notation it is!). For example, Leibniz introduced the integral sign ∫, representing an elongated S, from the Latin word *summa*, and the *d* used for differentials, from the Latin word *differentia*. This cleverly suggestive notation for calculus is probably his most enduring mathematical legacy. As Bertrand Russell once said, "*A good notation has a subtlety and suggestiveness which at times make it seem almost like a live teacher.*" However, don't demean Leibniz's mathematical expertise; he was brilliant and after all, the product rule of differential calculus is called the "Leibniz's law". In addition, the theorem that tells how and when to differentiate under the integral sign is called the Leibniz integral rule and we will use it to calculate $I(a)$ above.

Alright, now back to this wonderful evaluation of

$$I(a) = \int_0^\infty \frac{\cos ax}{x^2+b^2}\,dx.$$

If we integrate by parts with $u = 1/(x^2+b^2)$ and $dv = \cos(ax)dx$, we will get $du = -2x/(x^2+b^2)^2$ and $v = (1/a)\sin(ax)$. We therefore obtain

$$I(a) = \left[\frac{\sin(ax)}{a(x^2+b^2)^2}\right]_0^\infty + \frac{2}{a}\int_0^\infty \frac{x\sin(ax)}{(x^2+b^2)^2}\,dx$$

Since the 1st term on the right vanishes when evaluated at both the lower and upper limits, this last equation can be written as

$$aI(a) = 2\int_0^\infty \frac{x\sin(ax)}{(x^2+b^2)^2}\,dx.$$

At first glance, it appears as though we are going nowhere with this integration by parts because it certainly looks like we now have an integral that is more complex than that with which we started. However, I believe this last equation is key to the following thoughts. Differentiating with respect to the parameter *a* will create an integral in which the numerator of the integrand will contain a cosine function and, of course, the original integral contains a cosine function in the numerator. So, after differentiating, if we split the integrand into partial fractions, one of the fractions should be able to be written as a function of *I*(*a*) itself (because of the cosine function) and so, the differential equation needed to use this differentiation technique will emerge. So, with that in mind, we continue with the differentiation and obtain

$$a\frac{dI(a)}{da} + I(a) = 2\int_0^\infty \frac{x^2\cos(ax)}{(x^2+b^2)^2}dx.$$

Now, expanding this integrand into partial fractions, we have

$$\frac{x^2\cos(ax)}{(x^2+b^2)^2} = \frac{\cos(ax)}{x^2+b^2} - \frac{b^2\cos(ax)}{(x^2+b^2)^2}.$$

Hence,

$$a\frac{dI(a)}{da} + I(a) = 2\int_0^\infty \frac{x^2\cos(ax)}{(x^2+b^2)^2}dx = 2\int_0^\infty \frac{\cos(ax)}{x^2+b^2}dx - 2b^2\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}dx$$

Now note that the first term on the right of the second equals sign is exactly 2*I*(*a*). So we can rewrite this last equation to be

$$a\frac{dI(a)}{da} + I(a) = 2I(a) - 2b^2\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}dx.$$

Of course, this simplifies to the following expression

$$a\frac{dI(a)}{da} - I(a) = -2b^2\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}dx.$$

Certainly, the differential equation that we are seeking is starting to appear. But aren't we still stuck with a more complex integral? Yes, but I believe that now comes the aha moment and it's a dilly! Envision what happens if we differentiate again; the left side of the above will contain a second derivative which means that our equation is becoming a second-order differential equation. Is that bad? Not if we can solve the equation because the value of our original integral will be the solution to that 2^{nd} order DE. What's the point in doing this though? Well, look what happens to the right side of the above equation upon differentiation. An *x* times sin(*ax*) will appear in the numerator of the integrand. That is the same integral we already saw when we integrated by parts back in the beginning and it equals *aI*(*a*). So we come to the realization that our differential equation will indeed emerge after differentiating a second time, albeit, a second-order one. So, upon differentiating again, we get

$$a\frac{d^2I(a)}{da^2} + \frac{dI(a)}{da} - \frac{dI(a)}{da} = 2b^2\int_0^\infty \frac{x\sin(ax)}{(x^2+b^2)^2}dx$$

Upon substituting *aI*(*a*) for the integral and then re-arranging, we have

$$\frac{d^2I(a)}{da^2} - b^2I(a) = 0$$

Such second-order differential equations of the above format are well-known to have exponential solutions. If we let $I(a) = Ce^{ka}$ where C and k are constants and then substitute that into our differential equation we get

$$Ck^2e^{ak} - b^2(Ce^{ak}) = 0$$

from which we see that $k = \pm b$. Therefore, we have two particular solutions to the differential equation; the general solution will simply be their sum. Thus

$$I(a) = C_1e^{ab} + C_2e^{-ab}$$

where C_1 and C_2 are different constants of integration that can be determined by looking for two different conditions on $I(a)$. In this case, the two conditions can be gotten from our two different expressions for $I(a)$, e.g.,

$$I(a) = \int_0^\infty \frac{\cos(ax)}{x^2+b^2}dx \quad \text{and} \quad I(a) = \frac{2}{a}\int_0^\infty \frac{x\sin(ax)}{(x^2+b^2)^2}dx.$$

From the 2nd of these two integrals, we see that

$$\lim_{a\to\infty} I(a) = \lim_{a\to\infty} \frac{2}{a}\int_0^\infty \frac{x\sin(ax)}{(x^2+b^2)^2}dx = \lim_{a\to\infty}(C_1e^{ab} + C_2e^{-ab}) = 0$$

The only way in which this last term can be zero is if C_1 is zero. Hence, our general solution reduces to

$$I(a) = C_2e^{-ab}.$$

Now examine the first of the two integrals and note that

$$I(0) = \int_0^\infty \frac{1}{x^2+b^2}dx = \left[\frac{1}{b}\tan^{-1}\frac{x}{b}\right]_0^\infty = \frac{\pi}{2b} = C_2.$$

So finally, we are able to write down the value of this integral due to this remarkable evaluation, that is,

$$\boxed{\int_0^\infty \frac{\cos(ax)}{x^2+b^2}dx = \frac{\pi}{2b}e^{-ab} \quad \text{Q.E.D.}}$$

8.5 A Power Series Approach

The next integral that I feel belongs in the crème de la crème chapter is the following:

$$I = \int_0^1 \log(1+x)\log(1-x)dx.$$

If you make the change of variable $x = -u$, $dx = -du$, and $(0, 1) \to (0, -1)$, you will see that the integrand doesn't change—the only difference is that the integration interval is now -1 to 0, e.g.,

$$I = -\int_0^{-1} \log(1-u)\log(1+u)du = \int_{-1}^0 \log(1+x)\log(1-x)dx.$$

Hence, we can integrate from -1 to 1 and take ½ the result and the integral will still have the same value. Thus,

$$I = \frac{1}{2}\int_{-1}^{1} \log(1+x)\log(1-x)dx.$$

Now we are going to make another change of variable. This time we will let $x = 2u - 1$, $dx = 2du$, and $(-1, 1) \rightarrow (0, 1)$. We thus obtain

$$I = \frac{1}{2}\int_{0}^{1} \log(2u)\log[2(1-u)](2du).$$

By the property of logarithms this can be written as

$$I = \int_{0}^{1}[\log(2) + \log(u)][\log(2) + \log(1-u)]du.$$

The next step is to multiply out the two square bracketed terms in the above integral. Doing that we have

$$I = \int_{0}^{1}[\log^2(2) + \log(2)\log(u) + \log(2)\log(1-u) + \log(u)\log(1-u)]du.$$

Combining the middle two terms into one gives us

$$I = \log^2(2)\int_{0}^{1} du + \log(2)\int_{0}^{1}[\log(u) + \log(1-u)]du + \int_{0}^{1}\log(u)\log(1-u)\,du.$$

In the middle integral above, $\log(u)$ and $\log(1-u)$ take on the same set of values over the integration interval (see accompanying diagram which is their graph) and therefore the integrand may be replaced by $2\log(u)$.

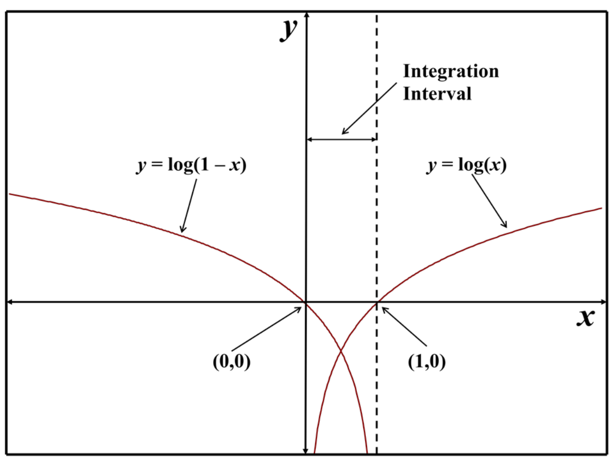


Figure 8-6. Graph of log(x) and log(1–x)

Hence,

$$I = \log^2(2)\int_{0}^{1} du + 2\log(2)\int_{0}^{1}\log(u)du + \int_{0}^{1}\log(u)\log(1-u)\,du.$$

The first two integrals now readily integrate—the second one by parts and it evaluates to –1, so we have

$$I = \log^2(2) - 2\log(2) + \int_{0}^{1}\log(x)\log(1-x)dx$$

Finally, this last integral can be cracked using a power series approach. We need a power series for $\log(1 - x)$. Here is a very good way to obtain such a power series. We have

$$\log(1-x) = -\int_0^x \frac{1}{1-t}dt.$$

However, if we actually perform the indicated division we see that

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots = \sum_{k=0}^{\infty} t^k.$$

Therefore,

$$\log(1-x) = -\int_0^x \left(\sum_{k=0}^{\infty} t^k\right) dt.$$

Reversing the order of integration and summation, this last expression becomes the desired power series.

$$\log(1-x) = -\sum_{k=0}^{\infty} \int_0^x t^k dt = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}.$$

So, the integral becomes

$$I = \log^2(2) - 2\log(2) - \int_0^1 \left[\log(x)\left(\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}\right)\right] dx.$$

Once again, we interchange the order of integration and summation and we obtain

$$I = \log^2(2) - 2\log(2) - \sum_{k=0}^{\infty} \left(\frac{1}{k+1}\right) \int_0^1 x^{k+1} \log(x) dx.$$

The remaining integral can now be evaluated using the integration by parts technique; it evaluates to $-1/(k+2)^2$. We are therefore left with

$$I = \log^2(2) - 2\log(2) + \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)^2}.$$

We now have to evaluate this sum in order to arrive at a final result, and you will notice that this sum is not included in our table of useful infinite series from Chapter 1. However (and here is what I believe must have been the aha moment), using the technique of partial fractions, this summation can be written as

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)^2} = \sum_{k=0}^{\infty} \left[\frac{1}{k+1} - \frac{1}{k+2} - \frac{1}{(k+2)^2}\right].$$

We now write this last expression as two separate sums, e.g.,

$$\sum_{k=0}^{\infty} \left[\frac{1}{k+1} - \frac{1}{k+2}\right] - \sum_{k=0}^{\infty} \frac{1}{(k+2)^2}.$$

Interestingly, here is a case of when the compact upper case sigma notation for a summation actually hinders us (at least it hinders me) from seeing our way forward. Look at the individual terms of the first sum in the above expression; they are $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots$; everything cancels except the leading term which is 1. Also notice that the second sum is merely useful infinite series #2 (Chapter 1) minus the first term of that series, in-other-words, it sums to $\pi^2/6 - 1$. So we have

$$\sum_{k=0}^{\infty}\left[\frac{1}{k+1} - \frac{1}{k+2}\right] - \sum_{k=0}^{\infty}\frac{1}{(k+2)^2} = 1 - \left(\frac{\pi^2}{6} - 1\right) = 2 - \frac{\pi^2}{6}.$$

We can now write down the final result

$$\boxed{\int_0^1 \log(1+x)\log(1-x)dx = \log^2(2) - 2\log(2) + 2 - \frac{\pi^2}{6}. \quad \text{Q.E.D.}}$$

8.6 A Recursion Relationship

In Chapter 2, example 7, we evaluated the following integral

$$I = \int_0^{\pi/2} \frac{d\theta}{a\sin^2(\theta) + b\cos^2(\theta)} = \frac{\pi}{2\sqrt{ab}}$$

