Playing With Dynamic Geometry

by Donald A. Cole

Cover Design: A three-dimensional image of the curve known as the Lemniscate of Bernoulli and its graph (see Chapter 15).

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Preface

There are some areas of mathematics that require us to cast aside practicality and allow ourselves the luxury of enjoying the pure aesthetic beauty without any need for further justification. Indeed, there must be an avenue where the free flowing lines of the art world find a crossroads with the analytical worlds of equation and computation. At that crossroads, we are transported to a world that is at once purely logical in function yet purely beautiful in form. Such a crossroads in geometry is the multitude of forms and shapes found in the plane algebraic and transcendental curves. With the advent of dynamic geometry software, such as Geometer's Sketchpad (GSP)*, this beauty can be visualized much more readily and in a variety of interesting ways.

This book, *Playing With Dynamic Geometry*, is essentially a summary of the classic plane curves of historical mathematics and, in this regard, it is no different from many other texts that have been produced in the past. What makes *Playing With Dynamic Geometry* unique is that for each curve treated in the text, many different GSP constructions are also given that will allow the reader to reconstruct the curve and watch it being drawn on the computer screen in all of its breathtaking wonder.

The book assumes that the reader has mastered analytic geometry as well as the differential and integral calculus. Further, it is also assumed that the reader is or will become familiar with the operation of Geometer's Sketchpad and can therefore reproduce each construction as it is encountered in the text. Such ability contributes significantly to one's understanding of the curve and, at the same time, is a great deal of fun.

With the exception of the first chapter, which contains review material, each chapter is devoted to one of the classic curves. The order of the chapters has been dictated by the author's interest in the curve and is therefore purely arbitrary. The dynamic geometry constructions are given in tabular form as a series of steps to perform. To my knowledge, no previous survey of the classic curves has ever included geometric constructions.

Also included in each chapter are:

- The equations of the curve and formulas for the curve's derivatives
- The equation of the curve's tangent
- Metric considerations such as the radial distance to the curve and the distance from the origin to the curve's tangent, associated areas, arc lengths, and/or surface area and volume of associated solids of revolution
- Formulas for the tangential-radial angle, the radial angle, the radius of curvature, and the coordinates for the curve's center of curvature.

All of the formulas presented, with the exception of areas, arc lengths, and volumes, have been verified by a method that is explained in the Appendix. This verification process (also unique to *Playing With Dynamic Geometry*) does not necessarily prove that the formula is correct, but the nature of the process is such that it gives a very high degree of confidence that the stated formula is correct. Interestingly enough, the verification process also makes use of GSP to validate the formula under consideration. Readers who are interested should refer to the Appendix for an explanation of this verification process and an example of the way it works.

Each chapter (again, with the exception of the first chapter) begins and ends with a colorful image, usually depicting a three-dimensional version of the curve that is the subject of that chapter. The caption on these figures explains what the image represents.

A brief word about how these three-dimensional images were produced: They were created with a wonderful piece of free software called Persistence of Vision Raytracer (POV-Ray)†, which is a ray tracing application that, in the hands of an expert, can produce magnificent computer art. This author's capability with POV-Ray is very rudimentary but—with the aid of many of the background scenes that are supplied as part of the freeware—I have been able to pull off the desired effect (or at least come close to it). That effect is a surrealistic view of the curve, such as floating over an infinite checkered plane or suspended in a cloud flecked sky. In my opinion, such surrealism adds to the mystery and majesty of the curve.

I hope that *Playing With Dynamic Geometry* helps many readers appreciate and enjoy the aesthetic, visual aspect of these famous curves—a blend of science, art, beauty, and balance!

Finally, I dedicate this book to Jo, my friend, my companion, my lover, and my wife. It would not have seen the light of day without her invaluable assistance.

Don Cole
October 3, 2010

*Geometer's Sketchpad™ is a product of Key Curriculum Press. A free trial version can be downloaded from their website at www.keypress.com.
† Persistence of Vision Raytracer (POV-Ray) can be downloaded free at the website www.povray.org.

Chapter 1 – Background

This book is essentially a reference book for the classic plane curves of mathematics. With the exception of this chapter, each ensuing chapter is devoted to one of the classic curves. This chapter, however, reviews the methodologies used to develop the material contained in subsequent chapters, while the format of this chapter sets the pattern for the format of all the subsequent chapters.

1.1 Introduction

The first section of each chapter will contain an introduction to the specific curve under consideration. This introduction may contain any interesting history of the curve, mathematicians and/or men of science who first worked with the curve, and any other tidbits of information about the curve that your author thinks worthy of note.

1.2 Equations and Graph

The second section of each chapter will contain the equations and graph of the curve under consideration. In order to derive or delineate equations, coordinate systems for representing plane curves need to be discussed. Of the various coordinate systems for representing plane curves, six are addressed here; however, any representation of a given curve in a subsequent chapter will not necessarily be given in all of these coordinate systems. The six are: Parametric, Cartesian, Polar, Pedal, Bipolar, and Intrinsic.

The first three (i.e., Parametric, Cartesian, and Polar) are very well known and therefore require very little discussion. Coordinate axes for the Cartesian and Parametric systems are a pair of mutually perpendicular lines usually drawn horizontally and vertically, where the horizontal line is called the x-axis and the vertical line is called the y-axis and the point where these two axes intersect is known as the *origin*. Further, the distance along the x-axis to the point under consideration is referred to as the *abscissa* and the distance along the y-axis to the point under consideration is referred to as the *ordinate*. Thus, a point of the curve is determined by a measure of the abscissa and ordinate of that point, usually notated by (x, y). The Polar coordinate system consists of a point (the *pole*) and a ray from this point (the *axis*) to the curve and here, a point of the curve is determined by the angle between the horizontal and the axis and the distance along the axis from the pole to the point of the curve, usually notated by (r, θ). The Cartesian and Polar coordinate systems are basically point concepts; given any point, P, of the curve, there is one and only one set of coordinates (x, y) or (r, θ) for P. In the Parametric system, coordinates of a curve are expressed independently as a function of a single variable, say t, such as $x = f(t)$ and $y = g(t)$. There may be (and usually are) many different and useful parametric representations.

In the Pedal coordinate system, coordinates are basically dependent on the curve, and a point of the curve, P, may have many different Pedal coordinates, usually denoted by (r, p), depending on the specific curve. Refer to Figure 1-1. Let O be a fixed point (the *pedal point*, or *pole*) lying at the origin, and let C be a differentiable curve (i.e., its tangent exists). At a point P (the point of the curve whose pedal coordinates are desired),

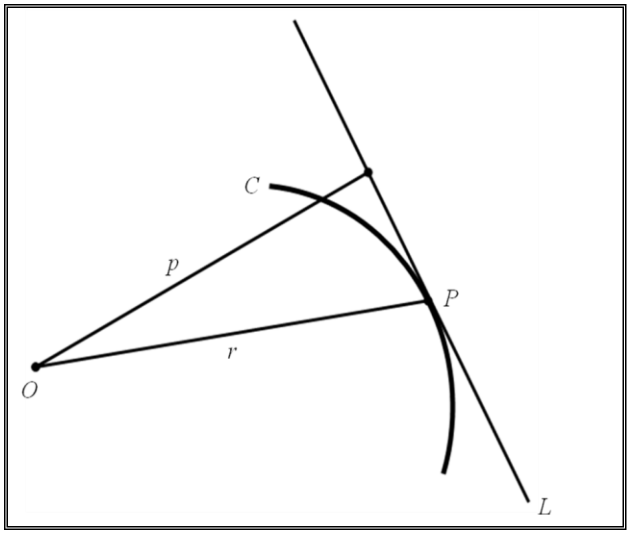

Figure 1-1: The Pedal Coordinate System

construct the tangent line L to C. Then, the Pedal coordinates of P with respect to C and O are the radial distance r from O to P and the perpendicular distance p from O to L. Note that for a different curve, say C_1, through point P, r is, of course, the same, but p may very well be different. Also note that if C does not have a tangent at point P, that is, if P is an isolated point or cusp, then the Pedal coordinates of P do not exist.

Now consider the bipolar coordinate system by referring to Figure 1-2. Let O_1 and O_2 be two fixed points (the *poles*) that are a distance $2c$ apart. The line segment $L = O_1O_2$ is termed the base line, and the bisector of L is known as the center. The

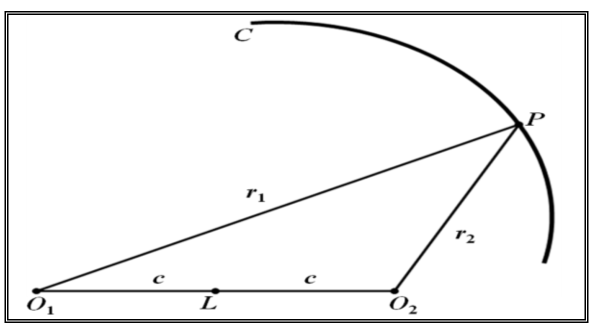

Figure 1-2: The Bipolar Coordinate System

Bipolar coordinates of a point P on a curve C are the distances r_1 and r_2 from O_1 and O_2, respectively, to point P. Note that points O_1, O_2, and P form a triangle, therefore r_1, r_2, and c must satisfy the inequalities $r_1 + r_2 \geq c$ and $|r_1 - r_2| \leq c$. Further, since r_1, r_2, and c are all assumed to be positive, any equation in Bipolar coordinates describes a locus that is symmetric about line L; conversely, a locus that is not symmetric about some line cannot have a bipolar equation.

The choice of which coordinate system to use to represent a specific curve is usually dictated by the curve's physical characteristics or by the particular information desired from the curve's properties. Thus, a system of rectangular coordinates (i.e., Cartesian) will be selected for curves in which slope is of primary importance. Curves which exhibit a central property (physical or geometrical) with respect to a point will be expressed in a polar system with the central point as the pole. (This is well illustrated in situations involving action under a central force; the path of the earth about the sun, for example.) Again, if an outstanding feature is the distance from a fixed point to the curve's tangent (as in the general problem of Caustics), a system of Pedal coordinates is most convenient.

The equations of curves in each of these systems, however, are for the most part "local" in character and are altered by certain transformations. Let a transformation (within a particular system or from system to system) be such that the measures of length and angle are preserved. Then area, arc length, curvature, number of singular points, etc., will be invariant. If a curve can be properly defined in terms of these invariants, its equation would be intrinsic in character and would express qualities of the curve which would not change from one system to another. In other words, an intrinsic property is one that depends only on the figure in question, and not its relation to a coordinate system or other external frame of reference. (For example, the fact that a rectangle has four equal angles is intrinsic to the rectangle, but the fact that a particular rectangle has two vertical sides is not, because an external frame of reference is required to determine which direction is vertical.) William Whewell[1] introduced a system involving arc length s and tangential angle ϕ, while Ernesto Cesáro[2] gave a system involving arc length s and radius of curvature ρ. Since $ds = \rho \cdot d\phi$, by definition, it is evident that these two systems are related. They are known as intrinsic coordinate systems.

