The Mystery of the SOR

by Donald A. Cole

Falling In Love with Non-Conventional Integration

As a college freshman Calculus student (long ago), I was dumbfounded one day when I saw in a table of definite integrals the Gaussian integral over the interval $(0, \infty)$ with a value of $\sqrt{\pi}/2$. I was knowledgeable enough at that time to know that the function in the integrand of that definite integral did not have an anti-derivative. How then, could that integral be evaluated? When I finally stumbled upon an answer, I was hooked: I've been a fan ever since of definite integrals that do not integrate in the conventional sense.

I later learned that I'm evidently in good company. G.H. Hardy (1877-1947), the greatest English mathematician of the first half of the 20th century was quoted as saying, "I could never resist an integral," and his reputation for doing non-conventional integration was reputed to be phenomenal. In fact, Hardy brought Srinivasa Ramanujan (the genius and self-taught Indian mathematician) all the way to Cambridge from India based on a letter that Ramanujan sent to Hardy in January of 1913. Attached to that letter were about 120 theorems, many of which involved the solution of definite integrals that completely astounded Hardy. Until that time, Ramanujan was unknown to the mathematical community.

Over the many years of my professional career as a cryptographer and mathematician, I have never lost my love for the methods of non-conventional integration. After my retirement, I decided to write a book on the subject (still in progress), and that has led me on a delightful and mysterious side-trip that is the basis for this story.

The Story

During the enjoyable process of writing my manuscript, I began adding historical vignettes or interesting facts about the men and women of science and mathematics who were responsible for creative solutions to the integrals I chose to include or whose methodology had been responsible for their solution. I've always contended that mathematics would be a much more popular subject to a greater number of people if the history of math were taught along with the math itself, because mathematics through the centuries has been densely populated with crazy stories, zany geniuses, and clever anecdotes.

For example, in relation to one of the integrals I intended to include in the book I wrote a short biography of Maria Gaetana Agnesi (1718-1799). Appointed to the University of Bologna by Pope Benedict XIV at the age of 32, Maria Agnesi became the first female professor of mathematics on a faculty anywhere in the world. Her connection to the integral was the following: If you took the famous classic curve of mathematics bearing her name ("The Witch of Agnesi" [$y = a^3/(x^2 + a^2),\ a \in \mathbb{R}^+$]) and revolved that curve about the x-axis, you would get a solid of revolution (SOR), and the integral I was describing could be interpreted as the volume of that solid.

That started me thinking about other classic curves that I was familiar with, one of which is my favorite: “The Folium of Descartes” [$x^3 + y^3 = 3axy, a \epsilon \mathbb{R}^+$]. This classic curve has a fascinating history and, to me, it is where the world of art, beauty, and balance intersects with the world of computation (see Figure 1).