It would be quite interesting to generalize this integral by raising the entire denominator to the n^{th} power (where $n \in \mathbb{N}^+$) and see if the integral's value can be calculated. If so, an entire family of integrals can be solved in one derivation. So, the integral we are now interested in is

$$I_n(a,b) = \int_0^{\pi/2} \frac{d\theta}{[a\sin^2(\theta) + b\cos^2(\theta)]^n} \quad n \in \mathbb{N}^+$$

Look what happens if we differentiate $I_n(a,b)$ with respect to the parameter a.

$$\frac{dI_n(a,b)}{da} = \int_0^{\pi/2} -n[a\sin^2(\theta) + b\cos^2(\theta)]^{-n-1}\sin^2(\theta)d\theta = -n\int_0^{\pi/2} \frac{\sin^2\theta d\theta}{[a\sin^2(\theta) + b\cos^2(\theta)]^{n+1}}$$

Similarly, we can now easily write down the result of differentiating with respect to b, i.e.,

$$\frac{dI_n(a,b)}{db} = -n\int_0^{\pi/2} \frac{\cos^2(\theta)d\theta}{[a\sin^2(\theta) + b\cos^2(\theta)]^{n+1}}$$

Now notice that the sum of these two derivatives gives us a numerator of $\sin^2(\theta) + \cos^2(\theta) = 1$. That is,

$$\frac{dI_n(a,b)}{da} + \frac{dI_n(a,b)}{db} = -n\int_0^{\pi/2} \frac{\sin^2(\theta) + \cos^2(\theta)}{[a\sin^2(\theta) + b\cos^2(\theta)]^{n+1}} d\theta = -nI_{n+1}(a,b)$$

If we now replace n with $n - 1$, we get the wonderful recursion relationship of

$$I_n(a,b) = -\frac{1}{n-1}\left[\frac{dI_{n-1}(a,b)}{da} + \frac{dI_{n-1}(a,b)}{db}\right].$$

As alluded to earlier, $I_1(a,b) = \frac{\pi}{2\sqrt{ab}}$ (see Chapter 2, example 7). To calculate I_2, all we have to do is plug $n = 2$ into the recursion and carry out the indicated operations. That is,

$$\frac{dI_1(a,b)}{da} = \frac{d\left(\frac{\pi}{2\sqrt{ab}}\right)}{da} = \frac{\pi}{2\sqrt{b}}\frac{d\left(a^{-\frac{1}{2}}\right)}{da} = -\frac{\pi}{4a^{3/2}b^{1/2}}$$

$$\frac{dI_1(a,b)}{db} = \frac{\pi}{2\sqrt{a}}\frac{d\left(b^{-\frac{1}{2}}\right)}{db} = -\frac{\pi}{4a^{1/2}b^{3/2}}$$

So, we finally have the value of I_2, namely,

$$\boldsymbol{I_2(a,b) = \int_0^{\pi/2} \frac{d\theta}{[a\sin^2(\theta) + b\cos^2(\theta)]^2} = \frac{\pi}{4\sqrt{ab}}\left(\frac{1}{a} + \frac{1}{b}\right)} \quad \textbf{Q.E.D.}$$

Oh, but we aren't done yet. Now that we have the value of I_2, we can use it to obtain I_3, then I_3 begets I_4, I_4 begets I_5, . . . ad nausea.

$$\boldsymbol{I_3(a,b) = \int_0^{\pi/2} \frac{d\theta}{[a\sin^2(\theta) + b\cos^2(\theta)]^3} = \frac{\pi}{16\sqrt{ab}}\left(\frac{3}{a^2} + \frac{2}{ab} + \frac{3}{b^2}\right)} \quad \textbf{Q.E.D.}$$

However, one finds that when computing I_4 from I_3, I_5 from I_4, etc., the algebra gets very messy, boring, and tiresome. Nevertheless, in "A Treatise of the Integral Calculus" by Joseph Edwards, Volume 2, Chapter 26, page 192, a general expression is derived for I_{n+1}. Here is what Edwards has to say about it: Since

$$\left(\frac{d}{da}\right)^p \left(\frac{d}{db}\right)^q a^{-1/2}b^{-1/2} = \frac{(-1)^{p+q}}{2^{p+q}}\left(1 \cdot 3 \cdots \overline{2p-1}\right)\left(1 \cdot 3 \cdots \overline{2q-1}\right)\frac{1}{\sqrt{ab}} \cdot \frac{1}{a^p b^q}$$

Which, in turn equates to

$$\frac{(-1)^{p+q}}{2^{2(p+q)}}\frac{(2p)!(2q)!}{p!q!}\frac{1}{\sqrt{ab}}\frac{1}{a^p b^q}.$$

Edwards then says, "The general result is"

$$I_{n+1} = \frac{\pi}{2}\frac{(-1)^n}{n!}\left[\frac{d}{da} + \frac{d}{db}\right]^n \frac{1}{\sqrt{ab}}$$

$$I_{n+1} = \frac{\pi}{2^{2n+1}}\frac{1}{\sqrt{ab}}\sum_0^n \frac{(2p)!\,(2q)!}{(p!)^2(q!)^2} \cdot \frac{1}{a^p b^q}, \quad \text{where } p + q = n$$

(I'm glad I didn't have to figure it out!)

8.7 Ahmed's Integral

The definite integral

$$I = \int_0^1 \frac{\tan^{-1}\left(\sqrt{2+x^2}\right)}{(1+x^2)\sqrt{2+x^2}}dx$$

has become known as Ahmed's Integral since it was proposed in 2001 to the American Mathematical Monthly by Zafar Ahmed. Zafar Ahmed is a nuclear physicist who works at the Nuclear Physics Division, Bhabha Atomic Research Centre in Mumbai, India. Since its proposal, this integral has been mentioned in mathematical encyclopedias and dictionaries and further, been cited and discussed in several books and journals. I'm not sure why it has received so much attention; however, its solution is clever enough in my mind for it to be in this crème de la crème chapter. Meanwhile, I have a thought about this integral. Back in Chapter 5, the DUI chapter, I discussed nuclear physicist Richard Feynman's use of DUI and told the little story about how, during the development of the atomic bomb in Los Alamos, Feynman talked a colleague into using the DUI technique to solve an integral needed to continue their work. Since Ahmed's integral presumably came out of Ahmed's work on the Indian atomic bomb project, my flight-of-imagination is that the two integrals are one and the same. (Probably not, but it's my fantasy!) Oh, and by-the-way, Ahmed's integral can be solved using DUI—as was Feynman's.

As you can see, Ahmed's integral contains no parameters—therefore we need to insert one and we will do so as an argument to the inverse tangent function in the integrand's numerator, i.e.,

$$I(q) = \int_0^1 \frac{\tan^{-1}\left(q\sqrt{2+x^2}\right)}{(1+x^2)\sqrt{2+x^2}}dx.$$

Note that $I(1)$ is Ahmed's integral and also note the following:

$$I(\infty) = \lim_{q\to\infty} I(q) = \lim_{q\to\infty} \int_0^1 \frac{\tan^{-1}\left(q\sqrt{2+x^2}\right)}{(1+x^2)\sqrt{2+x^2}}dx = \frac{\pi}{2}\int_0^1 \frac{dx}{(1+x^2)\sqrt{2+x^2}}dx.$$

If this last integral can be evaluated (and it can), it will provide the initial condition needed to determine the constant of integration that results from solving the differential equation that will occur after we differentiate the Ahmed integral with respect to the inserted parameter. As a-matter-of-fact, let's go ahead and attack $I(\infty)$ right now before we apply DUI to $I(q)$.

$$I(\infty) = \frac{\pi}{2}\int_0^1 \frac{dx}{(1+x^2)\sqrt{2+x^2}}.$$

A change of variable is what is called for here. Let $u = x/\sqrt{2+x^2}$. That's a weird CV isn't it? How is that arrived at? Good question! And I can't answer that satisfactorily. I'd like to say that in an aha moment, I realized that the CV would do such-and-such and therefore permit a solution, but that never happened. The answer to the question about the CV is the following. The CV was arrived at through experimentation. I worked on this initial condition integral for the better part of a week, trying all sorts of different CVs before I tried $u = x/\sqrt{2+x^2}$ which finally yielded the solution (lucky). Anyway, under this CV $(0,1) \to \left(0,\ 1/\sqrt{3}\right)$ and, in order to calculate dx we need to solve for x as a function of u. Doing that yields $x = \sqrt{2}u(1-u^2)^{-1/2}$. As a result,

$$dx = \frac{\sqrt{2}du}{(1-u^2)\sqrt{1-u^2}} \text{ and } 1+x^2 = \frac{1+u^2}{1-u^2} \text{ and } \sqrt{2+x^2} = \frac{\sqrt{2}}{\sqrt{1-u^2}}$$

Now, making all of these substitutions into $I(\infty)$ produces the following integral

$$I(\infty) = \frac{\pi}{2}\int_0^{1/\sqrt{3}} \frac{\frac{\sqrt{2}du}{(1-u^2)\sqrt{1-u^2}}}{\left(\frac{1+u^2}{1-u^2}\right)\frac{\sqrt{2}}{\sqrt{1-u^2}}}$$

Look at the integrand of this last integral. Everything cancels except the $1 + u^2$ in the denominator leaving us with the recognizable form of the inverse tangent. What an amazing CV—it may have been arrived at by sheer luck, but it certainly does the job. So, we have

$$I(\infty)=\frac{\pi}{2}\int_0^{1/\sqrt{3}}\frac{du}{1+u^2}=\left[\frac{\pi}{2}\tan^{-1}(u)\right]_0^{1/\sqrt{3}}=\frac{\pi}{2}\cdot\frac{\pi}{6}=\frac{\pi^2}{12}$$

Now we can turn our attention to Ahmed's integral itself. Don't forget, what we just accomplished is only an initial condition that we will undoubtedly need to put the finishing touches on Ahmed's integral. Using DUI, we first need to differentiate with respect to the inserted parameter, q.

$$\frac{dI(q)}{dq}=\int_0^1\frac{\sqrt{2+x^2}}{[1+q^2(2+x^2)](1+x^2)\sqrt{2+x^2}}dx=\int_0^1\frac{dx}{(1+x^2)(1+2q^2+q^2x^2)}$$

Now, breaking this last integral up into partial fractions, we obtain

$$\frac{dI(q)}{dq}=\int_0^1\left[\frac{\frac{1}{1+q^2}}{1+x^2}-\frac{\frac{q^2}{1+q^2}}{1+2q^2+q^2x^2}\right]dx=\frac{1}{1+q^2}\left[\int_0^1\frac{dx}{1+x^2}-\int_0^1\frac{q^2dx}{1+2q^2+q^2x^2}\right]$$

We can now perform the integration, recognizing that the two integrals are inverse tangent functions. If you don't see this in the last integral, divide both numerator and denominator by q^2,

$$\frac{dI(q)}{dq}=\frac{1}{1+q^2}\int_0^1\frac{dx}{1+x^2}-\frac{1}{1+q^2}\int_0^1\frac{dx}{\frac{1+2q^2}{q^2}+x^2}=\left[\frac{1}{1+q^2}\tan^{-1}(x)\right]_0^1-\left[\frac{1}{1+q^2}\frac{q}{\sqrt{1+2q^2}}\tan^{-1}\left(\frac{qx}{\sqrt{1+2q^2}}\right)\right]_0^1$$

Evaluating, we finally have the differential equation that when solved, will produce the value of Ahmed's integral. That is,

$$\frac{dI(q)}{dq}=\frac{1}{1+q^2}\left(\frac{\pi}{4}\right)-\frac{q}{(1+q^2)\sqrt{1+2q^2}}\tan^{-1}\left(\frac{q}{\sqrt{1+2q^2}}\right)$$

That's quite a horrendous looking differential equation. Then comes the aha moment and it's what makes this derivation so damn clever. Instead of solving the differential equation using indefinite integration, use definite integration—specifically, integrate over $(1,\infty)$. Why this particular interval? Look what happens to the left side of the differential equation

$$\int_1^\infty\frac{dI(q)}{dq}dq=[I(q)]_1^\infty=I(\infty)-I(1)$$

$I(\infty)$ is the integral that we solved at the beginning of this derivation and $I(1)$ is Ahmed's integral, the integral we are trying to solve. We therefore have,

$$I(\infty)-I(1)=\frac{\pi}{4}\int_1^\infty\frac{dq}{1+q^2}-\int_1^\infty\frac{q\tan^{-1}\left(q/\sqrt{1+2q^2}\right)}{(1+q^2)\sqrt{1+2q^2}}dq.$$

The 1st integral above is easy, however, the 2nd integral requires a change of variable. Let $q=1/z$ so that $dq=-dz/z^2$ and $(1,\infty)\to(1,0)$.