Not only may equations of the curve for the various coordinate systems discussed above be delineated in this section, but the equation of the tangent line will also be included. It will always be derived using the following methodology, assuming a parametric representation for the curve, that is, $x = f(t)$ and $y = g(t)$. The reader may recall the point-intercept equation for a straight line, namely, $y = mx + b$, where m is the slope of the line and b is the point where the line intercepts the y-axis. That is, if we use the "dot" notation to stand for derivatives with respect to the parameter t, we have

$$g(q) = \frac{\dot{g}(q)}{\dot{f}(q)} f(q) + b .$$

Hence, solving for b, we have

$$b = g(q) - \frac{\dot{g}(q)}{\dot{f}(q)} f(q) .$$

[1] William Whewell (1794-1866) was one of the most important philosophers in nineteenth-century Britain. Whewell is most known today for his massive works on the history and philosophy of science.
[2] Italian mathematician (1859-1906) who made important contributions to Intrinsic Geometry.

Therefore, the Cartesian equation of the tangent line to the curve under consideration at the point $t = q$ will always be

$$\dot{f}(q)\cdot y = \dot{g}(q)\cdot x + \dot{f}(q)\cdot g(q) - f(q)\cdot \dot{g}(q) \quad \textbf{Equation 1-1}$$

1.3 Analytical and Physical Properties

The third section of each chapter will contain subsections that deal with the analytical and physical properties of the curve under consideration.

1.3.1 Derivatives of the Curve

Given that the curve has a parametric representation, the first and second derivatives of the curve will be delineated using this parametric form. That is, if the parametric representation is $x = f(t)$ and $y = g(t)$, then the derivatives will include

- $\dfrac{dx}{dt} = \dot{x} = \dfrac{df(t)}{dt} = \dot{f}$
- $\dfrac{dy}{dt} = \dot{y} = \dfrac{dg(t)}{dt} = \dot{g}$
- $\dfrac{dy}{dx} = y' = \dfrac{\dot{g}}{\dot{f}}$
- $\dfrac{d^2x}{dt^2} = \ddot{x} = \dfrac{d^2f(t)}{dt^2} = \ddot{f}$
- $\dfrac{d^2y}{dt^2} = \ddot{y} = \dfrac{d^2g(t)}{dt^2} = \ddot{g}$
- $\dfrac{d^2y}{dt^2} = y'' = \dfrac{\dot{y}'}{\dot{f}}$

1.3.2 Metric Properties of the Curve

For some of the curves, the following quantities may be calculated: arc length, plane area, volume of the solid of revolution of the curve, surface area of the solid of revolution, distance from the origin to the curve, and distance from the origin to the curve's tangent.

Three different expressions for arc length may be used in the calculation of the length of a given curve. The three expressions are, respectively, for Cartesian, Parametric, and Polar representations of the curve. They are:

$$s=\begin{cases}\int\sqrt{1+y'^2}\,dx\\ \int\sqrt{\dot{x}^2+\dot{y}^2}\,dt\\ \int\left[\left(\dfrac{dr}{d\theta}\right)^2+r^2\right]^{1/2}d\theta\end{cases}\quad \textbf{Equation 1-2}$$

Three different expressions for the calculation of the plane area bounded by a given curve will be used and they each correspond to the Cartesian, Parametric, and Polar representations. They are:

$$\begin{aligned}A&=\int_a^b y\cdot dx\\ A&=\int_{t_1}^{t_2} g(t)\dot{f}(t)dt \quad \textbf{Equation 1-3}\\ A&=\tfrac{1}{2}\int_\alpha^\beta r^2 d\theta\end{aligned}$$

When a plane curve is revolved about one of the coordinate axes, the resulting figure is called a solid of revolution. For some curves, the volume and/or the surface area of this solid of revolution can be calculated. When this is possible in subsequent chapters, the following formulas will be used for the volume calculation depending upon whether the curve is represented by the Cartesian or Parametric systems, respectively. The notation, V_x, indicates the revolution takes place about the x-axis.

$$V_x=\begin{cases}\pi\int_a^b y^2dx\\ \pi\int_{t_1}^{t_2}[g(t)]^2\,\dot{f}(t)dt\end{cases}\quad \textbf{Equation 1-4}$$

Similarly, the following formulas will be used for the surface area calculation, again depending upon whether the curve is represented by the Cartesian or Parametric systems, respectively. The notation A_x not only indicates the axis about which the revolution takes place (in this case, the x-axis), but the mere presence of the subscript also distinguishes the calculation from the calculation of plane area. Note that when subsequent chapters include any of these four metrics (i.e., arc length, plane area, volume, or surface area), not only will the results of the calculation be given, but the integrals used in those calculations will be shown with some indication of how to perform the integration.

$$A_x = \begin{cases} 2\pi \int_a^b y\sqrt{1+y'^2}\,dx \\ 2\pi \int_{t_1}^{t_2} g(t)\sqrt{\dot{f}^2+\dot{g}^2}\,dt \end{cases} \quad \textbf{Equation 1-5}$$

Two distance calculations make up the final two metric properties for the given curve. They are the radial distance (i.e., the distance from the origin to the curve) and the tangential distance (i.e., the distance form the origin to the tangent). The formulas are:

$$r = \sqrt{x^2+y^2} \quad \textbf{Equation 1-6}$$

$$p = \frac{\dot{f}\cdot g - f\cdot\dot{g}}{\sqrt{\dot{f}^2+\dot{g}^2}} \quad \textbf{Equation 1-7}$$

1.3.3 Curvature

This section will contain two calculations, that is, the radius of curvature of the curve and the coordinates for the center of curvature. For the radius of curvature ρ we have

$$\rho = \frac{\left(\dot{f}^2+\dot{g}^2\right)^{3/2}}{\dot{f}\cdot\ddot{g}-\ddot{f}\cdot\dot{g}} \quad \textbf{Equation 1-8}$$

If (α,β) is the notation used to represent the coordinates of the center of curvature, then

$$\begin{aligned} \alpha &= f - \frac{\left(\dot{f}^2+\dot{g}^2\right)\dot{g}}{\dot{f}\cdot\ddot{g}-\ddot{f}\cdot\dot{g}} \\ \beta &= g + \frac{\left(\dot{f}^2+\dot{g}^2\right)\dot{f}}{\dot{f}\cdot\ddot{g}-\ddot{f}\cdot\dot{g}} \end{aligned} \quad \textbf{Equation 1-9}$$

Alternately, the coordinates of the center of curvature may also be calculated using

$$\begin{aligned} \alpha &= x - \frac{y'\left[1+(y')^2\right]}{y''} \\ \beta &= y + \frac{1+(y')^2}{y''} \end{aligned} \quad \textbf{Equation 1-10}$$

1.3.4 Angles

Three angles are of consequence: the slope angle, ϕ, the radial angle, θ, and the tangential-radial angle, ψ. The slope angle ϕ of a line is defined as the angle formed by that line and any horizontal line, such as the *x*-axis (see Figure 1-3), that intersects the given line; if the given line is parallel to the *x*-axis, then the slope angle is assumed to be zero. The radial angle θ is the angle between the radial line that connects the origin and point P, a point of the curve, and the x-axis (taken clockwise from the radial line). Finally, the tangential-radial angle ψ is defined as the angle between the radial line and the line L, again taken clockwise but this time from L. Note that all three angles are in the semi-closed interval $[0,\pi)$.

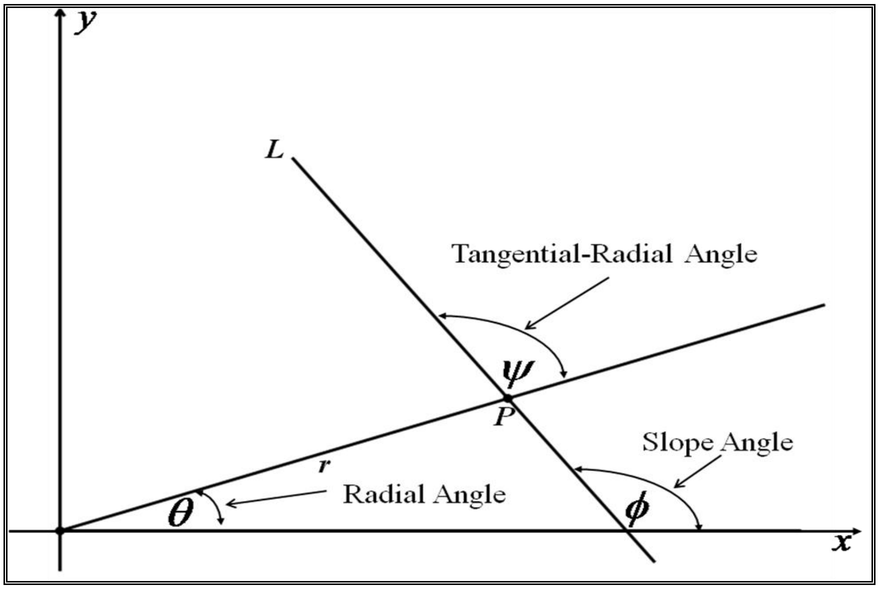

Figure 1-3: The Slope, Radial, and Tangential-Radial Angles

A formula for the tangent of the slope angle ϕ has already been given in the delineation of the derivatives in section 1.3.1, namely, y'. The radial angle θ is, of course, one of the coordinate variables in the Polar coordinate system and its tangent is given as $\tan\theta = y/x$. Finally, the tangential-radial angle ψ is related to the other two angles by $\psi = \phi - \theta$. However, another relationship exists for the tangential-radial angle that may be more useful, namely

$$\tan\psi = \frac{xy' - y}{x + yy'} \quad \textbf{Equation 1-11}$$

1.4 Geometric Properties

Ten different geometrical properties may be addressed in this fourth section. They are any asymptotes that the curve may possess, possible branches that the curve may have, critical points on the curve, if the curve includes any discontinuities, any envelopes to the curve, intercepts with the coordinate axes, the extent or range of the curve, loops that the curve may undergo, any singularities the curve may have, and finally any symmetries exhibited by the curve.

1.5 Types of Derived Curves

Once a curve has been defined, it is possible to use some of its properties together with auxiliary points, lines, and/or other curves, to obtain a new curve. Although, the subsequent chapters of this text do not make a point of addressing the derived curves for the curve covered by that specific chapter, sometimes a derived curve is mentioned or plays a role in one of the dynamic geometry constructions. For this reason, a short exposition of derived curves is included here.

1.5.1 Evolute

The idea of evolutes purportedly originated with Christian Huygens in 1673 in connection with his studies on light. However, the concept can actually be traced back to Apollonius (circa 200 BC) where it appears in the fifth book of his Conic Sections. Simply put, the evolute of a curve, C, is merely the locus of its center of curvature. As such, one can use Equation 1-9, the expressions for the coordinates of the center of curvature, for the parametric equation for the evolute. It can be shown that all tangents to an evolute are normals to the given curve. Hence, an alternate definition of the evolute is as the envelope of normals to the given curve.