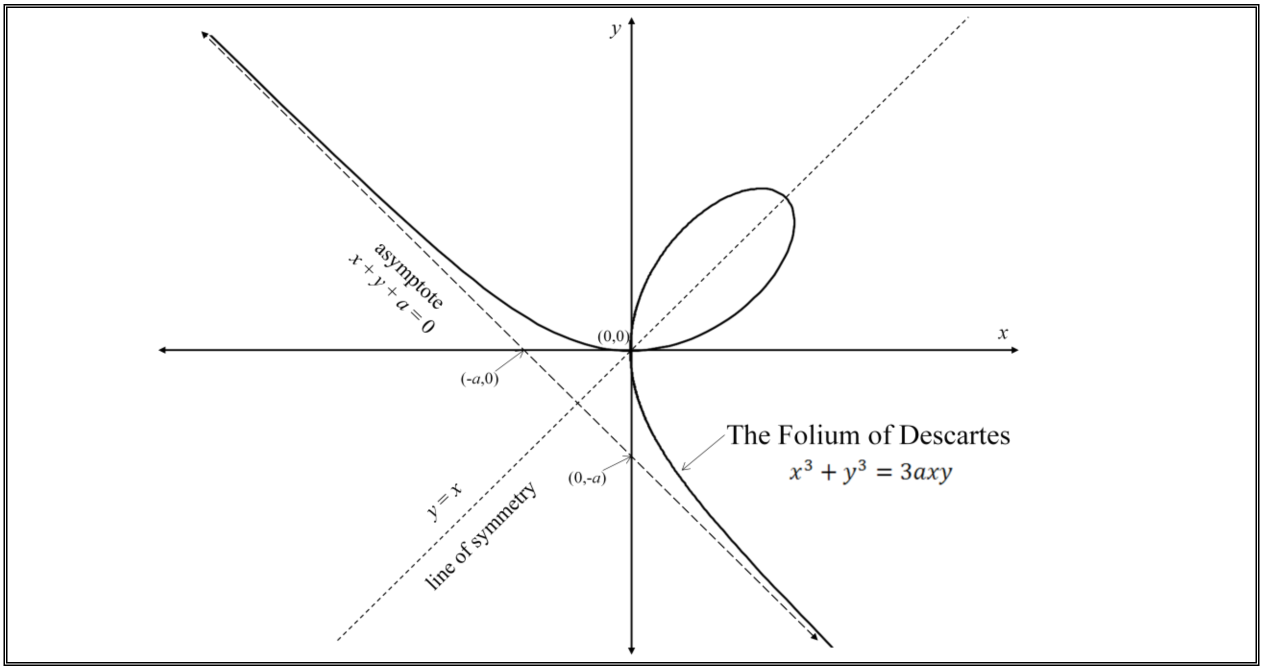

Figure 1. The Folium of Descartes

As a result, I decided to include in my book some kind of integral that was associated with the Folium of Descartes so that I could write about its intriguing history and also about Descartes himself—one of the giants of the mathematical world (I’ll refer to it simply as the Folium from here on). That endeavor subsequently led me to two ideas for computation, one for my book and another that possibly has not been done before, which is the mysterious side-trip that I alluded to earlier.

<u>The Ideas</u>

The Folium, when graphed as in Figure 1, displays a loop that extends from the origin into the first quadrant and then back to the origin and is bisected by a line with a slope of 45º. It is well known that the area of the loop is $3a^2/2$. If I were to rotate (within the *x*-*y* plane) the Folium about the origin by 45º clockwise, the loop would then be bisected by the positive *x*-axis; however, its area would still be the same. In other words, the curve’s orientation would change, as well as its equation, but not the shape or size of the loop; the loop area would remain as is—invariant under the rotation. If I could figure out what the new equation would be for the rotated curve, I might be able to set up an area integral for the loop based on this new equation; and I already know the area’s value, namely $3a^2/2$. Now if that’s not non-conventional integration, I don’t know what is! It would also be a perfect example for my book.

I was able to do that area computation, and that area integral and its derivation are shown in a following section. Now for the second idea that has produced a mystery.

When I originally finished deriving this area integral, I had another thought and it was directly related to the invariance of the loop's area. The loop's area is not the only invariant. Revolve that loop about its axis of symmetry, i.e., the line $y = x$, and I get an SOR, and the volume of that SOR is also invariant under the rotation mentioned in the previous paragraph. It should be easy to set up a volume integral for that SOR based on the rotated curve and, if that volume integral is tractable, I will have calculated the volume of the SOR formed when the loop of the Folium of Descartes is revolved about its axis of symmetry. Well, not only is the volume integral tractable, it is a relatively simple integral to evaluate—a first year calculus student could do it (see the final section where this SOR volume calculation is derived).

The Mystery

I have read many papers and visited many web sites that deal with the Folium, but in all my research into the Folium I have never come across even a mention—let alone a computation—of this SOR volume. Of course, I realize that this is certainly not cutting-edge mathematics or perhaps very important knowledge even if new, but I also believe that anything that adds to the knowledge base of this famous curve is worth considering.

Is this SOR volume computation new knowledge? The Folium is a classic, 400-year-old curve, which seems a good argument against that. Hasn't everything about it already been discovered? Maybe not.

Computation of the Area Integral

If we let the coordinates of the rotated Folium be denoted by x' and y', then rotation of the Folium of Descartes by 45° clockwise is equivalent to $x = \frac{x'+y'}{\sqrt{2}}$ and $y = \frac{x'-y'}{\sqrt{2}}$. So, substituting these values of x and y into the equation for the Folium, one obtains, after simplification

$$y' = \pm x' \sqrt{\frac{3a\sqrt{2}-2x'}{6x'+3a\sqrt{2}}}$$

as the equation of the rotated Folium (see Figure 2 where the prime notation has been discarded, i.e., x', y', and just x, y is used).

If we think of the portion of the loop above the x-axis as composed of a multitude of very thin rectangles of height y, width dx and therefore of area $dA = ydx$, the integral we are trying to obtain is then

$$\int_0^{\frac{3a\sqrt{2}}{2}} x\sqrt{\frac{3a\sqrt{2}-2x}{6x+3a\sqrt{2}}}\,dx.$$