$$I(\infty)-I(1)=\left[\frac{\pi}{4}\tan^{-1}(q)\right]_1^\infty-\int_1^0\frac{\frac{1}{z}\tan^{-1}\left(\frac{1/z}{\sqrt{1+2/z^2}}\right)\left(-\frac{dz}{z^2}\right)}{(1+1/z^2)\sqrt{1+2/z^2}}$$

This simplifies to the following

$$I(\infty)-I(1)=\frac{\pi}{4}\left[\frac{\pi}{2}-\frac{\pi}{4}\right]-\int_0^1\frac{\tan^{-1}\left(\frac{1}{\sqrt{z^2+2}}\right)}{(z^2+1)\sqrt{z^2+2}}dz=\frac{\pi^2}{16}-\int_0^1\frac{\tan^{-1}\left(\frac{1}{\sqrt{z^2+2}}\right)}{(z^2+1)\sqrt{z^2+2}}dz$$

Now, from the table of useful trigonometric identities in Chapter 1, entry #11, we have

$$\tan^{-1}\left(\frac{1}{\sqrt{z^2+2}}\right) = \frac{\pi}{2} - \tan^{-1}\left(\sqrt{z^2+2}\right)$$

Hence, we can write

$$I(\infty) - I(1) = \frac{\pi^2}{16} - \int_0^1 \frac{\pi/2 - \tan^{-1}\left(\sqrt{z^2+2}\right)}{(z^2+1)\sqrt{z^2+2}} dz = \frac{\pi^2}{16} - \frac{\pi}{2}\int_0^1 \frac{dz}{(z^2+1)\sqrt{z^2+2}} + \int_0^1 \frac{\tan^{-1}\left(\sqrt{z^2+2}\right)}{(z^2+1)\sqrt{z^2+2}} dz$$

Notice anything remarkable? The 1st integral above is again $I(\infty)$ and the 2nd integral above is again Ahmed's integral, $I(1)$. So we finally have a solution by simply solving for $I(1)$ since we already know $I(\infty)$. Thus,

$$I(\infty) - I(1) = \frac{\pi^2}{16} - I(\infty) + I(1)$$

$$\boxed{\int_0^1 \frac{\tan^{-1}\left(\sqrt{2+x^2}\right)}{(1+x^2)\sqrt{2+x^2}} dx = \frac{5\pi^2}{96} \quad \text{Q.E.D.}}$$

I'd like to show an alternate derivation of Ahmed's integral. The following is Ahmed's own derivation of his integral. Interestingly, one of the last steps in the previous derivation was to make use of the trigonometric identity

$$\tan^{-1}\left(\frac{1}{\sqrt{z^2+2}}\right) = \frac{\pi}{2} - \tan^{-1}\left(\sqrt{z^2+2}\right)$$

Ahmed makes use of this same identity as his first step. That is,

$$I = \int_0^1 \frac{\tan^{-1}\left(\sqrt{2+x^2}\right)}{(1+x^2)\sqrt{2+x^2}} dx = \int_0^1 \frac{\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{\sqrt{2+x^2}}\right)}{(1+x^2)\sqrt{2+x^2}} dx = \frac{\pi}{2}\int_0^1 \frac{dx}{(1+x^2)\sqrt{2+x^2}} - \int_0^1 \frac{\tan^{-1}\left(\frac{1}{\sqrt{2+x^2}}\right)}{(1+x^2)\sqrt{2+x^2}} dx.$$

Ahmed calls the 1st integral on the right I_1 and the 2nd integral I_2. He then solves I_1 by making the following change of variable. Let $x = \tan(\theta)$ so that $dx = \sec^2(\theta)d\theta$ and $(0, 1) \to (0, \pi/4)$. Under that CV, we have

$$I_1 = \frac{\pi}{2}\int_0^{\pi/4} \frac{\sec^2(\theta)d\theta}{(1+\tan^2\theta)\sqrt{2+\tan^2\theta}} = \frac{\pi}{2}\int_0^{\pi/4} \frac{\sec^2(\theta)d\theta}{\sec^2(\theta)\sqrt{2+\tan^2\theta}} = \frac{\pi}{2}\int_0^{\pi/4} \frac{d\theta}{\sqrt{2+\tan^2(\theta)}}.$$

This last integral can now be further manipulated to give

$$I_1 = \frac{\pi}{2}\int_0^{\pi/4} \frac{d\theta}{\sqrt{2+\frac{\sin^2(\theta)}{\cos^2(\theta)}}} = \frac{\pi}{2}\int_0^{\pi/4} \frac{\cos(\theta)d\theta}{\sqrt{2\cos^2(\theta)+\sin^2(\theta)}} = \frac{\pi}{2}\int_0^{\pi/4} \frac{\cos(\theta)d\theta}{\sqrt{2-\sin^2(\theta)}}$$

Now make another change of variable, i.e., let $\sin(\theta) = \sqrt{2}\sin(\phi)$. Of course, under that CV the $\cos(\theta)d\theta = \sqrt{2}\cos(\phi)\,d\phi$ and $(0, \pi/4) \to (0, \pi/6)$. And we have

$$I_1 = \frac{\pi}{2}\int_0^{\pi/6} \frac{\sqrt{2}\cos(\phi)d\phi}{\sqrt{2(1-\sin^2\phi)}} = \frac{\pi}{2}\int_0^{\pi/6} \frac{\sqrt{2}\cos(\phi)d\phi}{\sqrt{2}\cos(\phi)} = \frac{\pi^2}{12}.$$

Next, Ahmed goes about solving I_2 in the following manner. Remember, I_2 is the following integral

$$I_2 = \int_0^1 \frac{\tan^{-1}\left(\frac{1}{\sqrt{2+x^2}}\right)dx}{(1+x^2)\sqrt{2+x^2}}\,.$$

Ahmed says we can replace the inverse tangent function in the numerator of I_2 with another integral, turning I_2 into a double integral. An inverse tangent function can be represented in the following manner:

$$\frac{1}{a}\tan^{-1}\left(\frac{1}{a}\right) = \int_0^1 \frac{dy}{a^2+y^2}.$$

Note that this is simply the recognizable form of the inverse tangent but over (0, 1). Ahmed says let $a = \sqrt{2+x^2}$ and substitute that value of a into the above. If we do that we get

$$\frac{1}{\sqrt{2+x^2}}\tan^{-1}\left(\frac{1}{\sqrt{2+x^2}}\right) = \int_0^1 \frac{dy}{(2+x^2)+y^2} \;\Rightarrow\; \tan^{-1}\left(\frac{1}{\sqrt{2+x^2}}\right) = \sqrt{2+x^2}\int_0^1 \frac{dy}{2+x^2+y^2}$$

Now take this form of the inverse tangent function and substitute it for the inverse tangent function in I_2, obtaining

$$I_2 = \int_0^1 \frac{\sqrt{2+x^2}\left(\int_0^1 \frac{dy}{2+x^2+y^2}\right)}{(1+x^2)\sqrt{2+x^2}}dx = \int_0^1\int_0^1 \frac{dxdy}{(1+x^2)(2+x^2+y^2)}\,.$$

How clever, the radicals cancel!!!! That substitution must have been Ahmed's aha moment. We now have the promised double integral. Ahmed's next step is to use partial fractions on the integrand of the double integral.

$$I_2 = \int_0^1\int_0^1 \frac{1}{(1+y^2)}\left[\frac{1}{(1+x^2)} - \frac{1}{(2+x^2+y^2)}\right]dxdy$$

This, of course, can be split into two double integrals

$$I_2 = \int_0^1\int_0^1 \frac{dxdy}{(1+y^2)(1+x^2)} - \int_0^1\int_0^1 \frac{dxdy}{(1+y^2)(2+x^2+y^2)}$$

The 1st integral is easy, it evaluates to $(\pi/4)(\pi/4) = \pi^2/16$. The 2nd integral is even easier, it's I_2, given that y is a dummy variable. Hence,

$$2I_2 = \frac{\pi^2}{16} \quad \text{or} \quad I_2 = \frac{\pi^2}{32}$$

Therefore, we get the expected result of

$$\boxed{\boldsymbol{I = I_1 - I_2 = \frac{\pi^2}{12} - \frac{\pi^2}{32} = \frac{5\pi^2}{96}} \quad \textbf{Q.E.D.}}$$

8.8 Coxeter's Integrals

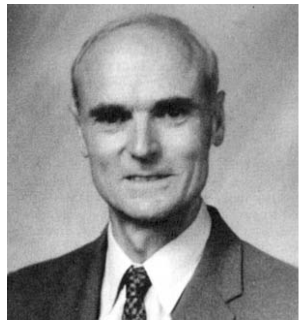


Figure 8-7. British/Canadian Mathematician Harold Scott MacDonald Coxeter (1907-2003)

"In our times, geometers are still exploring those new Wonderlands, partly for the sake of their applications to cosmology and other branches of science, but much more for the sheer joy of passing through the looking glass into a land where the familiar lines, planes, triangles, circles, and spheres are seen to behave in strange but precisely determined ways."—Harold Scott MacDonald Coxeter

Donald Coxeter was always known as Donald which came from his third name MacDonald. This needs a little explanation. He was first given the name MacDonald Scott Coxeter, but a godparent suggested that his father's name should be added, so Harold was added at the front. Another relative noted that H M S Coxeter made him sound like a British warship. A permutation of the names resulted in Harold Scott MacDonald Coxeter. Although Coxeter was born and educated in Great Britain, in 1936 he took up an appointment at the University of Toronto and remained on the faculty at Toronto until his death. This explains the dual British/Canadian tag in the caption of his picture.

Although not as well-known a name as most of the other mathematicians profiled in this book, Coxeter was very well-known within the mathematical community. He became one of the world's eminent geometers and is considered as the world's greatest geometer of the 20^{th} century. In particular he made major contributions in the theory of polytopes, non-euclidean geometry, group theory and combinatorics. His interest in geometry stems from a very early age. As a 19 year old teen-ager and an undergraduate at Trinity College, Cambridge, Coxeter undertook a study of various four dimensional shapes. By geometrical considerations and evidently verified graphically, the study suggested to him several quite "interesting" definite integrals, two of which we will evaluate shortly. However, before doing so, there is a little story associated with Coxeter and his integrals that I find quite interesting.

In 1926, Coxeter is 19 years old and is very busy with his study of four dimensional shapes. And he also has these definite integrals and would like very much to solve them (evaluate them). Evidently, he needs help. Well, here is how an enterprising young man (and a brilliant one also) gets the help he needs. He writes a letter to the *Mathematical Gazette* (i.e., the British analog of the *American Mathematical Monthly*) and asks if any reader can suggest how to solve these integrals. And of course, he provides a list of the integrals. He was inundated with responses. Every math buff who subscribed to the Gazette probably sent him their solutions. Decades later, in the Preface to his book *Twelve Geometric Essays*, Coxeter writes "I can still recall the thrill of receiving a solution from G.H. Hardy (see the preface to this book) during my second month as a freshman at Cambridge." Accompanying Hardy's solutions (which if your Coxeter, you know they are correct) was a note scribbled in a margin stating that "I tried very hard not to spend time on your integrals, but to me the challenge of a definite integral is irresistible." What an interesting young man Coxeter was. It was probably as thrilling hearing from Hardy as if one were to have gotten a letter from Albert Einstein discussing some common piece of work. WOW!

Coxeter maintained that kind of attitude throughout his career; here is what a colleague wrote about him. *Modern science is often driven by fads and fashions, and mathematics is no exception. Coxeter's style, I would say, is singularly unfashionable. He is guided, I think, almost completely by a profound sense of what is beautiful.*

His 12 books and 167 published articles cover more than mathematical research. Coxeter met Escher in 1954 and the two became lifelong friends. Coxeter's work helped inspire some of Escher's works, specifically, the "Circle Limit" series based on hyperbolic tessellations. Another friend, R Buckminister Fuller, used Coxeter's ideas in his architecture. Coxeter edited the *Canadian Journal of Mathematics* from 1949 till 1958. He also served as Vice President of the *American Mathematical Association*. Coxeter had many artistic gifts, particularly in music. In fact before he became a mathematician he wanted to become a composer. However his interest in symmetry took him towards mathematics and into a career which he loved throughout his life. Coxeter wrote: *I am extremely fortunate for being paid for what I would have done anyway.* Enough about this fascinating human being; let's get to the pleasurable job at hand and look at one of his integrals. The first one we will evaluate we will call I_1.

$$I_1 = \int_0^{\pi/4} \tan^{-1}\left[\sqrt{\frac{\cos(2\theta)}{2\cos^2(\theta)}}\right] d\theta$$

I don't know who was responsible for the following solution; I would love to know, particularly, if the solution was communicated to Coxeter as a result of his request for help in the Mathematical Gazette. That information does not appear to be available. It would really be something if the solution was what G.H. Hardy sent Coxeter (my fantasy). Anyway, we will push on.