1.5.2 Involute

Twenty years after Huygens addressed the evolute, he discussed and utilized the involute of a circle (circa 1693) in connection with his study of clocks without pendulums for service on ships at sea. An involute of a curve, C, is the trace of a selected point on a line that rolls as a tangent upon the given curve C. Obviously, since the selected point is an arbitrary point on the tangent line, there can be many different involutes to the given curve C; however, they are all parallel. It can be shown that if C_1 is an evolute of C_2, then C_2 is an involute of C_1. If n is the distance from the tracing point to the point of tangency, then the equations of the involute for the curve C in parametric form are given by

$$\begin{aligned} x &= f - \frac{n\dot{f}}{\sqrt{\dot{f}^2 + \dot{g}^2}} \\ y &= g - \frac{n\dot{g}}{\sqrt{\dot{f}^2 + \dot{g}^2}} \end{aligned} \qquad \textbf{Equation 1-12}$$

1.5.3 Parallel Curves

Leibnitz was the first to consider parallel curves in the years 1692 to 1694. He was prompted, no doubt, by the involutes of Huygens. Let P be a variable point on a given curve C. The locus of point Q_1 and Q_2 located $\pm n$ units distance from P as measured along the normal through P to C is defined to be the parallel curve; obviously, there are two branches. For some values on n, a parallel curve may not be unlike the given curve in appearance, but for other values of n it may be totally different. Note that since parallel curves have common normals, they have a common evolute. Equation 1-13 gives the parametric equations for the parallels in parametric form; note the similarity to the equations for the involute.

$$x = f \pm \frac{n\dot{g}}{\sqrt{\dot{f}^2 + \dot{g}^2}}$$
$$y = g \mp \frac{n\dot{f}}{\sqrt{\dot{f}^2 + \dot{g}^2}}$$
Equation 1-13

1.5.4 Inversion

Inversion in geometry is a transformation. Let *P* be a given point and let C be a circle centered at point *O* with a radius of *r*. The inverse of the point *P* with respect to C is a point *Q* on the radial line *OP* such that the distance *OP* multiplied by the distance *OQ* is equal to r^2. From this definition, two properties are readily evident. First, the point *Q* is an inverse of the point *P* if and only if *P* is an inverse of *Q*. Second, points inside the circle are mapped to the outside and vice-versa. Points on the circumference of the circle are, of course, fixed; that is, the inversion of any point on the inversion circle is mapped to itself. As *P* moves farther away from *O*, its image *Q* moves closer to *O*. From this observation, we may then define the inverse of the center of the inversion circle to be a point at infinity, and vice-versa. With such a definition, we have obtained a transformation on a plane that has a point at infinity, an important concept used in stereographic projection. It basically shows that such a plane is topologically equivalent to a sphere. Further, inversion can also be regarded as a generalization of reflection, where a normal reflection is simply an inversion where the inversion circle has an infinitely large radius. With this concept of the inversion of a point, the inversion of a curve is simply the inversion of every point on the curve and can be construed as a way to derive a new curve from the given curve. If curve A is the inverse of curve B, then curve B is the inverse of curve A with respect to the same inversion circle. The center of the inversion circle is sometimes referred to as the pole point. One property that is readily obvious from this definition of inversion is that the radius of the inversion circle affects the scale of the inverted curve, but does not affect its shape. Curves that invert into themselves are called anallagmatic curves. Circles, lines, and Cassinian ovals are all anallagmatic curves. Asymptotes to a curve C invert into a curve that is tangent to the inverse of C.

1.5.5 Pedal Curves

The idea of Pedal curves first occurred to Colin Maclaurin in 1718. Given a curve C and an arbitrary point *P* on C, construct the tangent to C through *P*. Then from a fixed point *O* (the pole point) located anywhere in the plane, drop a perpendicular to the tangent. Let the intersection of the perpendicular and the tangent be point *S*. The locus of point *S* as *P* moves along C is defined to be the first positive pedal curve of C with respect to the point *O*. If C is represented in parametric form and if (x_0, y_0) are the coordinates of the pole point, then the equations of the first positive pedal are

$$x = \frac{x_0 \dot{f}^2 + f \cdot \dot{g}^2 + (y_0 - g)\dot{f} \cdot \dot{g}}{\dot{f}^2 + \dot{g}^2}$$
$$y = \frac{y_0 \dot{g}^2 + g \cdot \dot{f}^2 + (x_0 - f)\dot{f} \cdot \dot{g}}{\dot{f}^2 + \dot{g}^2}$$
Equation 1-14

1.5.6 Conchoid

Let point *O* be fixed and let line *L* be a line through point *O* intersecting the given curve, curve C, at point *Q*. The locus of points P_1 and P_2 on *L* such that $P_1Q = QP_2 = a$,

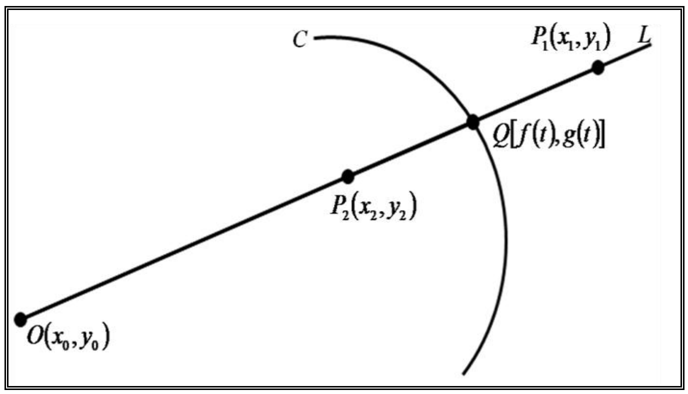

Figure 1-4: Conchoid

where *a* is a given constant is called a Conchoid of curve C with respect to point *O* (the pole point). Refer to Figure 1-4. In general, the locus of the point P_1 does not connect with the locus of P_2 and therefore the conchoid has two branches. If (x_0, y_0) are the coordinates of the pole point, then the parametric equations of the conchoid are given by

$$x = f \pm \frac{a(f - x_0)}{\sqrt{(f - x_0)^2 + (g - y_0)^2}}$$
$$y = g \pm \frac{a(g - y_0)}{\sqrt{(f - x_0)^2 + (g - y_0)^2}}$$
Equation 1-15

1.5.7 Strophoid

Given a curve C and two fixed points, *O* and *A*, as shown in Figure 1-5. Now let there be a line *L* through point *O* intersecting the curve C in point *Q*. Further, let P_1 and P_2 be two points on line *L* such that $P_1Q = QP_2 = QA$. The locus of P_1 and P_2 as *Q* varies over C is the Strophoid of C with respect to points *O* and *A*.

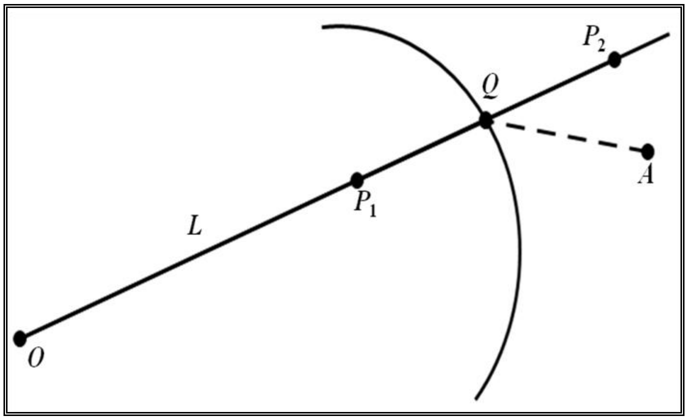

Figure 1-5: Strophoid

If the coordinates of points O and A are (x_0, y_0) and (x_1, y_1) respectively, and if C is represented in parametric form, then the parametric representation of the strophoid is given by

$$x = f \pm \sqrt{\frac{(x_1 - f)^2 + (y_1 - g)^2}{1 + k^2}}$$
$$y = g \pm k \cdot \sqrt{\frac{(x_1 - f)^2 + (y_1 - g)^2}{1 + k^2}} \qquad \textbf{Equation 1-16}$$
$$\text{where } k = \frac{g - y_0}{f - x_0}$$

1.5.8 Cissoid

Let C_1 and C_2 be two curves and let O be a fixed point. Let the line L through point O intersect the two curves C_1 and C_2 in points Q_1 and Q_2 respectively. Further, let P be a point on line L such that $OP = OQ_2 - OQ_1 = Q_2Q_1$. The locus of points P on all such lines L is called the Cissoid of C_1 and C_2 with respect to point O. See Figure 1-6.

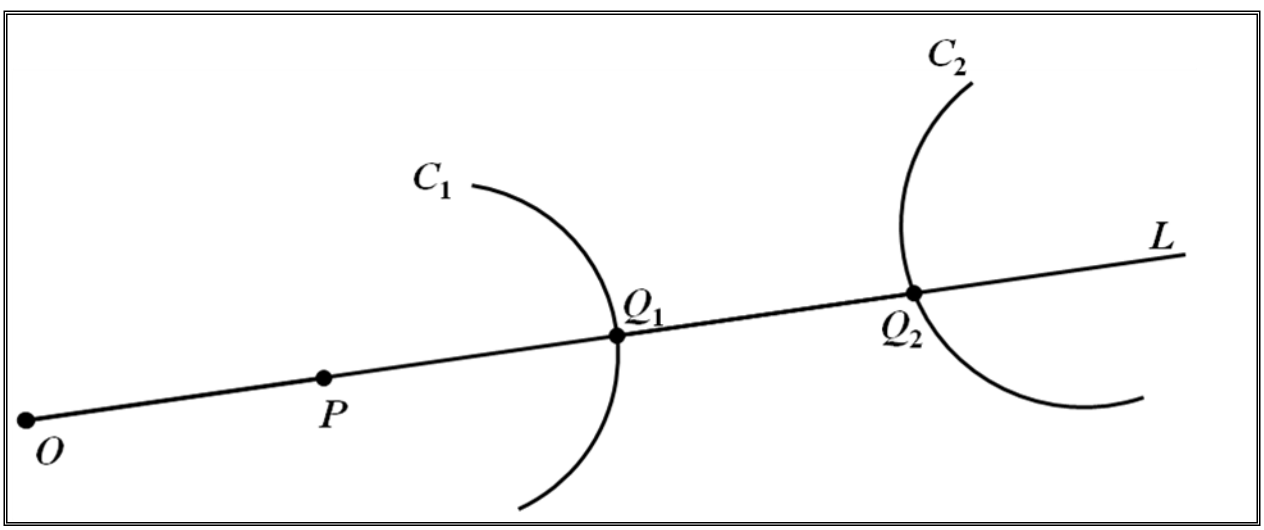

Figure 1-6: Cissoid

1.5.9 Roulette and Glissette

If a curve C_1 rolls, without slipping, along another fixed curve C_2, any fixed point *P* attached to C_1 describes a Roulette. The term is also sometimes applied to the envelope of a fixed line attached to C_1. A curve similar to the Roulette is the Glissette, which is defined to be the locus of a point carried by a curve C as it slides between two given curves C_1 and C_2, or slides tangent to a given curve C_1 at a point. It can be shown that any Glissette may also be defined as a Roulette.

1.5.10 Isoptic and Orthoptic

The locus of the intersection point of tangents to a curve C meeting at a constant angle α is an Isoptic; if $\alpha = \pi / 2$, the Isoptic is termed an Orthoptic.

1.5.11 Caustic

A Caustic of a given curve C is the envelope of light rays emitted from a point source *S* after reflection or refraction at C. If the light rays are reflected, the curve is called a Catacaustic; if the light rays are refracted, the curve is called Diacaustic.

1.6 Special Considerations

This section will contain any other interesting problems, facts, or associated discussion relating to the curve under consideration that your author deems worthy of including.

1.7 Dynamic Geometry Construction

This section will, of course, contain the main motive for this text—namely the dynamic geometry constructions. The number per curve will vary; many different constructions exist for some curves while only a few are known for other curves. Regardless of the number, they will be in a tabular form with easy-to-follow steps so that the reader can reproduce the construction, execute the animation, and see the result unfold in all of its beauty. Some of the constructions will be illustrated with a snapshot of what the final construction should look like.