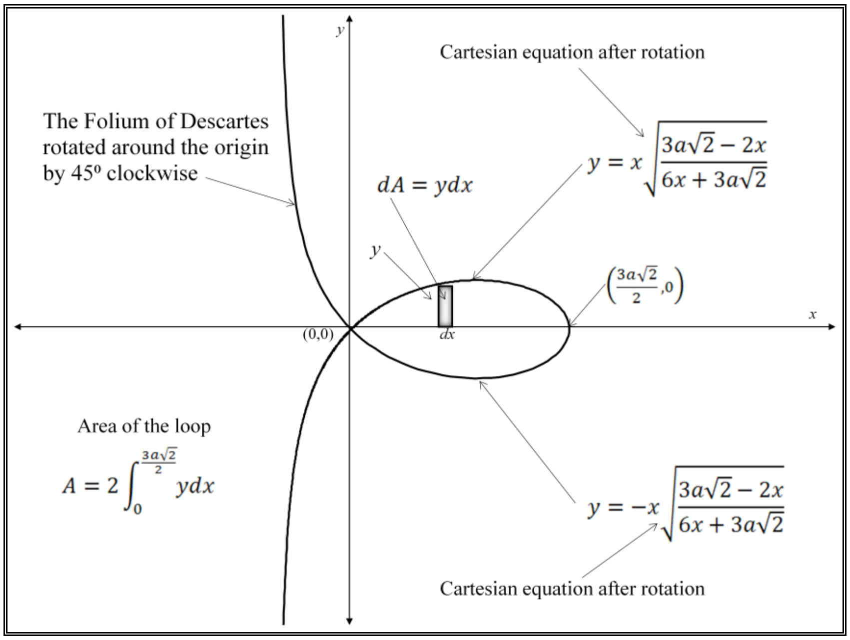

Figure 2. The Folium of Descartes Rotated by 45º Clockwise

However, this is only the area of the loop above the *x*-axis; the total area is simply twice the integral due to symmetry. As a result, we have this very exotic integral for which we know the value, namely:

$$\boxed{\int_0^{\frac{3a\sqrt{2}}{2}} x\sqrt{\frac{3a\sqrt{2}-2x}{6x+3a\sqrt{2}}}\,dx = \frac{3a^2}{4} \quad \text{Q.E.D.}}$$

Interestingly enough, this area integral can be integrated directly but algebraically it is rather a tedious calculation.

<u>Computation of the Volume Integral</u>

See Figure 3 for the set-up of the volume of this solid of revolution.

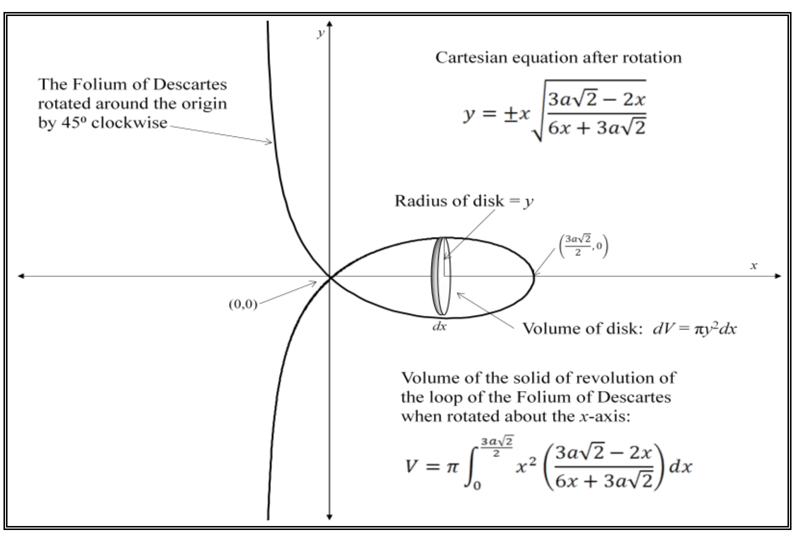

Figure 3. Volume Integral

$$V = \pi \int_0^{\frac{3a\sqrt{2}}{2}} x^2 \left(\frac{3a\sqrt{2}-2x}{6x+3a\sqrt{2}}\right) dx$$

Evaluation of this integral is a bit easier if we temporarily let the constant $3a\sqrt{2}/2 = b$ thereby giving the following

$$V = \pi \int_0^b x^2 \left(\frac{b-x}{3x+b}\right) dx = \pi b \int_0^b \frac{x^2}{3x+b} dx - \pi \int_0^b \frac{x^3}{3x+b} dx\,.$$

The change of variable of $u = 3x + b$ in both of these last two integrals makes them become quite tractable, i.e.,

$$V = \frac{\pi b}{3^3} \int_b^{4b} \frac{(u-b)^2}{u} du - \frac{\pi}{3^4} \int_b^{4b} \frac{(u-b)^3}{u} du = \frac{b^3 \pi}{2 \cdot 3^3} [3 + 4 \log_e(2)] - \frac{b^3 \pi}{2 \cdot 3^4} [15 - 4 \log_e(2)]$$

These last two terms can be combined to give

$$V = \frac{b^3 \pi}{3^4} [8 \log_e(2) - 3].$$

But, $b = \frac{3a\sqrt{2}}{2}$ and therefore $b^3 = \frac{3^3\sqrt{2}a^3}{2^2}$ and the final result is

$$\boxed{V = \frac{a^3 \pi \sqrt{2}}{2^2 \cdot 3} [8 \log_e(2) - 3]. \quad \text{Q.E.D.}}$$