To do I_1, we need three preliminary results. I will number them ①, ②, and ③ and refer back to them by those numbers. They are:

① $\int_0^1 \frac{a\,dx}{1+a^2x^2} = \tan^{-1}(a)$

② $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$

③ $\int_0^1 \frac{dx}{(x^2+1)\sqrt{x^2+2}} = \frac{\pi}{6}$

All three of the above integrals are easy. Number ① is merely the recognizable form of the inverse tangent. If you don't see that, make the change of variable of $u = ax$, so that $dx = du/a$, and $(0, 1) \rightarrow (0, a)$, i.e.,

$$\int_0^1 \frac{a\,dx}{1+a^2x^2} = \int_0^a \frac{a\left(\frac{du}{a}\right)}{1+u^2} = [\tan^{-1}(u)]_0^a = \tan^{-1}(a).$$

Number ② is done by simply breaking the integrand up into partial fractions and integrating each fraction, i.e.,

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\left[\int_0^\infty \frac{dx}{x^2+b^2} - \int_0^\infty \frac{dx}{x^2+a^2}\right] = \frac{1}{a^2-b^2}\left[\frac{\tan^{-1}\left(\frac{x}{b}\right)}{b} - \frac{\tan^{-1}\left(\frac{x}{a}\right)}{a}\right]_0^\infty = \frac{\pi}{2ab(a+b)}.$$

Number ③ is easy because we've already done it, although at the time we did, it was difficult. In the previous section (Ahmed's integral—the 1st derivation) we evaluated this integral to obtain an initial condition, i.e., $I(\infty)$. You will recall, we made the very non-intuitive change of variable of $u = x(x^2+2)^{-1/2}$ so that $dx = \sqrt{2}du/(1-u^2)^{3/2}$ and $(0, 1) \rightarrow (0, 1/\sqrt{3})$. We have

$$\int_0^1 \frac{dx}{(x^2+1)\sqrt{x^2+2}} = \int_0^{1/\sqrt{3}} \frac{\frac{\sqrt{2}}{(1-u^2)^{3/2}}}{\frac{\sqrt{2}(1+u^2)}{(1-u^2)^{3/2}}} du = \int_0^{1/\sqrt{3}} \frac{du}{1+u^2} = [\tan^{-1}(u)]_0^{1/\sqrt{3}} = \frac{\pi}{6}.$$

Now we are all set to attack I_1. Using ① above, we have

$$I_1 = \int_0^{\pi/4} \tan^{-1}\left[\sqrt{\frac{\cos(2\theta)}{2\cos^2(\theta)}}\right] d\theta = \int_0^{\pi/4}\int_0^1 \frac{\sqrt{\frac{\cos(2\theta)}{2\cos^2(\theta)}}}{1+\left[\frac{\cos(2\theta)}{2\cos^2(\theta)}\right]x^2} dx d\theta.$$

Reversing the order of integration, we can write

$$I_1 = \int_0^1\int_0^{\pi/4} \frac{\sqrt{\frac{\cos(2\theta)}{2\cos^2(\theta)}}}{1+\left[\frac{\cos(2\theta)}{2\cos^2(\theta)}\right]x^2} d\theta dx$$

By pure trigonometric manipulation, the integrand can now be written as

$$I_1 = \int_0^1\int_0^{\pi/4} \frac{\sqrt{1-2\sin^2(\theta)}\cdot\sqrt{2}\cos(\theta)}{2-2\sin^2(\theta)+[1-2\sin^2(\theta)]x^2} d\theta dx$$

Now, a change of variable will help, that is, let $\sqrt{2}\sin(\theta) = \sin(\phi)$ so that $\sqrt{2}\cos(\theta)d\theta = \cos(\phi)d\phi$ and $2\sin^2(\theta) = \sin^2(\phi)$ while $(0, \pi/4) \rightarrow (0, \pi/2)$. Thus, this change of variable gives us

$$I_1 = \int_0^1\int_0^{\pi/2} \frac{\sqrt{1-\sin^2(\phi)}\cdot\cos(\phi)}{2-\sin^2(\phi)+[1-\sin^2(\phi)]x^2} d\phi dx$$

In this last integral, the integrand can be again manipulated using purely trigonometric identities to arrive at the following

$$I_1 = \int_0^1\int_0^{\pi/2} \frac{\cos^2(\phi)}{\sin^2(\phi)+(x^2+2)\cos^2(\phi)} d\phi dx$$

It took me a long time to make this last step. I said above, it's trig manipulation only. It took a long time for me to write the integer 2 in the denominator in the previous integrand as $2[\sin^2(\phi) + \cos^2(\phi)]$. However, once that aha comes to you, the previous integral does easily become the last integral. Now look what happens when you further manipulate the integrand by dividing both numerator and denominator by $\cos^2(\phi)$. You obtain

$$I_1 = \int_0^1\int_0^{\pi/2} \frac{d\phi dx}{\tan^2(\phi)+x^2+2}$$

Another change of variable where $y = \tan(\phi)$ so that $dy = \sec^2(\phi)d\phi$ and therefore $d\phi = dy/(y^2 + 1)$ and $(0, \pi/2) \rightarrow (0, \infty)$. Therefore

$$I_1 = \int_0^1 \int_0^\infty \frac{dydx}{(y^2+x^2+2)(y^2+1)}$$

Now use ② on the inner integral (i.e., the one with respect to the y variable). Remember the result of ② it was $\pi/[2ab(a + b)]$. In our case, $a^2 = x^2 + 2$ and $b^2 = 1$ and so the result of using ② to evaluate the inner integral, we have

$$I_1 = \frac{\pi}{2}\int_0^1 \frac{dx}{\sqrt{x^2+2}\left(\sqrt{x^2+2}+1\right)}$$

Stick with me, we're getting there. Next, multiply both numerator and denominator by $\sqrt{x^2+2} - 1$. Doing that gives

$$I_1 = \frac{\pi}{2}\int_0^1 \frac{\sqrt{x^2+2}-1}{\sqrt{x^2+2}\left(\sqrt{x^2+2}+1\right)\left(\sqrt{x^2+2}-1\right)}dx = \frac{\pi}{2}\int_0^1 \frac{\sqrt{x^2+2}-1}{\sqrt{x^2+2}(x^2+1)}dx = \frac{\pi}{2}\int_0^1 \left[\frac{1}{x^2+1} - \frac{1}{\sqrt{x^2+2}(x^2+1)}\right]dx$$

The first fraction in the integrand of I_1 is $\pi/4$ (recognizable form of the inverse tangent) and the second fraction in that integrand is $\pi/6$ by the use of ③ above. Hence we have our final result, namely

$$\boldsymbol{I_1 = \int_0^{\pi/4} \tan^{-1}\left[\sqrt{\frac{\cos(2\theta)}{2\cos^2(\theta)}}\right] d\theta = \frac{\pi}{2}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\pi^2}{24}} \quad \textbf{Q.E.D.}$$

Whew! Quite a derivation and worthy of the crème de la crème chapter!

Now, consider the second Coxeter integral that we are going to derive. We will call it I_2.

$$I_2 = \int_0^{\pi/2} \cos^{-1}\left[\frac{\cos(x)}{1+2\cos(x)}\right]dx$$

This derivation is so long and so complex, we are going to outline it as a series of steps (contained in the accompanying table) to perform before we actually do the mathematics. In that way, the outline (or steps) provides a roadmap that allows one to follow without bogging down in the details of the actual algebraic and/or trigonometric manipulations required to arrive at the ultimate value.

Step #	Action
1	Show that $I_2 = 4\int_0^{\pi/4} \tan^{-1}\left[\frac{\cos(u)}{\sqrt{2-3\sin^2(u)}}\right]du$
2	From step 1, show that $I_2 = 4\int_0^{\pi/4}\int_0^1 \frac{\cos(u)\sqrt{2-3\sin^2(u)}}{(x^2+2)-(3+x^2)\sin^2(u)}dxdu$
3	From step 2, show that $I_2 = 8\sqrt{3}\int_0^{\pi/3}\int_0^1 \frac{\cos^2(\theta)}{x^2+2\cos^2(\theta)(x^2+3)}dxd\theta$
4	From step 3, show that $I_2 = 4\sqrt{3}\int_0^1 \frac{1}{x^2+3}\left[\int_0^{\sqrt{3}}\frac{dz}{1+z^2} - \int_0^{\sqrt{3}}\frac{dz}{z^2+3+\frac{6}{x^2}}\right]dx$
5	From step 4, show that $I_2 = \frac{2\pi^2}{9} - 4\int_0^1 \frac{x}{(x^2+3)\sqrt{x^2+2}}\tan^{-1}\left(\frac{x}{\sqrt{x^2+2}}\right)dx$
6	From step 5, show that $I_2 = \frac{5\pi^2}{24}$

Alright, we are now ready to proceed with the details of step 1. As shown in the statement of step 1 above, we are going to attempt to show that

$$I_2 = 4\int_0^{\pi/4} \tan^{-1}\left[\frac{\cos(\theta)}{\sqrt{2-3\sin^2(\theta)}}\right] d\theta.$$

We will start with the simple trigonometric identity of the cosine of a double angle, that is,

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \cos^2(\theta) - [1-\cos^2(\theta)] = 2\cos^2(\theta) - 1$$

What we need to do first is to establish a new trigonometric identity that says

$$\cos^{-1}(2\theta^2 - 1) = 2\cos^{-1}(\theta)$$

Actually, that's pretty easy. In our double angle identity above, temporarily let $\cos(\theta) = k$. Doing this implies that

$$\cos(2\theta) = 2k^2 - 1 \quad \text{and} \quad \theta = \cos^{-1}(k) \quad \text{as well as} \quad 2\theta = \cos^{-1}(2k^2 - 1)$$

Now, substituting for θ in the last expression, we get $2\cos^{-1}(k) = \cos^{-1}(2k^2-1)$.

Since k is just an arbitrary symbol (or dummy variable), we can call it whatever we want. The point is, we have now established the identity $\cos^{-1}(2\theta^2-1) = 2\cos^{-1}(\theta)$. Why did we need this new trigonometric identity? Well, notice that the integrand of I_2 (the integral we want to solve) has an inverse cosine function with an argument of $\cos(x)/[1 + 2\cos(x)]$. We are going to eventually use the new trigonometric identity to manipulate that integrand. Let's get to it. To do so, we also need a new dummy variable. This time let's call it p. Thus,

$$p = 2\theta^2 - 1 \Rightarrow \theta = \sqrt{\frac{1+p}{2}}$$

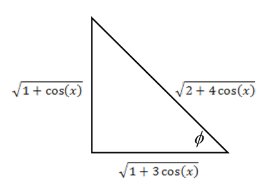
And from our trigonometric identity we get

$$\cos^{-1}(p) = 2\cos^{-1}\left(\sqrt{\frac{1+p}{2}}\right)$$

And our integral becomes

$$I_2 = \int_0^{\pi/2} \cos^{-1}\left[\frac{\cos(x)}{1+2\cos(x)}\right] dx = 2\int_0^{\pi/2} \cos^{-1}\left[\sqrt{\frac{1+\frac{\cos(x)}{1+2\cos(x)}}{2}}\right] dx = 2\int_0^{\pi/2} \cos^{-1}\left[\sqrt{\frac{1+3\cos(x)}{2+4\cos(x)}}\right] dx = 2\int_0^{\pi/2} \phi dx$$

Now the value of an inverse trigonometric function, such as our inverse cosine in the integrand of I_2 above, merely represents some angle which I have called ϕ. Here is the aha moment for step 1.



A right triangle with an acute angle of ϕ by the good old Pythagorean theorem would have sides as shown above. As can be seen, ϕ could also be represented as

$$\phi = \tan^{-1}\left[\frac{\sqrt{1+\cos(x)}}{1+3\cos(x)}\right]$$

Therefore, we are justified in writing

$$I_2 = 2\int_0^{\pi/2} \tan^{-1}\left[\frac{\sqrt{1+\cos(x)}}{1+3\cos(x)}\right] dx$$

Now make the simple change of variable $x = 2u$, so that $dx = 2du$, and $(0, \pi/2) \rightarrow (0, \pi/4)$ and we have

$$I_2 = 4\int_0^{\pi/4} \tan^{-1}\left[\sqrt{\frac{1+\cos(2u)}{1+3\cos(2u)}}\right] du = 4\int_0^{\pi/4} \tan^{-1}\left[\frac{\cos(u)}{\sqrt{2-3\sin^2(u)}}\right] du$$

This completes step 1. The last integral is arrived at by using the trigonometric identity that we started with, i.e., the cosine of a double angle.