Chapter 2 – The Cissoid of Diocles

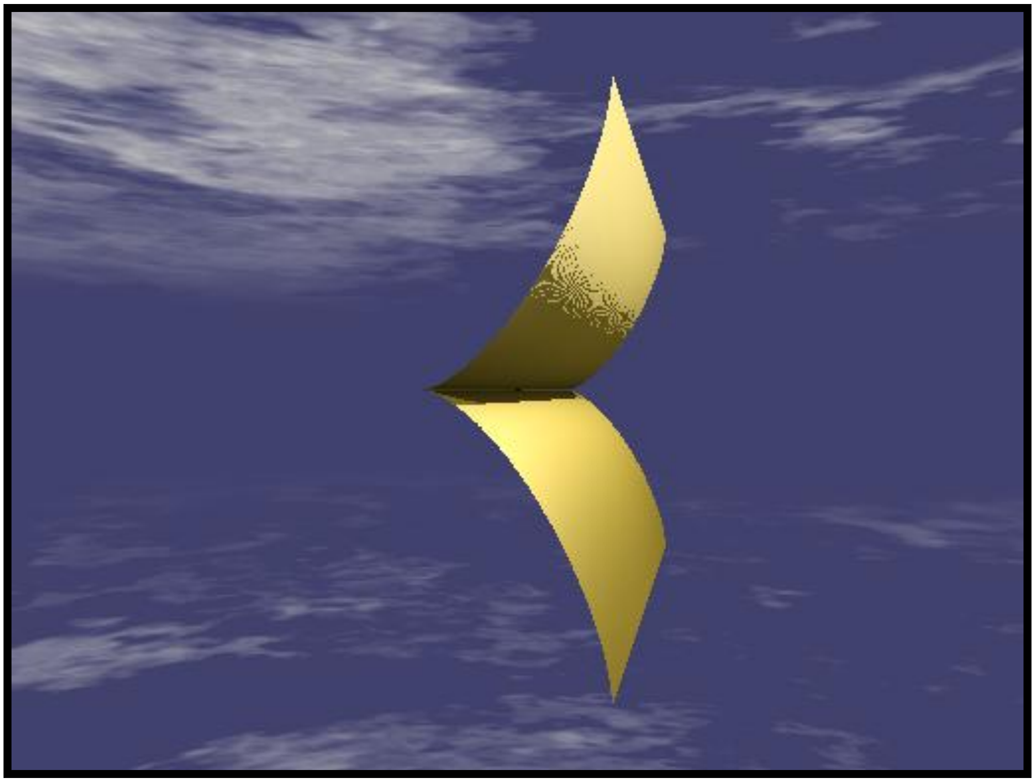

Figure 2-1: A Three-Dimensional Version of the Cissoid of Diocles

The Cissoid of Diocles is rendered as a three-dimensional object. It has been extruded into the third dimension, i.e., normal to the plane of the paper, and then truncated along its asymptote. The surface of the object has been given a golden-metallic appearance. The background has been made to appear as though the object is floating in a bluish-purple sky with white clouds randomly scattered. Lighting has been placed so as to put the cusp of the object and a portion of the upper branch in shadow.

2.1 Introduction

Supposedly, in 430 BC, the Athenians were suffering from a terrible plague that was causing much death and misery. In order to appease the gods and stop the plague, the oracle of the god Apollo at Delos was consulted. The Athenians were instructed by the oracle to double the size of their altar, which at the time, was a cube. They then proceeded to double every edge, but, of course, the ravages of the plague increased. After some thought, they came to the observation that the problem consists of constructing a length that is the cube root of 2 times the length of an edge of the original altar. As a result, the problem of duplicating the cube using only compass and straightedge became known as the Delian problem. As it turns out, one can never find a solution requiring only the straightedge and compass, as was proven some centuries later. However, the Greeks obtained many solutions using a variety of clever techniques. Special curves were invented and investigated for this purpose, which themselves could not be constructed by using only a straightedge and compass. One such curve investigated by Diocles (circa 180 BC) in connection with this Delian problem is a curve that is today called the Cissoid of Diocles; cissoid means ivy-shaped.

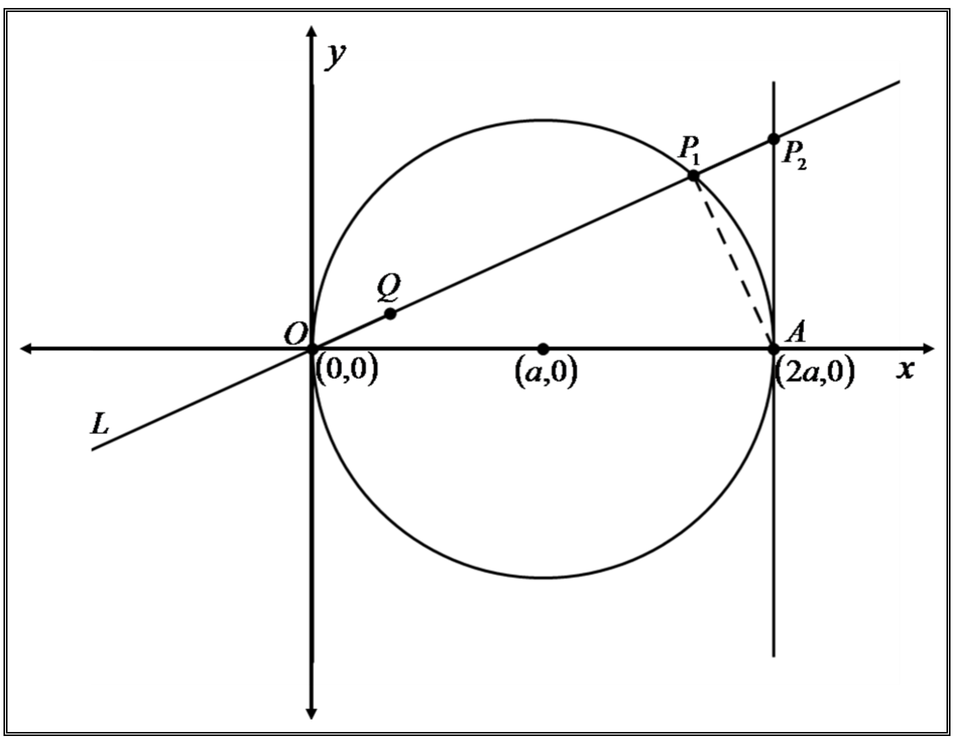

Figure 2-2: The Cissoid of Diocles as a Locus of Points

Refer to Figure 2-2, which depicts an origin *O*, a circle of radius *a* passing through the origin, and a tangent to the circle at the point (2*a*, 0). Let any line, *L*, passing through the origin intersect the circle in the point P_1 and intersect the tangent line in P_2. Let the point *Q* be a point on *L* such that the distance from the origin to *Q* equals the distance between points P_1 and P_2. The Cissoid of Diocles is then defined as the locus of point *Q* for all possible lines *L* through the origin which intersect the circle and tangent as shown. In section 2.5, it will be shown how this curve can be used to solve the Delian problem. First, however, let us derive the equations of this curve.

2.2 Equations and Graph of the Cissoid of Diocles

It is relatively straightforward to derive the polar equation for the Cissoid from the geometric relationships depicted in Figure 2-2. From elementary geometry, ΔAOP_1 is a right triangle since one side is the diameter of a circle and the opposite vertex lies on the circle's circumference. Therefore, the $\cos\theta = OP_1 / 2a$, or $OP_1 = 2a\cos\theta$. Similarly, ΔAOP_2 is also a right triangle because the line $x = 2a$ is tangent to the circle at point A. Therefore, $\cos\theta = 2a / OP_2$ or $OP_2 = 2a / \cos\theta$. Now, since $OQ = r = OP_2 - OP_1$, we have

$$r = \frac{2a}{\cos\theta} - 2a\cos\theta = \frac{2a - 2a\cos^2\theta}{\cos\theta} = \frac{2a\left(1-\cos^2\theta\right)}{\cos\theta} = \frac{2a\sin^2\theta}{\cos\theta}.$$

Hence,

$$r = 2a\sin\theta\tan\theta \quad \textbf{Equation 2-1}$$

From this polar representation, it is a simple matter to derive the Cartesian form of the Cissoid by substituting the familiar polar-to-rectangular coordinate transformations, i.e., $x = r\cos\theta$, $y = r\sin\theta$, and $r = (x^2 + y^2)^{½}$. Making that substitution in Equation 2-1, we get

$$\sqrt{x^2 + y^2} = \frac{2ay}{\sqrt{x^2 + y^2}} \cdot \frac{y}{x}.$$

Clearing the fractions and rearranging, we obtain

$$2ay^2 - xy^2 = x^3$$

or,

$$y^2 = \frac{x^3}{2a - x} \quad \textbf{Equation 2-2}$$

From the Cartesian form of the equation, we can derive a parametric form by considering the line $y = tx$. Making this substitution in Equation 2-2 and solving for x, one obtains

$$x = \frac{2at^2}{1+t^2}.$$

Similarly, eliminating x, and solving for y,

$$y = \frac{2at^3}{1+t^2}.$$

Hence, a parametric representation of the Cissoid of Diocles is

$$(x, y) = \frac{2at^2}{1+t^2}(1, t), \quad -\infty < t < +\infty \qquad \textbf{Equation 2-3}$$

An alternative and useful parametric representation of the Cissoid of Diocles can be obtained by setting tan $\theta = t$ in Equation 2-1. A little algebraic manipulation will then yield the following alternative parametric representation

$$(x, y) = 2a\sin^2 t(1, \tan t), \quad -\pi/2 < t < \pi/2 \qquad \textbf{Equation 2-4}$$

Intrinsic equations can also be derived for the Cissoid of Diocles. The Whewell and Cesáro equations are, respectively

$$s = a(\sec^3 \varphi - 1) \qquad \textbf{Equation 2-5}$$

$$729(s+a)^8 = a^2\left[9(s+a)^2 + \rho^2\right]^3 \qquad \textbf{Equation 2-6}$$

Graphing the Cissoid of Diocles, we get the curve shown in Figure 2-3. As can be seen from this plot, the vertical line $x = 2a$ is an asymptote, and the derivative at the origin, i.e., (0, 0), does not exist although the curve is continuous at that point.

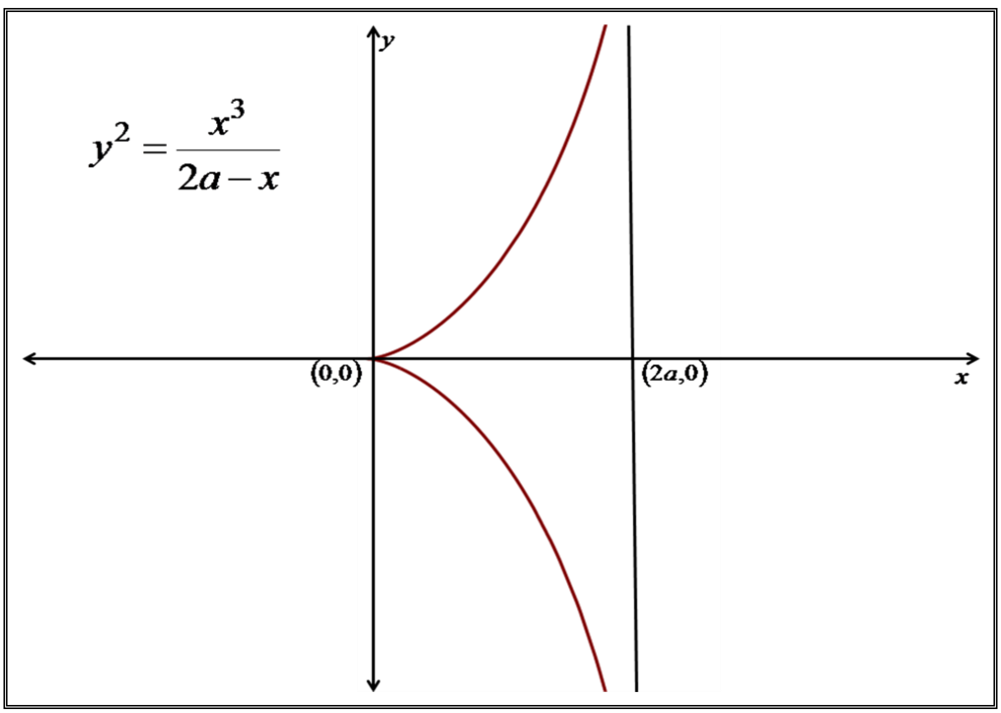

Figure 2-3: Graph of the Cissoid of Diocles

The equation of the tangent at the point $t = q$ is

$$2y = \tan q \sec^2 q(1 + 2\cos^2 q)x - 2a\tan^3 q. \qquad \textbf{Equation 2-7}$$

2.3 Analytical and Physical Properties of the Cissoid of Diocles

Using the parametric representation of the Cissoid of Diocles given in Equation 2-4, i.e., $x = 2a\sin^2 t$ and $y = 2a\tan t \cdot \sin^2 t$, the following paragraphs delineate the relevant analytical and physical properties of the Cissoid of Diocles.

2.3.1 Derivatives of the Cissoid of Diocles

- $\dot{x} = 4a\sin t \cdot \cos t.$
- $\ddot{x} = 4a\left(\cos^2 t - \sin^2 t\right).$
- $\dot{y} = 2a\tan^2 t \cdot \left(1 + 2\cos^2 t\right).$
- $\ddot{y} = 4a\tan t \cdot \sec^2 t \cdot \left(1 + 2\cos^4 t\right).$
- $y' = \dfrac{1}{2}\tan t \cdot \sec^2 t \cdot \left(1 + 2\cos^2 t\right).$
- $y'' = \dfrac{3}{8a\sin t\cos^5 t}.$

2.3.2 Metric Properties of the Cissoid of Diocles

If A is the area between the Cissoid of Diocles and its asymptote, then $A = 3\pi a^2$. This result is easily obtained using the Cartesian form of the Cissoid and the appropriate equation for plane area which yields the following integral for the area under consideration

$$A = 2\int_0^{2a} \frac{x^{3/2}\,dx}{\sqrt{2a - x}}.$$

This integral is most easily evaluated by making the substitution $x = 2a\sin^2\theta$. Under this substitution, we have

$$A = 16a^2 \int_0^{\pi/2} \sin^4\theta \cdot d\theta.$$

This resulting integral can be evaluated by elementary methods using the trigonometric identity $\sin^2\theta = ½ - ½ \cdot \cos 2\theta$, so that we finally get the result indicated above, that is, $A = 3\pi a^2$.

If V is the volume of the solid of revolution about the asymptote, then $V = 2\pi^2 a^3$. This result may be obtained by considering an incremental cylindrical shell. The volume of this incremental element is the circumference of its circular portion times its height times its thickness, i.e., $dV = 2\pi(2a - x) \cdot 2y \cdot dx$. By integrating between $x = 0$ and $x = 2a$ we can calculate the total volume, that is,

$$V = 4\pi\int_0^{2a}\frac{(2a-x)x^{3/2}dx}{\sqrt{2a-x}} = 4\pi\int_0^{2a}\sqrt{x^3}\sqrt{2a-x}dx.$$

Again, the substitution of $x = 2a\sin^2\theta$ yields the following integral which can be evaluated in a manner similar to that used in the previous area calculation.

$$V = 64\pi a^3\int_0^{\pi/2}\left(\sin^4\theta - \sin^6\theta\right)d\theta.$$

That is, writing each of the two portions of the integrand in terms of the $\cos 2\theta$, expanding the resulting binomials, evaluating the simple integrals that result, and continuing this process until all remaining integrals can be calculated, we get the result indicated above of $V = 2\pi^2 a^3$.

If r is the distance from the origin to the curve, then

$$r = 2a\sin t\tan t.$$

If p is the distance from the origin to the tangent, then

$$p = \frac{-2a\sin^3 t}{\sqrt{1+3\cos^2 t}}.$$

2.3.3 Curvature of the Cissoid of Diocles

If ρ represents the radius of curvature of the Cissoid of Diocles, then

$$\rho = \frac{a}{3}\sin t\sec^4 t\left(1+3\cos^2 t\right)^{3/2}.$$

If (α, β) are the coordinates of the center of curvature, then

$$\alpha = -\frac{a}{3}\sin^2 t\sec^4 t\left(1+5\cos^2 t\right) \quad \text{and} \quad \beta = \frac{8a}{3}\tan t.$$

2.3.4 Angles for the Cissoid of Diocles

If ϕ is the slope angle, then

$$\tan\phi = \frac{1}{2}\tan t\sec^2 t\left(1+2\cos^2 t\right).$$

If θ denotes the radial angle, then

$$\theta = t.$$

If ψ is the angle between the tangent and the radius vector at the point of tangency, then

$$\tan\psi = \frac{\sin t \cos t}{1+\cos^2 t}.$$

2.4 Geometric Properties of the Cissoid of Diocles

- Intercepts: (0, 0).
- Extrema: (0, 0), x-minimum.
- Extent: $0 \le x < 2a$; $-\infty < y < +\infty$.
- Symmetry: $y = 0$.
- Asymptote: $x = 2a$.
- Cusp: (0, 0).

2.5 Doubling the Cube

In order to double a cube of volume V whose side is s, one must be able to construct the side of a second cube with length $\sqrt[3]{2}\,s$. Given a segment s = CB, one can use the Cissoid of Diocles to construct a segment CM such that $CM^3 = 2CB^3$. It is done in the following manner (refer to Figure 2-4).

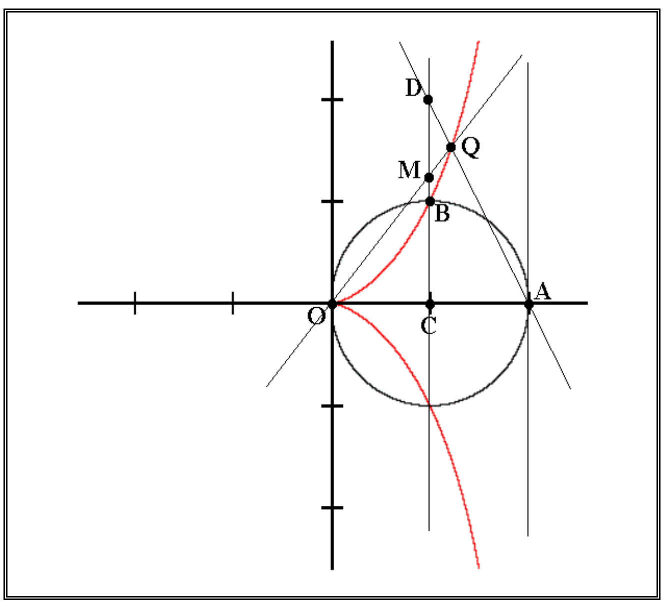

Figure 2-4: Doubling the Cube

1. Given two points C and B, construct a circle with center at point C and passing through point B.
2. Construct points O and A such that OA is perpendicular to CB.
3. Construct the tangent to the circle at point A.
4. Construct the Cissoid with origin at point O.

5. Construct point D such that point B is the midpoint of segment CD.
6. Construct the line AD.
7. Let the intersection of line AD and the Cissoid be point Q.[3]
8. Construct the line OQ.
9. Let the intersection of lines CD and OQ be the point M.

If one follows the steps outlined above, upon completion of step 9, the cube of segment CM will equal twice the cube of segment CB, i.e., $CM^3 = 2CB^3$.

2.6 Dynamic Geometry of the Cissoid of Diocles

Dynamic geometry applications, such as the Geometer's Sketchpad (GSP), can be used to generate the Cissoid of Diocles in a variety of entertaining ways, as the next eleven subsections illustrate.

2.6.1 A Construction Based on the Definition of the Cissoid of Diocles

This construction, which can be found in Table 2-1, follows directly from the definition of the Cissoid of Diocles as addressed in Section 2-1.

Table 2-1: The Cissoid of Diocles by Definition

1. Draw horizontal line AB (with point B to the right of point A)	7. Let point E be the intersection of line CD and P_1.
2. Draw circle AB with center at A and passing through B	8. Draw line segment DE
3. Let point C be on the circumference opposite point B	9. Construct a circle centered at C with radius = segment DE
4. Construct $P_1 \perp$ to line AB through point B	10. Let point F be the intersection of line CD and the 2nd circle
5. Let D be a random point on circle AB (on the circumference)	11. Trace point F and change its color
6. Draw line CD	12. Animate point D around circle AB

As point D moves around circle AB, point F will trace the Cissoid of Diocles. Note that the distances DE and FC remain equal to each other even though the values change due to the movement of point D. Also note that in step 7, where point E becomes the intersection of P_1 and line CD, if CD does not appear to intersect P_1 in your particular GSP construction, simply drag point D around circle AB until the desired intersection becomes evident. Further, note also that in step 10 of the construction, the 2nd circle and the line CD actually intersect in two points; either point will trace the Cissoid of Diocles, however, the second point (call it point G) will generate a curve that opens in the opposite direction from that of point F, although both will have cusps at point C.

2.6.2 Diocles' Method

By some modern accounts, Diocles constructed his Cissoid using a methodology similar to that delineated below in Table 2-2.

Table 2-2: Diocles' Method

1. Draw horizontal line AB	7. Let point F′ be the reflection of F across line CD
2. Draw circle AB centered at point A and through point B	8. Draw line EF′
3. Construct $P_1 \perp$ to line AB through point A	9. Construct $P_2 \perp$ to line AB through point F
4. Let points C and D be the intersections of circle AB and P_1.	10. Let point G be the intersection of line EF′ and P_2
5. Let point E be on circle AB opposite from point B.	11. Trace point G and change its color
6. Let F be a random point on circle AB (on the circumference)	12. Animate point F around circle AB

[3] This intersection point cannot be found with a straightedge and compass only.

As point F moves around circle AB in the construction above, point G will trace the Cissoid of Diocles. If one also constructs a perpendicular line to AB through point B and then lets point H be the intersection of line EF' and this new perpendicular, it can be shown that although distances F'H and EG do not remain constant as point F travels around circle AB, they do however remain equal.

2.6.3 Newton's Method

Newton also had a method of generating the Cissoid of Diocles. He used two line segments of equal length at right angles to each other. If they are moved so that one line segment always passes through a fixed point and the end of the other segment slides along a straight line, then the midpoint of the sliding segment traces out the Cissoid. This method is often referred to as the Carpenter Square (T-square) method of construction. See Figure 2-5.