We are now ready to begin step 2, and we do so by recognizing the following

$$\int_0^1 \frac{dx}{1+\alpha^2x^2} = \frac{1}{\alpha^2}\int_0^1 \frac{dx}{\frac{1}{\alpha^2}+x^2} = \frac{1}{\alpha^2}[\alpha\tan^{-1}(\alpha x)]_0^1 = \frac{1}{\alpha}\tan^{-1}(\alpha)$$

Now, if we let $\alpha = \frac{\cos(u)}{\sqrt{2-3\sin^2(u)}}$ which is the argument of the inverse tangent function in the latest incarnation of I_2, we therefore have

$$\int_0^1 \frac{dx}{1+\left[\frac{\cos^2(u)}{2-3\sin^2(u)}\right]x^2} = \frac{\sqrt{2-3\sin^2(u)}}{\cos(u)}\tan^{-1}\left[\frac{\cos(u)}{\sqrt{2-3\sin^2(u)}}\right]$$

Or, in-other-words, the integrand of I_2 can be replaced with this formulation and I_2, itself, becomes the following double integral

$$I_2 = 4\int_0^{\pi/4} \frac{\cos(u)}{\sqrt{2-3\sin^2(u)}}\left\{\int_0^1 \frac{1}{1+\left[\frac{\cos^2(u)}{2-3\sin^2(u)}\right]x^2}dx\right\}du = 4\int_0^{\pi/4}\int_0^1 \frac{\cos(u)[2-3\sin^2(u)]}{\sqrt{2-3\sin^2(u)}[2-3\sin^2(u)+x^2\cos^2(u)]}dxdu$$

This last integral can be written as the following which brings us to the end of step 2.

$$I_2 = \int_0^{\pi/4}\int_0^1 \frac{\cos(u)\sqrt{2-3\sin^2(u)}}{2-3\sin^2(u)+x^2[1-\sin^2(u)]}dxdu = 4\int_0^{\pi/4}\int_0^1 \frac{\cos(u)\sqrt{2-3\sin^2(u)}}{(x^2+2)-(3+x^2)\sin^2(u)}dxdu$$

To begin step 3, make a change of variable to the outer integral in the double integral of the previous step. Let $\sin(u) = (⅔)^{½}\sin(\theta)$. Under this CV, we have $\cos(u)du = (⅔)^{½}\cos(\theta)d\theta$ and $(0, \pi/4) \rightarrow (0, \pi/3)$. As a result, we obtain

$$I_2 = 4\int_0^{\pi/3}\int_0^1 \frac{\left(\frac{2}{3}\right)^{½}\cos(\theta)\sqrt{2-3(⅔)\sin^2(\theta)}}{(x^2+2)-⅔(3+x^2)\sin^2(\theta)}dxd\theta = \frac{8}{\sqrt{3}}\int_0^{\pi/3}\int_0^1 \frac{\cos^2(\theta)}{(x^2+2)-⅔(3+x^2)\sin^2(\theta)}dxd\theta$$

This can be further simplified to arrive at the result we want for the conclusion of step 3.

$$I_2 = 8\sqrt{3}\int_0^{\pi/3}\int_0^1 \frac{\cos^2(\theta)}{3x^2+6-2[x^2+3-x^2\cos^2(\theta)-3\cos^2(\theta)]}dxd\theta = 8\sqrt{3}\int_0^{\pi/3}\int_0^1 \frac{\cos^2(\theta)}{x^2+2\cos^2(\theta)(x^2+3)}dxd\theta$$

As the initial action of step 4, make the following change of variable to the outer integral of I_2. Let $z = \tan(\theta)$ so that $dz = \sec^2(\theta)d\theta = d\theta/\cos^2(\theta)$ and $(0, \pi/3) \to (0, \sqrt{3})$. Further, $1 + z^2 = \tan^2(\theta) = \sec^2(\theta) = 1/\cos^2(\theta) \Rightarrow \cos^2(\theta) = 1/(1 + z^2) \Rightarrow d\theta = dz/(1 + z^2)$. So, under this change of variable, I_2 becomes

$$I_2 = 8\sqrt{3}\int_0^{\sqrt{3}}\int_0^1 \frac{\frac{1}{1+z^2}}{x^2+2(x^2+3)\frac{1}{1+z^2}}\left(\frac{dxdz}{1+z^2}\right) = 8\sqrt{3}\int_0^{\sqrt{3}}\int_0^1 \frac{dxdz}{x^2(1+z^2)^2+2(x^2+3)(1+z^2)}$$

Factoring the common $1 + z^2$ term from the denominator, we obtain

$$I_2 = 8\sqrt{3}\int_0^{\sqrt{3}}\int_0^1 \frac{dxdz}{(1+z^2)(x^2z^2+3x^2+6)}$$

Separating the integrand into partial fractions gives us the following expression for I_2,

$$I_2 = 8\sqrt{3}\int_0^{\sqrt{3}}\int_0^1 \left[\frac{\frac{1}{2(x^2+3)}}{1+z^2} - \frac{\frac{x^2}{2(x^2+3)}}{x^2z^2+3x^2+6}\right]dxdz$$

A bit of straight-forward algebraic manipulation along with a reversal of the order of integration gives us the following and the end of step 4.

$$I_2 = 4\sqrt{3}\int_0^1 \frac{1}{x^2+3}\left[\int_0^{\sqrt{3}} \frac{dz}{1+z^2} - \int_0^{\sqrt{3}} \frac{dz}{z^2+3+\frac{6}{x^2}}\right]dx$$

The 1st inner integral is actually something that can finally be integrated. It is the recognizable form of the inverse tangent function and therefore, we have

$$I_2 = 4\sqrt{3}\int_0^1 \frac{1}{x^2+3}\left\{[\tan^{-1}(z)]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{dz}{z^2+3+\frac{6}{x^2}}\right\}dx = 4\sqrt{3}\int_0^1 \frac{1}{x^2+3}\left\{\frac{\pi}{3} - \int_0^{\sqrt{3}} \frac{dz}{z^2+3+\frac{6}{x^2}}\right\}dx$$

The 2nd inner integral is also an inverse tangent form, it's just not as recognizable as the 1st was. However, if we write it as

$$I_2 = 4\sqrt{3}\int_0^1 \frac{1}{x^2+3}\left\{\frac{\pi}{3} - \int_0^{\sqrt{3}} \frac{dz}{\left(\sqrt{3+\frac{6}{x^2}}\right)^2+z^2}\right\}dx$$

We can now see that it is, indeed, an inverse tangent form. We then have

$$I_2 = 4\sqrt{3}\int_0^1 \frac{1}{x^2+3}\left\{\frac{\pi}{3} - \frac{1}{\sqrt{3+\frac{6}{x^2}}}\left[\tan^{-1}\left(\frac{z}{\sqrt{3+\frac{6}{x^2}}}\right)\right]_0^{\sqrt{3}}\right\}dx = 4\sqrt{3}\int_0^1 \frac{1}{x^2+3}\left\{\frac{\pi}{3} - \frac{x}{\sqrt{3}\sqrt{x^2+2}}\tan^{-1}\left(\frac{x}{\sqrt{x^2+2}}\right)\right\}dx$$

Well, we now have eliminated the double integral and reduced it back to a single integral which can now be written as two single integrals, ala

$$I_2 = \frac{4\pi\sqrt{3}}{3}\int_0^1 \frac{dx}{x^2+3} - 4\int_0^1 \frac{x}{(x^2+3)\sqrt{x^2+2}}\tan^{-1}\left(\frac{x}{\sqrt{x^2+2}}\right)dx$$

And, the first of these two integrals is again the recognizable form of an inverse tangent. Thus,

$$I_2 = \frac{4\pi\sqrt{3}}{3}\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_0^1 - 4\int_0^1 \frac{x}{(x^2+3)\sqrt{x^2+2}}\tan^{-1}\left(\frac{x}{\sqrt{x^2+2}}\right)dx$$

And this, of course, simplifies to the following, which brings us to the end of step 5.

$$I_2 = \frac{2\pi^2}{9} - 4\int_0^1 \frac{x}{(x^2+3)\sqrt{x^2+2}} \tan^{-1}\left(\frac{x}{\sqrt{x^2+2}}\right) dx$$

So all that is left, is to now evaluate this last messy-looking integral, which, if possible will bring this stupendous derivation to a conclusion. Fortunately, it can be done and to do so we simply have to invoke our old stand-by methodology of integration by parts. Let

$$u = \tan^{-1}\left(\frac{x}{\sqrt{x^2+2}}\right) \Rightarrow du = \frac{dx}{(x^2+1)\sqrt{x^2+2}} \text{ and } dv = \frac{xdx}{(x^2+3)\sqrt{x^2+2}} \Rightarrow v = \tan^{-1}\left(\sqrt{x^2+2}\right)$$

If you, dear reader, have trouble calculating *du* from *u* and/or *v* from *dv*, I suggest that you work backwards—that is, differentiate the expression for *du* and/or *v* to see that it/they is/are algebraically equivalent to *u* and/or *dv*. (I certainly had trouble writing this last sentence but I can't work that backwards.) My point is that you, dear reader, will at least be satisfied that *du* and *dv* are indeed correct. Given that they are, the integration by parts gives us

$$I_2 = \frac{2\pi^2}{9} - 4\left[\tan^{-1}\left(\frac{1}{\sqrt{x^2+2}}\right)\tan^{-1}\left(\sqrt{x^2+2}\right)\right]_0^1 + 4\int_0^1 \frac{\tan^{-1}\left(\sqrt{x^2+2}\right)}{(x^2+1)\left(\sqrt{x^2+2}\right)} dx$$

Now, the value of the square-bracketed expression above is $\pi^2/18$ and when multiplied by the 4 we get $2\pi^2/9$ which exactly cancels with the like term in the expression for I_2, and so we obtain

$$I_2 = 4\int_0^1 \frac{\tan^{-1}\left(\sqrt{x^2+2}\right)}{(x^2+1)\left(\sqrt{x^2+2}\right)} dx$$

Oh-my-gosh you might say, this remaining integral is much too difficult. Not so! We already did it! Its Ahmed's integral. In the previous section of this chapter we evaluated that precise integral and found its value to be $5\pi^2/96$ and so we have

$$I_2 = 4\left(\frac{5\pi^2}{96}\right) = \frac{5\pi^2}{24}$$

Or, ta-da

$$\boxed{\boldsymbol{I_2 = \int_0^{\pi/2} \cos^{-1}\left[\frac{\cos(x)}{1+2\cos(x)}\right] dx = \frac{5\pi^2}{24}} \quad \textbf{Q.E.D.}}$$

8.9 Rene Descartes

We are going to start out this section with a very simple integral, one that can be evaluated so easily that it can be done in one's head. Here it is

$$I = \int_0^\infty \frac{x^2}{(1+x^3)^2} dx$$

A CV of $u = 1 + x^3$ is enough to bring this integral to its knees: That CV implies $du = 3x^2dx$ and $(0, \infty) \rightarrow (1, \infty)$. Under the stated CV, $= \frac{1}{3}\int_1^\infty \frac{du}{u^2} = \frac{1}{3}\int_1^\infty u^{-2}du = \frac{1}{3}\left[-\frac{1}{u}\right]_1^\infty = \frac{1}{3}$. Why does this simple integral merit inclusion in the Crème de la Crème chapter? It's the origin of this integral and what this integral leads us to that deserves this chapter; its origin and what follows is, I think, quite interesting. You may be familiar with a classic curve of antiquity called The Folium of Descartes. The figure below (figure 8-8) depicts a graph of this famous curve and gives its Cartesian equation.