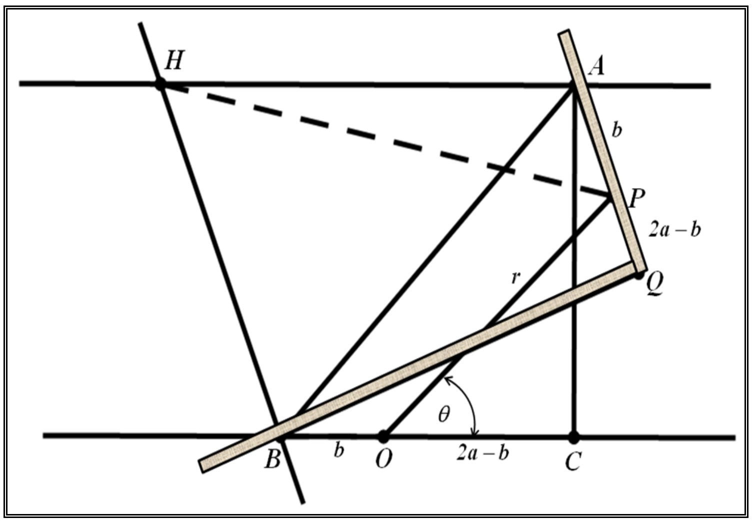

Figure 2-5: The Carpenter Square Method

With a right angle at Q, the fixed point A of the T-square moves along CA while the other edge of the T-square passes through B, a fixed point on the line BC perpendicular to AC. The path of P, a fixed point on AQ describes the curve. Two items are of interest here: (1) Let AP = OB = b, and BC = AQ = $2a$, with O the origin of coordinates. Then AB = $2a \cdot \sec\theta$ and $r = 2a \cdot \sec\theta - 2b \cdot \cos\theta$. The point Q describes a Strophoid (see Chapter 3). (2) Point A has the direction of the line CA while the point of the T-square at B moves in the direction BQ. Normals to AC and BQ at A and B respectively meet in H the center of rotation. HP is thus normal to the path of P. A perpendicular to that normal through P will thus be a tangent. Table 2-3 contains the construction steps for Newton's method.

Table 2-3: Newton's Method

1. Draw horizontal line segment AB With A to the right of B	8. Draw line segment AF
2. Construct P_1 $\perp$to line segment AB passing through point A	9. Construct circle C_2 centered at B and radius = segment AF
3. To the right of P_1, draw circle CD centered at C through D	10. Let points G and H be the intersection of circles C_1 and C_2
4. Let E be a random point on the circumference of circle CD	11. Draw line segments FG and FH
5. Draw ray CE starting at point C and passing through point E	12. Let I and J be the midpoints of segments FG and FH, resp.
6. Let point F be the intersection of ray CE and perpendicular P_1	13. Trace points I and J and change their color
7. Construct circle C_1 centered at F and radius = segment AB	14. Animate point E around circle CD

Note that this construction (if using GSP) traces not only the Cissoid, but also a straight line that intersects the Cissoid at its cusp; the two traced points share in the production of the straight line. The straight line is not part of the Cissoid. If, in step 6, ray CE does not intersect P_1, simply drag point E around circle CD until the intersection occurs.

2.6.4 A Construction Based on Dividing a Circle's Diameter

The Cissoid of Diocles can also be generated by measuring the ratio into which a point divides the diameter of a circle. This construction is delineated below in Table 2-4. In this construction, the ratio that is calculated of course changes as point F moves around circle CD. However, an examination of another ratio, namely that of segment F'G to segment FG, will always be equal to that of segment EG to segment DG.

Table 2-4: Construction by Dividing a Diameter

1. Draw horizontal line AB	10. Let point G be the intersection of P_1 and the parallel line
2. Let C be a random point <u>not</u> on line AB	11. Draw line segment EG
3. Rotate line AB around point C by 180º	12. Measure the length of line segment EG
4. Construct P_1 $\perp$ to line AB through point C	13. Draw line segment DG
5. Let point D be the intersection of P_1 and the rotated line	14. Measure the length of line segment DG
6. Let point E be the intersection of line AB and P_1	15. Calculate the ratio of segment EG to segment DG
7. Draw circle CD with center at C and passing through point D	16. Let F' be the image when F is dilated about G by the ratio
8. Let F be a random point on circle CD	17. Trace point F' and change its color
9. Construct the parallel line to line AB through point F	18. Animate point F around circle CD

2.6.5 A Construction Based on Three Lines

This construction for the Cissoid of Diocles opens up or down depending on how you draw the vertical line asked for in the first step of the construction. If you draw the vertical line with point A above point B, the curve opens down. If you draw the vertical line with point A below point B, the curve opens upward. Table 2-5 contains the steps for this construction which, if you don't count the perpendiculars, requires only three straight lines.

Table 2-5: Construction Based on Three Lines

1. Draw vertical line AB	7. Construct P_2 $\perp$ to line AC through point C
2. Draw circle AB centered at A and passing through point B	8. Draw line BD
3. Let C be a random point on circle AB (on the circumference)	9. Let point E be the intersection of line BD and P_2.
4. Construct P_1 $\perp$ to line AB passing through point B	10. Trace point E and change its color.
5. Draw line AC	11. Animate point C around circle AB.
6. Let point D be on Circle AB diametrically opposite point C	

2.6.6 The Cissoid of Diocles as the Inversion of a Parabola

In Chapter 1, we learned how inversion can be used to derive a new curve from a given curve. This concept of inversion of a curve forms the basis for the construction shown here. It turns out that if the vertex of a parabola is used as the center of inversion, then the parabola will invert into the Cissoid of Diocles. Table 2-6 contains this construction.

Table 2-6: The Cissoid of Diocles as the Inversion of a Parabola

1. Draw horizontal line AB	15. Trace point J and change its color
2. Let C be a random point on line AB	16. Let K be the midpoint of line segment CD
3. Construct $P_1 \perp$ to line AB through point C.	17. Draw line JK
4. Let D be a random point on perpendicular P_1.	18. Measure distance JK
5. Draw line segment CD and hide perpendicular P_1.	19. Draw a circle centered at point K of any radius, say KL
6. Draw circle EF anywhere below line AB (centered at E).	20. Measure distance KL
7. Let G be a random point on the circumference of circle EF	21. Calculate KL^2 / JK
8. Draw line EG	22. Mark the distance calculated in the previous step
9. Let point H be the intersection of lines AB and EG	23. Let K′ be the image as point K is translated by KL^2 / JK
10. Draw line segment DH	24. Draw circle KK′ centered at K and passing through K′
11. Let I be the midpoint of line segment DH	25. Let point M an intersection of line JK and circle KK′
12. Construct $P_2 \perp$ to line segment DH through point I.	26. Trace point M and change its color
13. Construct $P_3 \perp$ to line AB through point H.	27. Animate point G around circle EF
14. Let point J be the intersection of perpendiculars P_2 and P_3	

Note that line JK obviously intersects circle KK' in two points (i.e., step 25 instructs one to label the intersection point M). The trace of point M is that of a Cissoid of Diocles which has a cusp at point K; if the other point of intersection is chosen instead, it will also trace a Cissoid of Diocles with a cusp at point K but it will open in the opposite direction.

Obviously, the inversion process, in general, requires three steps. First, the curve which is to be inverted must be constructed—in this case, the Parabola which is done in steps 1 to 15. Second, the pole point (or inversion point) must be located—in this case, the vertex of the Parabola. This is the midpoint of line segment CD and is identified in step 16. Third and finally, the inversion process must be executed—in this case, steps 17 to 27.

2.6.7 The Cissoid of Diocles as the Pedal Curve of a Parabola

In Chapter 1 we learned that if C is a curve and O is a point (referred to as the pedal point), the locus of the foot of the perpendicular from point O to a variable tangent to C is called the pedal curve of C with respect to the pedal point O. It so happens that the pedal curve of a parabola when the pedal point is on the vertex of the parabola is a Cissoid of Diocles, as can be seen from the following construction (Table 2-7).

Table 2-7: The Cissoid of Diocles as the Pedal of a Parabola

1. Draw horizontal line AB	11. Let I be the midpoint of line segment DH
2. Let C be a random point on line AB	12. Construct $P_2 \perp$ to line segment DH through point I
3. Construct $P_1 \perp$ to line AB through point C.	13. Construct $P_3 \perp$ to line AB through point H.
4. Let D be a random point on perpendicular P_1.	14. Let point J be the intersection of perpendiculars P_2 and P_3
5. Draw line segment CD and hide perpendicular P_1.	15. Trace point J and change its color
6. Draw circle EF anywhere below line AB (centered at E).	16. Let K be the midpoint of line segment CD
7. Let G be a random point on the circumference of circle EF	17. Construct $P_4 \perp$ to P_2 through point K
8. Draw line EG	18. Let point L be the intersection of perpendiculars P_2 and P_4.
9. Let point H be the intersection of lines EG and AB	19. Trace point L and change its color
10. Draw line segment DH	20. Animate point G around circle EF

The construction used here for the Parabola is the same as that used in the previous section (i.e., section 2.6.6), that is, steps 1 to 15. Step 16 is, of course, the identification of the vertex of the Parabola and the remaining steps are the execution of the Parabola's Pedal using the vertex as the pole point. Perpendicular P_2 is the tangent to the Parabola traced by point J. P_4 is perpendicular to this tangent and passes through point K, the vertex of the Parabola. Therefore, the intersection of these two perpendiculars (point L) is, by definition, a point on the pedal curve when the vertex is used as the pedal (or pole) point. Notice how the Cissoid's cusp and the Parabola's vertex coincide. Fascinating!

2.6.8 The Tangent to the Cissoid of Diocles

One of the geometric elements that it is always desirable to construct to a curve is its tangent. Here is a "sweet" little construction for the tangent to the Cissoid of Diocles (Table 2-8).

Table 2-8: The Cissoid of Diocles and Its Tangent

1. Create *x-y* coordinate axes with origin A and unit point B	11. Construct $P_4 \perp$ to line AC through point E
2. Draw circle AB centered at A and passing through point B	12. Construct $P_5 \perp$ to P_3 through point D.
3. Let C be a random point on circle AB's circumference	13. Let point F be the intersection of perpendiculars P_4 and P_5
4. Draw line AC	14. Let point G be the intersection of perpendiculars P_2 and P_4
5. Construct $P_1 \perp$ to line AC through point A	15. Draw line segment FG
6. Construct $P_2 \perp$ to P_1 through point B	16. Let H be the midpoint of line segment FG
7. Let point D be the intersection of perpendiculars P_1 and P_2	17. Let point H' be the image of H translated by vector E → H
8. Construct $P_3 \perp$ to the *x*-axis through point D	18. Draw line AH'
9. Let point E be the intersection of line AC and P_3	19. Construct P_6 (the tangent) $\perp$ to line AH' through point E
10. Construct the locus of point E as point C traverses circle AB	20. Animate point C around circle AB

Dynamic geometry applications often support the capability of making certain geometric elements of the construction stand out by using thicker lines and different colors. GSP supports such a capability. In the construction above, if one thickens and colors (say blue) perpendicular P_6 (the tangent) and thickens and colors (say red) the locus (step 10), when the animation is run, it is easier to focus on the elements one is interested in observing.

2.6.9 The Osculating Circle of the Cissoid of Diocles

As mentioned above, a curve's tangent is desirable; so too is the curve's osculating circle. The osculating circle is simply the circle with center point coincident with the curve's center of curvature and tangent to the curve. In order to construct this circle, one must construct the curve's center of curvature (usually not an easy task). But it's like a "five-in-one" deal. Once accomplished, one has a construction that gives not only the osculating circle, the radius of curvature, and the center of curvature, but also the curve's evolute and the curve's normal. Table 2-9 gives such a construction (albeit a very complex one) for the Cissoid of Diocles.

Drawing line segment E_3E gives, of course, the radius of curvature for the Cissoid of Diocles. Drawing line E_3E gives the normal to the Cissoid of Diocles, point E_3 is the center of curvature, and finally, if one traces point E_3 and reruns the animation, point E_3 will trace the evolute to the Cissoid of Diocles.

2.6.10 The Generalized Concept of the Cissoid

In Chapter 1 we learned that a concept termed the Cissoid can be used to derive other curves, and in point-of-fact, we learned that what was required were two curves, which we called C_1 and C_2 and a fixed point O. Let the line L through point O intersect the two curves in Q_1 and Q_2 respectively. Further, let P be a point on line L, such that

Table 2-9: The Osculating Circle for the Cissoid of Diocles

1. Create *x-y* coordinate axes with origin A and unit point B	27. Draw line segment IJ
2. Draw circle AB with center at A and passing through point B	28. Let K be the midpoint of line segment IJ
3. Let C be a random point on the circumference of circle AB	29. Let K' be the image when K is translated by vector A → K
4. Draw line AC	30. Draw line segment AE
5. Construct $P_1 \perp$ to line AC through point A	31. Let L be the midpoint of line segment AE
6. Construct $P_2 \perp$ to P_1 through point B	32. Let L' be the image when L is translated by vector K' → L
7. Let point D be the intersection of perpendiculars P_1 and P_2	33. Let A' be the image when A is translated by vector L' → A
8. Construct $P_3 \perp$ to the *x*-axis through point D	34. Let point M be the intersection of the *x*-axis and P_4
9. Let point E be the intersection of P_3 and line AC	35. Construct $P_9 \perp$ to the *x*-axis through point M
10. Construct the locus of point E as point C traverses circle AB	36. Let point N be the intersection of P_9 and line AC.
11. Construct $P_4 \perp$ to line AC through point E	37. Let N' be the image when N is translated by vector A → N
12. Construct $P_5 \perp$ to P_3 through point D	38. Draw line segment A'N'
13. Let point F be the intersection of perpendiculars P_2 and P_4	39. Let O be the midpoint of line segment A'N'
14. Let point G be the intersection of perpendiculars P_4 and P_5	40. Let O' be the image when O is translated by vector A → O
15. Draw line segment FG	41. Construct $P_{10} \perp$ to line AC through point O'
16. Let H be the midpoint of line segment FG	42. Let point P be the intersection of perpendiculars P_6 and P_{10}
17. Let H_1 be the image when H is translated by vector E → H	43. Construct $P_{11} \perp$ to line segment AH_2 through point A
18. Let H_2 be the image when H_1 is rotated about point A by 90°	44. Construct $P_{12} \perp$ to P_{11} through point P
19. Draw line segment AH_2	45. Let point Q be the intersection of perpendiculars P_{11} and P_{12}
20. Let E_1 be the image when E is translated by vector H_1 → E	46. Draw line segment H_2Q
21. Let E_2 be the image when E_1 is translated by vector E → E_1	47. Construct $P_{13} \perp$ to line segment H_2Q through point H_2
22. Construct perpendicular P_6 to P_4 through point E_2	48. Let point R be the intersection of perpendiculars P_{11} and P_{13}
23. Construct $P_7 \perp$ to P_6 through point B	49. Let E_3 be the image when E is translated by vector R → A
24. Construct $P_8 \perp$ to the *x*-axis through point B	50. Draw circle E_3E with center at E_3 and passing through point E
25. Let point I be the intersection of line AC and P_7	51. Make circle E_3E thick and change its color
26. Let point J be the intersection of P_8 and line AC	52. Animate point C around circle AB

$OP = OQ_2 – OQ_1 = Q_2Q_1$. The locus of points P on all such lines L is called the Cissoid of C_1 and C_2 with respect to the point O (see Figure 1-6). We can therefore generalize the concept of the Cissoid wherein the Cissoid of Diocles becomes a specific case of the generalized Cissoid. In Figure 2-2, where we defined the Cissoid of Diocles, if we replace the circle by any curve, C_1, and replace the tangent line by any other curve, C_2, then the resulting locus of Q as P_1 moves on C_1 is called the Cissoid of C_1 and C_2 with respect to the pole, O. Note that there are two points on L such that the distances OQ and P_1P_2 are equal (the one shown and a similar point below the *x*-axis). The two points are symmetric around point O on L, so that either one can be used to generate the (same) Cissoid. As alluded to above, if C_1 is a circle and C_2 is a line tangent to C_1 at point A and point O is the point on C_1 opposite point A, then the Cissoid of C_1, C_2, and the pole O is called the Cissoid of Diocles. If O is an arbitrary point on the circle, the curve is termed an Oblique Cissoid. A GSP construction for an Oblique Cissoid is given in Table 2-10, below.

Table 2-10: The Oblique Cissoid

1. Draw vertical line AB	6. Draw line CD
2. Draw circle AB centered at A and passing through point B	7. Let point E be the intersection of line CD and P_1.
3. Construct $P_1 \perp$ to line AB through point B	8. Let C′ be the image when C is translated by vector D → E
4. Let C be a random point on the circumference of circle AB	9. Trace point C′ and change its color
5. Let D be a 2nd random point on the circumference of circle AB	10. Animate point D around circle AB

If the line passes through the center of the circle (as opposed to being tangent to the circle), and the pole is on the circle's circumference, then the resulting curve is called a Strophoid; if the pole is a point on the circumference and farthest from the line, the curve is a special case of the Strophoid, namely a Right Strophoid (see Chapter 3). The Cissoid of a line and a circle, with pole at the center of the circle, is any member of the family known as the Conchoid of Nicomedes (see Chapter 5). It can be shown that when a Cissoid is based on curves C_1, C_2, and pole point O, where C_1 and C_2 intersect at point P, then the line OP will be tangent to the Cissoid at point O. Note that if random point C in the construction shown above is moved to be diametrically opposite point B, then the Oblique Cissoid becomes the Cissoid of Diocles.

2.6.11 An Alternate Construction for the Osculating Circle

At the risk of being redundant, here is an alternate construction for the osculating circle of the Cissoid of Diocles. This shows that even the very complex constructions often have many distinct constructions. Refer to Table 2-11.

Table 2-11: An Alternate Construction of the Osculating Circle for the Cissoid of Diocles

1. Draw horizontal line AB	18. Let point J be the intersection of perpendiculars P_4 and P_6
2. Draw circle AB with center at A and passing through point B	19. Construct $P_7 \perp$ to P_5 through point A
3. Let C be a random point on the circumference of circle AB	20. Let point K be the intersection of perpendiculars P_2 and P_7
4. Let D be the point diametrically opposite point B	21. Construct the locus of point K as point C traverses circle AB
5. Draw line AC	22. Draw line segment AG
6. Construct $P_1 \perp$ to line AB through point D	23. Let L be the midpoint of line segment AG
7. Let E be the intersection of line AC and perpendicular P_1	24. Draw line KL
8. Draw line segment BE	25. Let point M be the intersection of line KL with P_4
9. Let F be the midpoint of line segment BE	26. Construct $P_8 \perp$ to line KL through point M
10. Construct $P_2 \perp$ to line segment BE through point F	27. Let point N be the intersection of perpendiculars P_7 and P_8
11. Construct $P_3 \perp$ to P_1 through point E	28. Draw line segment AJ
12. Let point G be the intersection of perpendiculars P_2 and P_3	29. Let O be the midpoint of line segment AJ
13. Construct $P_4 \perp$ to P_2 through point G	30. Draw line NO
14. Let point H be the intersection of line AB and P_4	31. Let point P be the intersection of lines KL and NO
15. Construct $P_5 \perp$ to P_4 through point H	32. Draw circle PK centered at P and passing through point K
16. Let point I be the intersection of perpendiculars P_3 and P_5	33. Make circle PK thick and change its color
17. Construct $P_6 \perp$ to P_3 through point I	34. Animate point C around circle AB

If you trace point G, you will see point G sweep out a parabola; if you draw circle JG (i.e., the circle centered at point J and passing through point G) and rerun the animation, you will find that circle JG is the osculating circle to that parabola. A very nice construction!

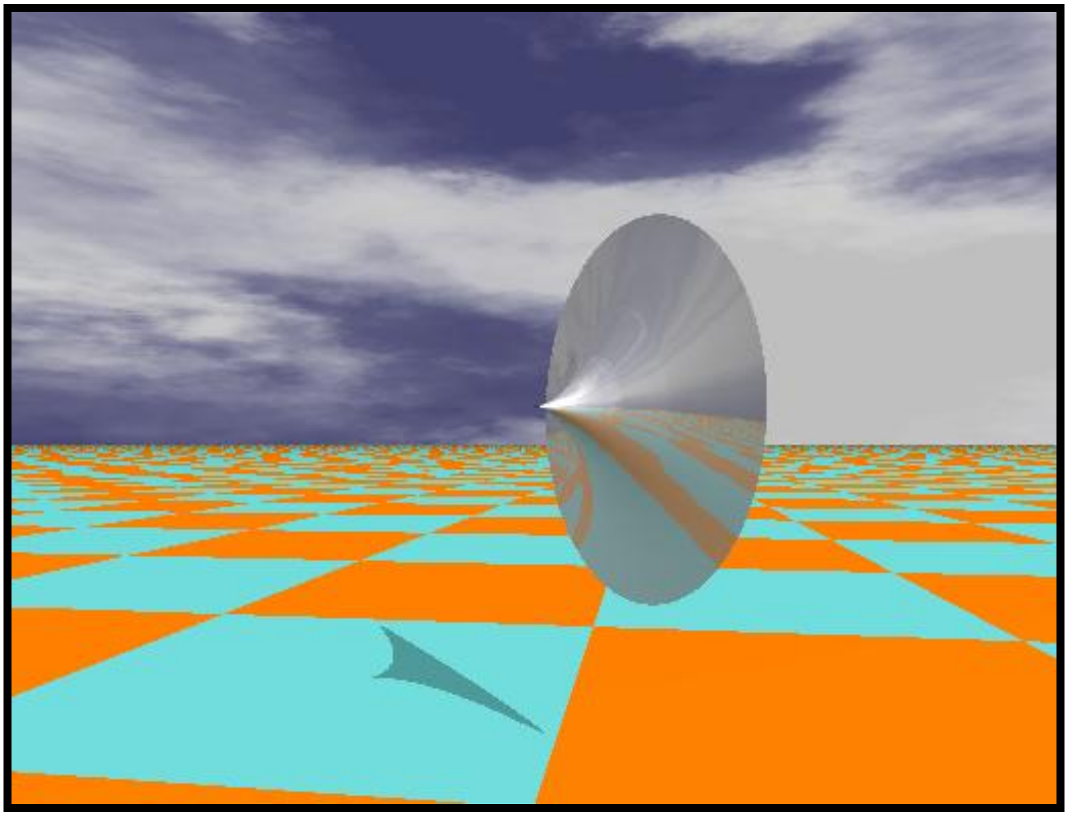

Figure 2-6: The Solid of Revolution Formed from the Cissoid of Diocles

This image was created by truncating the Cissoid of Diocles along its asymptote and then taking that result and rotating it around the x-axis. The solid of revolution was then placed over the infinite checkered plane which meets a cloudy sky at the horizon. The solid of revolution was then given a silvery-metallic surface so as to reflect its immediate environment, and one can see that it reflects the plane in its lower half and reflects the clouds in its upper half. A light source was placed so as to cast the solid's shadow onto the checkered plane.