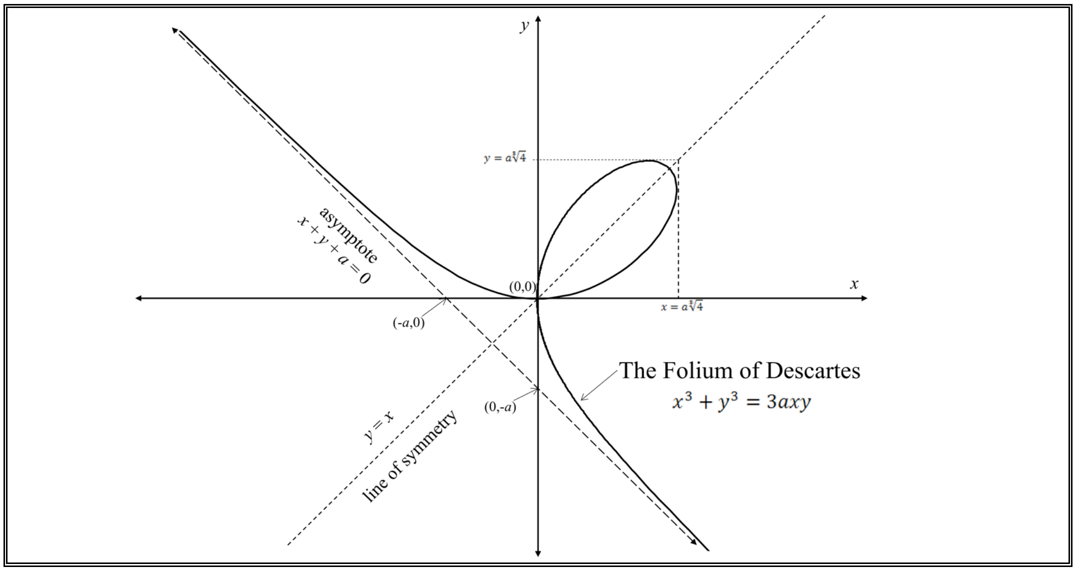


Figure 8-8. The Folium of Descartes

This curve was first discussed by Descartes in 1638; Descartes devised this curve to challenge Fermat's tangent-finding techniques. The challenge itself is another interesting bit of mathematical history which I will discuss subsequently. But before we go any further with the origin of this simple integral and Descartes' challenge, let's talk a little bit about Descartes himself, certainly one of the titans of the mathematical past.

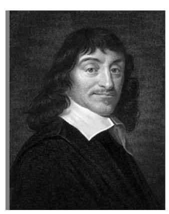


Figure 8-9. French Philosopher/Mathematician Rene Descartes (1596–1650)

"It is not enough to have a good mind; the main thing is to use it well."—Rene Descartes

Modern mathematics began with two great advances, analytic geometry and the calculus. The man who finally crystallized the method of wedding algebra to geometrical proof was René Descartes. Moreover, his persistent rational skepticism, his questioning of how one could ever know truth, has led to what we generally today call "the scientific method," i.e., controlled experiments based on the application of rigid mathematical reasoning.

His life spanned one of the greatest intellectual periods in the history of all civilization. To mention only a few of the giants, Fermat and Pascal were his contemporaries in mathematics. Shakespeare died when Descartes was twenty; Descartes outlived Galileo by eight years; and Newton was eight when Descartes died. Descartes was twelve when Milton was born and

Harvey outlived Descartes by seven years. Father Mersenne, the famous amateur of science and mathematics (famous for the Mersenne primes), was Descartes' older chum, schoolmate, and life-long friend. Cardinal Richelieu was his supporter. Descartes was a very well-rounded individual for this period of history.

The concept of analytic geometry came to Descartes in a dream on November 10, 1619; thus, this day marks the official birthday of modern mathematics. Its formal debut to his contemporaries came on June 8, 1637, with the publication of *La Géométrie as* an appendix to his now famous *Discours de la Méthode.* Descartes was then forty-one years old. Descartes' ground-breaking work of the invention of analytic geometry or coordinate geometry as it is sometimes called had the effect of allowing the conversion of geometry into algebra (and vice versa). It allowed the development of Newton's and Leibniz's subsequent discoveries of calculus. It also unlocked the possibility of navigating geometries of higher dimensions, impossible to physically visualize -- a concept which was to become central to modern technology and physics -- thus transforming mathematics forever.

As Alfred North Whitehead said, "*It is impossible not to feel stirred at the thought of the emotions of men at certain historic moments of adventure and discovery—Columbus when he first saw the Western shore, Pizarro when he stared at the Pacific Ocean, Franklin when the electric spark came from the string of the kite, Galileo when he first turned his telescope to the heavens. Such moments are also granted to students in the abstract regions of thought, and high among them must be placed the morning when Descartes lay in bed and invented the method of coordinate geometry*."

As a soldier, Descartes joined armies and survived fierce battles. As a gentleman and traveler, he visited most of the major sites of late Renaissance Europe. As a teacher, he enjoyed the companionship of royalty. He died in Stockholm of complications acquired while delivering 5 AM instruction to Queen Christina of Sweden, the daughter of Gustavus Adolphus.

Now, let's get back to the origin of the simple integral, designated *I*. In a nut-shell, *I* arises as we attempt to compute the plane loop area of the Folium of Descartes. In order to compute that area, we need a further understanding of the Folium of Descartes. Recollect that the plane area bounded by a given curve may be obtained by $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$, where r is the radial coordinate of a point on the curve, θ is the corresponding angular coordinate, and (α, β) is the angular interval of the plane where the curve area resides. Well, we don't have the polar form of the Folium of Descartes, but I know a quick way to get what we need. Incongruously, a parametric form of the curve will get us there and we can easily get that by letting $y = xt$ in the Cartesian form of the Folium (see figure 8-8).

$$y^3 + x^3 - 3axy = x^3t^3 + x^3 - 3ax^2t = x^3(t^3 + 1) - 3ax^2t = 0$$

This can now be solved for x as a function of t, giving us $x = \frac{3at}{1+t^3}$ and since $y = xt$, $y = \frac{3at^2}{1+t^3}$. Viola, we have derived a parametric relationship for the Folium of Descartes and it is

$$(x, y) = \frac{3at}{1+t^3}(1, t) \quad \text{where} \quad -\infty < t < \infty$$

Now recall that the square of the polar radial coordinate, i.e., r^2, is basically defined as $x^2 + y^2$. That is,

$$x^2 + y^2 = \left(\frac{3at}{1+t^3}\right)^2 + \left(\frac{3at^2}{1+t^3}\right)^2 = \frac{9a^2t^2+9a^2t^4}{(1+t^3)^2} = \frac{9a^2t^2(1+t^2)}{(1+t^3)^2}$$

Further, θ is defined as $\tan^{-1}\left(\frac{y}{x}\right)$. That is,

$$\theta = \tan^{-1}\left(\frac{\frac{3at^2}{1+t^3}}{\frac{3at}{1+t^3}}\right) = \tan^{-1}(t) \Rightarrow d\theta = \frac{dt}{1+t^2}$$

Hence, we have as the area of the loop of the Folium of Descartes

$$A = \frac{1}{2}\int_0^{\infty} \frac{9a^2t^2(1+t^2)}{(1+t^3)^2} \cdot \frac{dt}{1+t^2} = \frac{9a^2}{2}\int_0^{\infty} \frac{t^2dt}{(1+t^3)^2}$$

Does this final integral look familiar? It should because this integral is exactly I (with the exception of the $9a^2/2$ that pre-multiplies the integral). Since we have already determined that this integral has a value of 1/3, we can conclude that the area of the loop portion of the Folium of Descartes is $A = 3a^2/2$. Wait-a-minute, you might ask, where the hell did the $(0, \infty)$ interval of integration come from? Re-examine the graph of the curve and note the following:

The portion of the curve in the 4th quadrant is when $x > 0$ and $y < 0 \Rightarrow -\infty < t < -1$.
The portion of the curve in the 2nd quadrant is when $x < 0$ and $y > 0 \Rightarrow -1 < t < 0$.
The portion of the curve in the 1st quadrant (i.e., the loop) is when $x > 0$ and $y > 0 \Rightarrow 0 < t < \infty$.

Having computed the area of the loop of the Folium of Descartes allows one to immediately conclude that

$$\int_0^{\pi/2} \frac{\sin^2(\theta)\cos^2(\theta)}{[\sin^3(\theta)+\cos^3(\theta)]^2} d\theta = \frac{1}{3}$$

Where in the world does this come from? Transform the Cartesian equation of the folium to polar coordinates by substituting $x = r\cos(\theta)$ and $y = r\sin(\theta)$, solving for r. Doing so, one obtains

$$r = \frac{3a\sin(\theta)\cos(\theta)}{\sin^3(\theta)+\cos^3(\theta)}$$

Squaring that result and plugging it into the plane area formula and integrating over the 1st quadrant, we obtain

$$A = \frac{9a^2}{2}\int_0^{\pi/2} \frac{\sin^2(\theta)\cos^2(\theta)}{[\sin^3(\theta)+\cos^3(\theta)]^2} d\theta$$

However, we already know that the area of the loop is $3a^2/2$. Therefore, the integral is 1/3.

Alright, that's pretty interesting math and unconventional integration which, if you remember, is the subject of this book; however it's going to get even more interesting math-wise. But you will have to wait for that, because I feel that a slight digression is necessary here to talk about the Descartes/Fermat challenge.

The 17^{th} Century was a century rich in mathematical discoveries, but it was also rich in mathematical discussion and controversies. Today, mathematical discussion and controversies take the form of mathematical seminars and publications, but back in the 17^{th} century, mathematical controversies took the form of challenges. That is, one mathematician in possession of a problem and a solution to the problem, challenges a colleague or even the whole scientific community to solve the problem (the challenger may even pretend to not have the solution). One of the more famous confrontations to take place was between Descartes and Fermat over the problem of finding and/or constructing tangents to a curve. Remember, this is years before Newton and Leibniz solved this very problem analytically by inventing Differential Calculus. There evidently was a correspondence between Descartes and Fermat and in those letters, Fermat claimed to have a method that always worked. Descartes did not fully understand Fermat's method and thought his own method to be superior. So, following the protocol of the times, Descartes challenged Fermat to find tangents to an especially complicated curve that he (Descartes) had invented. Today, we know that curve as The Folium of Descartes. (Folium, of course, means leaf and that comes from the 1^{st} quadrant loop that is part of the curve. By-the-way, the rest of the curve is known as the wings, the left-upper wing in the 2^{nd} quadrant and the lower right wing in the 4^{th} quadrant.) Descartes was very proud of his own method of tangent construction and evidently felt confident that Fermat could not possibly answer the challenge correctly. After all, Fermat was not even a professional mathematician; he was an amateur whose avocation was mathematics; he was a judge by vocation. Well, Fermat accepted the challenge and when Fermat subsequently provided the required tangents not only at the folium's vertex, but at any other point on the curve, Descartes was obliged to acknowledge the superiority of Fermat's method. Descartes' method worked on the folium but only at the vertex. Well, as they say, "You win some and you lose some." Descartes was, of course very embarrassed, but his reputation was already so great that it certainly didn't damage him in any respect.

Now, let's get back to this fascinating mathematics. It's really good stuff! We know the area of the Folium's loop; remember, it is $3a^2/2$. In the Folium's original incarnation by Descartes, the loop is symmetric about the line $y = x$; this line bisects the 1^{st} quadrant with a slope of 45º. If we were to rotate (within the x-y plane) the Folium by 45º clockwise, the loop would then be bisected by the positive x-axis, however, its area would still be the same. In-other-words, the curve's orientation would change, as well as its equations, but not its shape or size; the loop area would remain as is—invariant with respect to the rotation. If we can figure out what the new equations will be, we may be able to set up an area integral for the loop based on this new equation for which we already know the value, namely $3a^2/2$. If that's not unconventional integration, I don't know what you call it.

If the coordinates of the rotated curve are denoted by x' and y', then rotation of the Folium of Descartes by 45º is equivalent to $x = \frac{x'+y'}{\sqrt{2}}$ and $y = \frac{x'-y'}{\sqrt{2}}$ where $\sin(45^0) = \cos(45^0) = \frac{1}{\sqrt{2}}$. So, substituting these values of x and y into the equation for the Folium, we obtain, after simplification

$$y' = \pm x' \sqrt{\frac{3a\sqrt{2}-2x'}{6x'+3a\sqrt{2}}}$$

as the equation of the rotated Folium (see figure 8-10 where we have done away with the prime notation, i.e., x', y', and use just x, y).

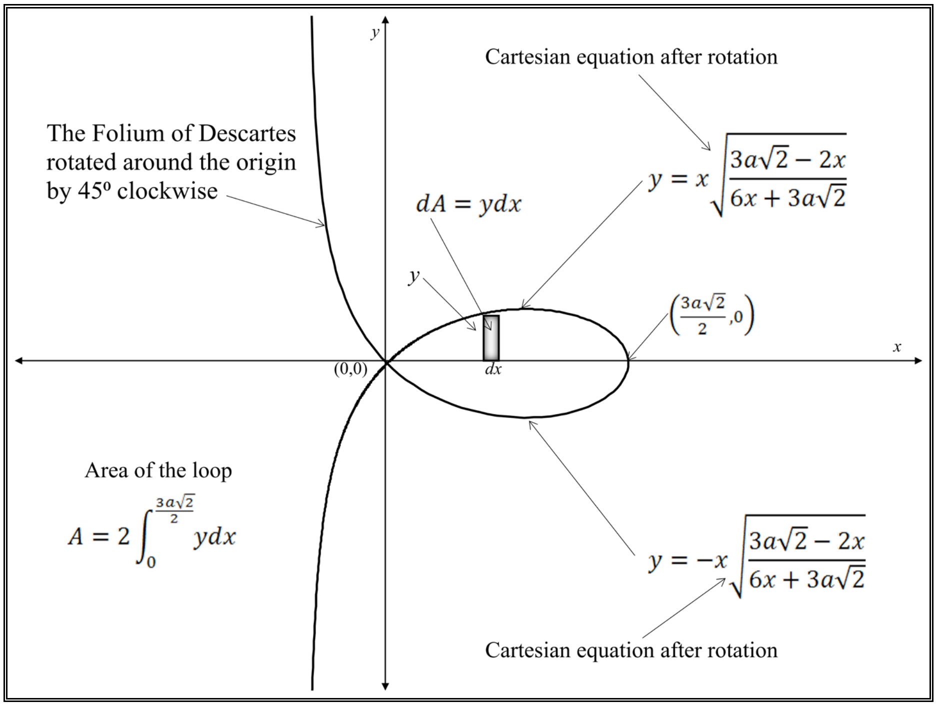


Figure 8-10. The Folium of Descartes rotated 45º Clockwise

If we think of the portion of the loop above the *x*-axis as composed of a multitude of very thin rectangles of height *y*, width *dx* and therefore of area *dA* = *ydx*, the integral we are trying to obtain is then

$$\int_0^{\frac{3a\sqrt{2}}{2}} x\sqrt{\frac{3a\sqrt{2}-2x}{6x+3a\sqrt{2}}}\,dx.$$

However, this is only the area of the loop above the *x*-axis; the total area is simply twice the integral due to symmetry. As a result, we have this very exotic integral for which we know the value, namely:

$$\boldsymbol{I = \int_0^{\frac{3a\sqrt{2}}{2}} x\sqrt{\frac{3a\sqrt{2}-2x}{6x+3a\sqrt{2}}}\,dx = \frac{3a^2}{4}} \quad \textbf{Q.E.D.}$$

It is interesting to attempt to derive the value of this exotic integral above by direct means; and it can be done but requires a great deal of perseverance. Let's do it! Dividing both numerator and denominator by 2, we obtain

$$I = \int_0^{\frac{3a\sqrt{2}}{2}} x\sqrt{\frac{\frac{3a\sqrt{2}}{2}-x}{3x+\frac{3a\sqrt{2}}{2}}}\,dx.$$

Now, let *b* temporarily denote the constant $3a\sqrt{2}/2$ which will save us a lot of writing as we proceed. When we are finished, we will replace *b* by its value. So, we have

$$I = \int_0^b x\sqrt{\frac{b-x}{3x+b}}\,dx$$

Making the radical portion of this integrand disappear would seem to be a prudent step and that can be done with a change of variable, i.e., let $u = \sqrt{\frac{b-x}{3x+b}}$. Thus, $x = \frac{b(1-u^2)}{1+3u^2}$ and after a long and tedious differentiation, $dx = \frac{-8bu}{(1+3u^2)^2}du$ and don't forget, $(0,\ b) \to (1,\ 0)$. Our integral I then becomes

$$I = 2^3b^2\int_0^1 \frac{u^2(1-u^2)}{(1+3u^2)^3}du$$

Breaking this integrand up into partial fractions results in the following

$$I = \frac{2^3b^2}{3^2}\int_0^1\left[\frac{5}{(1+3u^2)^2} - \frac{1}{1+3u^2} - \frac{4}{(1+3u^2)^3}\right]du = \frac{2^3{\cdot}5b^2}{3^2}\int_0^1\frac{du}{(1+3u^2)^2} - \frac{2^3b^2}{3^2}\int_0^1\frac{du}{1+3u^2} - \frac{2^5b^2}{3^2}\int_0^1\frac{du}{(1+3u^2)^3}$$

As we have learned in a previous chapter, denominators of this form generally call for a change of variable involving the tangent of the new variable, that is, in this case, $\sqrt{3}u = \tan(\theta)$. Under that CV, $1 + 3u^2 = 1 + \tan^2(\theta) = \sec^2(\theta)$, while $du = \frac{1}{\sqrt{3}}\sec^2(\theta)d\theta$ and $(0,\ 1) \to (0,\ \pi/3)$. We thus obtain,

$$I = \frac{2^3{\cdot}5b^2}{3^2\sqrt{3}}\int_0^{\pi/3}\cos^2(\theta)d\theta - \frac{2^3b^2}{3^2\sqrt{3}}\int_0^{\pi/3}d\theta - \frac{2^5b^2}{3^2\sqrt{3}}\int_0^{\pi/3}\cos^4(\theta)d\theta$$

I am going to call these last three integrals I_1, I_2, and I_3 respectively. We will evaluate each separately and then put the whole thing back together when done. Thus

$$I_1 = \int_0^{\pi/3}\cos^2(\theta)\,d\theta = \int_0^{\pi/3}\left[\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right]d\theta = \frac{\pi}{2{\cdot}3} + \frac{\sqrt{3}}{2^3} \quad \text{and} \quad I_2 = \int_0^{\pi/3}d\theta = \frac{\pi}{3} \quad \text{while}$$

$$I_3 = \int_0^{\pi/3}\cos^4(\theta)d\theta = \int_0^{\pi/3}\left[\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right]^2 d\theta = \int_0^{\pi/3}\left[\frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{4}\cos^2(2\theta)\right]d\theta$$

If you've been paying attention, you can see where I_3 is going; the first two terms above can be integrated by elementary means and the third term can be integrated the same way we did I_1 above. Having done that, one obtains

$$\frac{\pi}{2^3} + \frac{7\sqrt{3}}{2^6}$$

Now, of course, we must multiply each of these results by the leading coefficients of I_1, I_2, and I_3, respectively. As a result, we have

$$I = \frac{2^3{\cdot}5b^2}{3^2\sqrt{3}}\left(\frac{\pi}{2{\cdot}3} + \frac{\sqrt{3}}{2^3}\right) - \frac{2^3b^2}{3^2\sqrt{3}}\left(\frac{\pi}{3}\right) - \frac{2^5b^2}{3^2\sqrt{3}}\left(\frac{\pi}{2^3} + \frac{7\sqrt{3}}{2^6}\right)$$

The best way to proceed at this point is to factor as much as possible out of the three terms above, that is,

$$I = \frac{2^3b^2}{3^2\sqrt{3}}\left[5\left(\frac{\pi}{2{\cdot}3} + \frac{\sqrt{3}}{2^3}\right) - \frac{\pi}{3} - 2^2\left(\frac{\pi}{2^3} + \frac{7\sqrt{3}}{2^6}\right)\right] = \frac{2^3b^2}{3^2\sqrt{3}}\left(\frac{5\sqrt{3}}{2^3} - \frac{7\sqrt{3}}{2^4}\right) = \frac{2^3b^2}{3^2\sqrt{3}}\left(\frac{3\sqrt{3}}{2^4}\right) = \frac{b^2}{2{\cdot}3}$$

But remember, we temporarily set $b = 3a\sqrt{2}/2$. Thus, we get the final value and it agrees with what we know it should be:

$$\boxed{\boldsymbol{I = \frac{3a^2}{4}} \quad \textbf{Q.E.D.}}$$

Finally, before leaving the subject of this amazing curve known as the Folium of Descartes, there is at least one more calculation that I would like to address. It also deals with the loop of the Folium. It occurs to me, that since we have managed to rotate the folium so that its loop lies bisected by the *x*-axis and further, derived equations that represent the rotated curve, there is no reason why an integral can't be formulated that represents the volume of the SOR formed when that loop is revolved about the *x*-axis and, of course, if that integral can be evaluated, we will then have the actual volume of the solid when rotated about the line $y = x$, since just as the rotated loop's area is the same as the non-rotated loop, the same can be said for the volume—invariant with respect to the 45º clockwise rotation of the Folium of Descartes. In all the literature sources that I have perused in my studies of the folium, I have never come across mention of the volume of this solid. This may be new territory! See figure 8-11 for the set-up for the volume of this solid of revolution.

From figure 8-11, we see that the volume integral is

$$V = \pi \int_0^{\frac{3a\sqrt{2}}{2}} x^2 \left(\frac{3a\sqrt{2}-2x}{6x+3a\sqrt{2}}\right) dx$$

It turns out, this integral can indeed be evaluated thereby yielding up the volume, however, it is a rather messy calculation. Let's do it anyway. We can make our job a bit easier if we first divide both the numerator and denominator of the fraction in the integrand by 2, which gives

$$V = \pi \int_0^{\frac{3a\sqrt{2}}{2}} x^2 \left(\frac{\frac{3a\sqrt{2}}{2}-x}{3x+\frac{3a\sqrt{2}}{2}}\right) dx$$

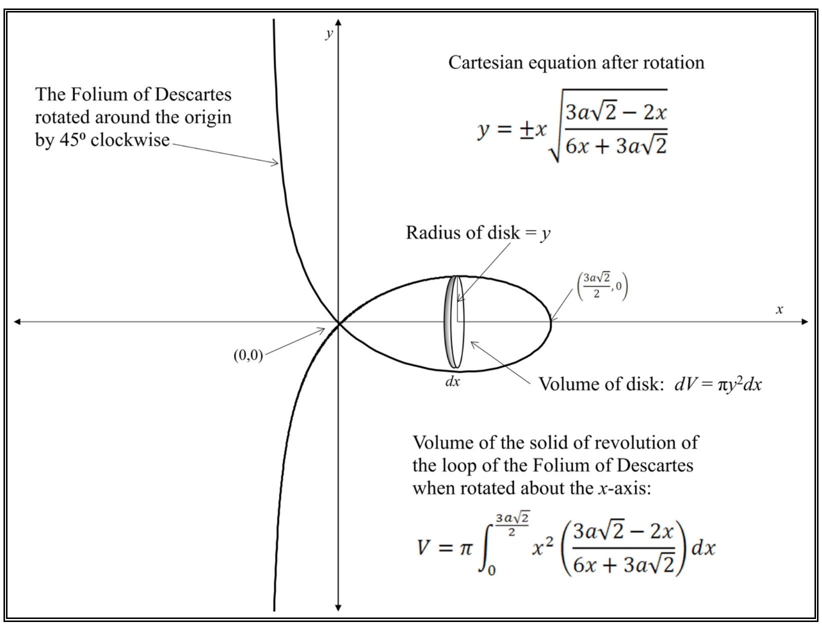


Figure 8-11. The Solid of Revolution from the loop of the Folium of Descartes

If we now let the constant $\frac{3a\sqrt{2}}{2}=n$, it will save a lot of unnecessary calculating and writing. When we are done, we will substitute back what we have set *n* equal to. So our volume integral is now

$$V = \pi \int_0^n x^2 \left(\frac{n-x}{3x+n}\right) dx = n\pi \int_0^n \frac{x^2 dx}{3x+n} - \pi \int_0^n \frac{x^3 dx}{3x+n} = I_1 - I_2$$

We are simply calling these last two integrals I_1 and I_2 and we will evaluate each separately. For I_1, make the change of variable $u = 3x + n$ so that $du = 3dx$ and $(0, n) \to (n, 4n)$. Further, $x = \frac{1}{3}(u - n)$. As a result, we have

$$I_1 = \frac{n\pi}{3^2}\int_n^{4n} \frac{(u-n)^2}{u}\frac{du}{3} = \frac{n\pi}{3^3}\int_n^{4n} \frac{u^2-2nu+n^2}{u}\,du = \frac{n\pi}{3^3}\int_n^{4n}\left(u - 2n + \frac{n^2}{u}\right)du$$

The last term can be broken into three integrals each of which can easily be evaluated. We get

$$I_1 = \frac{n\pi}{3^3}\left[\frac{1}{2}u^2\right]_n^{4n} - \left[\frac{2n^2\pi u}{3^3}\right]_n^{4n} + \left[\frac{n^3\pi\log(u)}{3^3}\right]_n^{4n}$$

Doing all the arithmetic and simplifying, we obtain the following expression for I_1

$$I_1 = \frac{n^3\pi}{2\cdot 3^3}(3 + 4\log 2).$$

Before going any further, let's now turn our attention to I_2.

$$I_2 = \pi\int_0^n \frac{x^3dx}{3x+n}$$

The same change of variable as used on I_1 will suffice for I_2, giving us

$$I_2 = \frac{\pi}{3^4}\int_n^{4n}\frac{u^3-3nu^2+3n^2u-n^3}{u}\,du = \frac{\pi}{3^4}\int_n^{4n}\left(u^2 - 3nu + 3n^2 - \frac{n^3}{u}\right)du$$

And, just like I_1, each term of I_2's integrand can be written as separate integrals that are easily evaluated, giving us

$$I_2 = \left[\frac{\pi}{3^5}u^3\right]_n^{4n} - \left[\frac{n\pi}{2\cdot 3^3}u^2\right]_n^{4n} + \left[\frac{n^2\pi}{3^3}u\right]_n^{4n} - \left[\frac{n^3\pi}{3^4}\log(u)\right]_n^{4n} = \frac{n^3\pi}{2\cdot 3^4}(15 - 4\log 2)$$

Now,

$$V = I_1 - I_2 = \frac{n^3\pi}{3^4}(8\log 2 - 3)$$

We are not done. Don't forget, we have to substitute back for n. We initially set $n = \frac{3a\sqrt{2}}{2}$. Therefore,

$$n^3 = \frac{3^3a^3\cdot 2\sqrt{2}}{2^3} = \frac{3^3\sqrt{2}a^3}{2^2}$$

Finally, the volume of the solid formed by the loop of the Folium of Descartes when rotated about its line of symmetry is

$$\boldsymbol{V = \pi\int_0^{\frac{3a\sqrt{2}}{2}} x^2\left(\frac{3a\sqrt{2} - 2x}{6x + 3a\sqrt{2}}\right)dx = \frac{a^3\pi\sqrt{2}}{2^2\cdot 3}[8\log(2) - 3]} \quad \textbf{Q.E.D.}$$

Appendix A

This appendix contains material related to the solution of I_4 of Chapter 6—the chapter where the solution to the integral on the title page resides. This integral was first solved by Laplace however I wrote about Gauss instead of Laplace when introducing the solution to this integral. Let's look at what Laplace did to evaluate I_4 of Chapter 6. He was absolutely brilliant and here is how he solved this integral. I will re-label this integral as I_1 in keeping with the book's format.

Example A-1. $I_1 = \int_0^\infty e^{-x^2} dx$ (the Laplace solution)

Laplace solved I_1 (above) in the following manner. He started, just as we did in Chapter 6 with

$$(I_1)^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$$

Now, however, Laplace deviates from what our thinking was in Chapter 6—Laplace thought change of variable of the outer integral (i.e., the integral with respect to the variable *y*). Let $y = xu$, so that $dy = xdu$. WHAT? Why isn't it $dy = xdu + udx$ after all, *x* is a variable? Yes, *x* is a variable, but for the integration with respect to *y*, *x* is kept constant. You can liken it to taking a partial derivative, but in reverse! (This has to be Laplace's aha moment.) Hence,

$$(I_1)^2 = \int_0^\infty \left(\int_0^\infty x e^{-x^2(1+u^2)} dx \right) du$$

Now the inner integral is tractable and we get

$$(I_1)^2 = \int_0^\infty \left[\frac{e^{-x^2(1+u^2)}}{-2(1+u^2)} \right]_0^\infty du = \frac{1}{2} \int_0^\infty \frac{du}{1+u^2}$$

Aha, this last integral is, of course, the recognizable form for the inverse tangent function, so we ultimately have,

$$(I_1)^2 = \left[\frac{\tan^{-1}(u)}{2} \right]_0^\infty = \frac{\pi}{4} \text{ and therefore, } I_1 = \frac{\sqrt{\pi}}{2} \text{ as expected.}$$

Not very complex, but it took a mind like Laplace to show us the way. By-the-way, note that the function e^{-x^2} is symmetric about the *y*-axis and, as a result, $\int_{-\infty}^\infty e^{-x^2} dx = 2I_1 = \sqrt{\pi}$

Let's address a bit about Laplace; a very interesting and brilliant scientist.

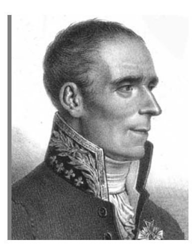


Figure A-1. French mathematician Pierre-Simon, marquis de Laplace (1749-1827)

Laplace was one of the greatest scientists of all time, sometimes referred to as the *French Newton* or *Newton of France*; he possessed a phenomenal natural mathematical faculty superior to that of any of his contemporaries. His work was important to the development of mathematics, statistics, physics, and astronomy. He summarized and extended the work of his predecessors in his five-volume *Mécanique Céleste* (Celestial Mechanics). This work translated the geometric study of classical mechanics to one based on calculus, opening up a broader range of problems. In statistics, the Bayesian interpretation of probability was developed mainly by Laplace. Laplace formulated Laplace's Equation, and pioneered what is today termed the Laplace transform. If he had done nothing else but this Laplace transform he would be remembered and famous. The Laplace transform is, among many other uses, a method of solving differential equations by transforming them to algebraic equations which are then easier to solve and then performing the inverse transform on the algebraic solution to arrive at the solution to the original differential equation. An absolutely amazing, magical idea! Oh, and by-the-way, the Laplace transform takes the form of a properly improper integral.

Example A-2. $\boldsymbol{I_2 = \int_0^1 \frac{dx}{\sqrt{-\log(x)}}}$

I know, I know, in the denominator of the integrand it looks like we are taking the square root of a negative number—look again. The interval of integration is (0, 1) and the natural logarithm of any number in that range is itself negative and since we are negating a negative number, all is copasetic.

Sometimes, one can also derive some spectacular looking results by simply making a substitution of the integration variable on an already known result. Take I_1 above and make the substitution $u = e^{-x^2}$. Under that substitution, $\log(u) = -x^2 \Rightarrow x = (-\log u)^{1/2}$ and

$dx = -\frac{du}{2u\sqrt{-\log(u)}}$. Also, (0, ∞) → (1, 0). Putting this all back under the integral sign, we have the stunning result

$$\boxed{\boldsymbol{I_2 = \int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}}}$$

Example A-3. $\boldsymbol{I_3 = \int_{-\infty}^{\infty} e^{-a(x+b)^2} dx \quad a, b \in \mathbb{R}}$

This integral looks a bit more complex than I_1; however, it literally falls apart under a CV of $\sqrt{a}(x+b) = u \Rightarrow u = \sqrt{a} \cdot x + \sqrt{a} \cdot b \Rightarrow du = \sqrt{a} \cdot dx$. Further, $u^2 = a(x+b)^2$ and the integration interval remains unchanged, i.e., (–∞, +∞) → (–∞, +∞). Therefore, we have a very interesting result. The value of I_3 is completely independent of the value of the parameter b.

$$I_3 = \int_{-\infty}^{\infty} \frac{e^{-u^2}}{\sqrt{a}} du = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{2}{\sqrt{a}} I_1 = \sqrt{\frac{\pi}{a}}$$

Hence, we know that

$$\boxed{I_3 = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \quad \text{Q.E.D.}}$$

Now, differentiate both sides of the result immediately above with respect to the parameter a and we have

$$\frac{d(I_3)}{da} = \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{2a^{3/2}}$$

Differentiate again, and

$$\frac{d^2(I_3)}{da^2} = \int_{-\infty}^{\infty} x^4 e^{-ax^2} dx = \frac{3\sqrt{\pi}}{2^2 a^{5/2}}$$

And again,

$$\frac{d^3(I_3)}{da^3} = \int_{-\infty}^{\infty} x^6 e^{-ax^2} dx = \frac{3 \cdot 5\sqrt{\pi}}{2^3 a^{7/2}}$$

Of course, we can keep doing this ad infinitum, but, at this point, I think we can write down a general result. That is

$$\frac{d^n(I_3)}{da^n} = \int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{(2n-1)!!\,\sqrt{\pi}}{2^n a^{(2n+1)/2}} \quad \text{where } n \in \mathbb{N}^+$$

What is the point of all of this? Well, take a good look at this last integral; it's quite complex and all as a result of having initially solved I_1, the integral responsible for this book. That's quite remarkable. The lesson to be learned is that one simple result can lead to many extraordinary findings. Let's continue with this delectable material.

Example A-4. $I_4 = \int_0^{\infty} e^{-ax^2 - b/x^2}\, dx$

Consider I_4 in the form $I_4 = \int_0^{\infty} e^{-(ax^2 + b/x^2)} dx$. Note that when $a = 1$ and $b = 0$, I_4 reduces to I_1. To start with, make a CV of $x = u/a^{1/2}$ so that $dx = du/a^{1/2}$ and $(0, \infty) \to (0, \infty)$. Thus, under this CV, we have

$$I_4 = \frac{1}{\sqrt{a}} \int_0^{\infty} e^{-(u^2 + ab/u^2)} du \ \text{ or } \ \sqrt{a} I_4 = \int_0^{\infty} e^{-(u^2 + ab/u^2)} du$$

Now we are going to make another CV, and you may ask, why not combine it with the previous CV and do it all in one-fell-swoop? The reason is that we want to preserve the last incarnation of $\sqrt{a}I_4$ immediately above so that we can add it to the result of this upcoming CV. Bear with me, and you will soon see how this develops (it's damn clever). Let $u = \frac{\sqrt{ab}}{y}$ so that $du = -\frac{\sqrt{ab}}{y^2} dy$ and $u^2 = \frac{ab}{y^2}$. Further, $(0, \infty) \to (\infty, 0)$. This gives us

$$\sqrt{a} I_4 = \int_{\infty}^{0} e^{-(ab/y^2 + y^2)} \left(-\frac{\sqrt{ab}}{y^2} dy\right) = \sqrt{ab} \int_0^{\infty} \frac{e^{-(y^2 + ab/y^2)}}{y^2} dy$$

Now comes the aforementioned addition

$$2\sqrt{a}I_4 = \int_0^\infty e^{-(u^2+ab/u^2)}du + \sqrt{ab}\int_0^\infty \frac{e^{-(y^2+ab/y^2)}}{y^2}dy$$

Now do you see what's happening? Look at the exponent in each integrand. Except for the fact that the first is in terms of the variable named *u* and the second is in terms of the variable named *y*, they are both the same. Can you now guess what's coming next? It's dummy variable time once again. Let's call the variable in the 2nd integral above *u* instead of *y*, and when we do we have,

$$2\sqrt{a}I_4 = \int_0^\infty e^{-(u^2+ab/u^2)}du + \sqrt{ab}\int_0^\infty \frac{e^{-(u^2+ab/u^2)}}{u^2}du = \int_0^\infty e^{-(u^2+ab/u^2)}\left(1+\frac{\sqrt{ab}}{u^2}\right)du$$

Aha, one last CV and we're home. Let $x = u - \frac{\sqrt{ab}}{u}$ so that $dx = \left(1+\frac{\sqrt{ab}}{u^2}\right)du$. And further, $(0, \infty) \to (-\infty, \infty)$. Note also that

$$x^2 = \left(u - \frac{\sqrt{ab}}{u}\right)^2 = u^2 - 2\sqrt{ab} + \frac{ab}{u^2} \Rightarrow u^2 + \frac{ab}{u^2} = x^2 + 2\sqrt{ab}$$

Which gives us the following

$$2\sqrt{a}I_4 = \int_{-\infty}^\infty e^{-(x^2+2\sqrt{ab})}dx = e^{-2\sqrt{ab}}\int_{-\infty}^\infty e^{-x^2}dx$$

The last integral, as we know, is simply $\sqrt{\pi}$. Lo and behold, as if by magic, we have our final value, namely

$$\boxed{\boldsymbol{I_4 = \int_0^\infty e^{-ax^2-b/x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}} \quad \text{Q.E.D.}}}$$

Afterword

If you have gone on the journey of using this book from front cover to I_4 above, I hope you enjoyed it as much as I did in creating it! I'm sure some might say that the development of integral calculators, many of which can be found on the Web, make this kind of book/knowledge completely useless. Well, I guess that's progress and I have no problem with that, however, it would be such a shame if the ability to do and enjoy the necessary calculations is eventually lost.

Don Cole