Chapter 3 – The Strophoid

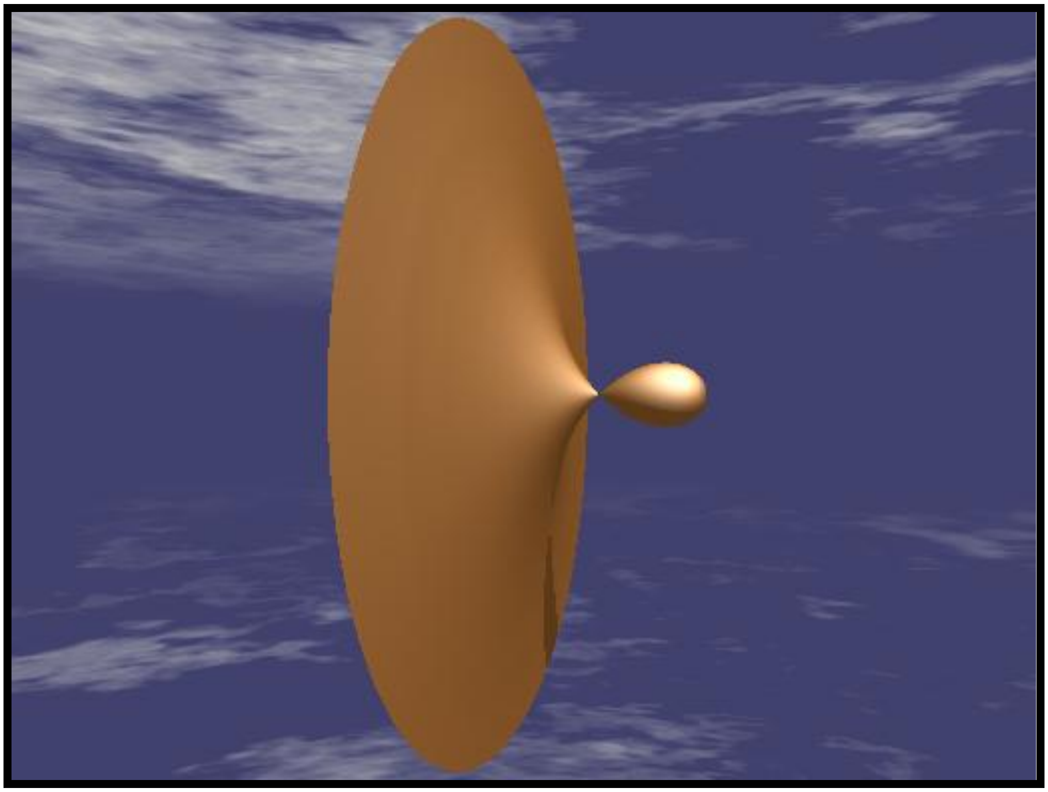

Figure 3-1: The Solid of Revolution Formed from the Right Strophoid

The Right Strophoid has been truncated along its asymptote and then revolved about the x-axis. The result is the solid of revolution seen in Figure 3-1. The knob protruding from the cusp is simply the loop portion of the Right Strophoid after revolution. The solid has been given a bronze-metallic surface texture and the entire figure has been placed in a cloud-flecked sky. Notice the lighting—it casts a partial glare on the knob and a portion of the cusp and the knob creates a shadow that shows up on the surface of the object itself.

3.1 Introduction

The Strophoid first appears in work by the English mathematician Isaac Barrow in 1670. (Barrow, incidentally, was Isaac Newton's teacher.) However, Torricelli actually describes the curve in his letters prior to Barrow's work—around 1645—and Roberval[4] found it as the locus of the focus of the conic obtained when the plane cutting the cone rotates about the tangent at its vertex. The name Strophoid, meaning a "belt with a twist, " was proposed by Montucci in 1846. The general Strophoid is a family of curves represented by the equation in polar coordinates

$$r = a(\cos\alpha \pm \sin\theta)\cdot\sec(\theta-\alpha) \quad \textbf{Equation 3-1}$$

Each value of the parameter α gives another member of the family. Figure 3-2 shows the graph of Equation 3-1 for some selected values of the parameter α, i.e., $\alpha =$ 0, $\pi / 6$, $\pi / 4$, and $\pi / 3$.

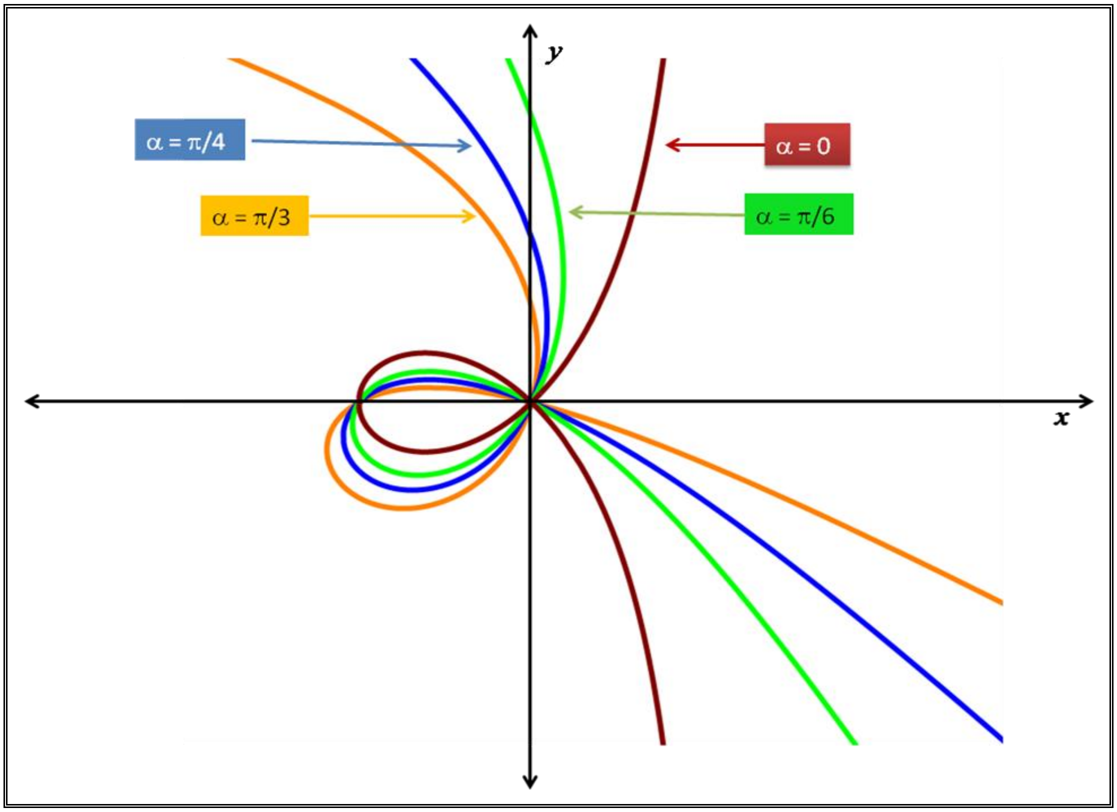

Figure 3-2: Members of the Family of Strophoids

[4] Gilles Roberval (French mathematician, 1602-1675) developed powerful methods in the early study of integration, writing *Traité des indivisibles.* He computed the definite integral of sin *x*, worked on the cycloid, and computed the arc length of a spiral. Roberval is important for his discoveries on plane curves and for his method for drawing the tangent to a curve, already suggested by Torricelli. This method of drawing tangents makes Roberval essentially the founder of kinematic geometry.

3.2 Equations and Graph of the Right Strophoid

When the parameter α is zero, i.e., $\alpha = 0$, Equation 3-1 reduces to

$$r = a(\sec\theta \pm \tan\theta) \quad \textbf{Equation 3-2}$$

a member of the family known as the Right Strophoid. The standard substitutions of $x = r\cos\theta$ and $y = r\sin\theta$ into Equation 3-2 will yield the Cartesian equation for the Right Strophoid, that is,

$$x(x-a)^2 = y^2(2a-x).$$

However, the more accepted form of the Right Strophoid has the point where the curve crosses itself at the origin and the other point on the x-axis at ($-a$, 0). This is, of course, just a translation to the left along the x-axis of a distance a. Hence, the Cartesian equation for the Right Strophoid becomes

$$y^2(a-x) = x^2(a+x) \quad \textbf{Equation 3-3}$$

If we now transform this back to polar form, we have

$$r = a(\sec\theta - 2\cos\theta) \quad \textbf{Equation 3-4}$$

This last equation may be easily transformed to parametric form by substituting the value of r into the equations $x = r\cos\theta$ and $y = r\sin\theta$, and letting $\theta = t$. This gives

$$(x, y) = a(1 - 2\cos^2 t)(1, \tan t), \; -\pi/2 < t < +\pi/2 \quad \textbf{Equation 3-5}$$

The equation of the tangent line at the point $t = q$ is

$$4\sin q\cos q \cdot y = (\sec^2 q + 4\sin^2 q - 2)x + a(1 - 2\cos^2 q)(2 - \sec^2 q). \quad \textbf{Equation 3-6}$$

Figure 3-3 displays the graph of the Right Strophoid.

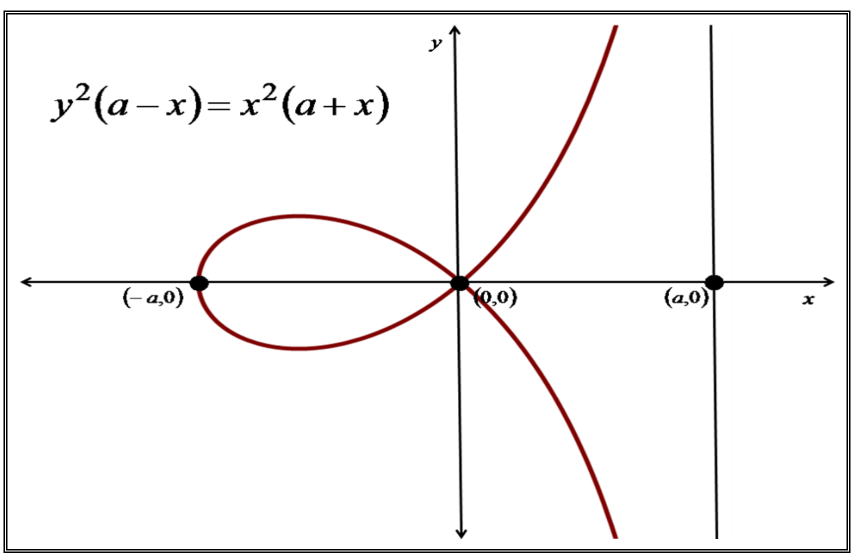

Figure 3-3: Graph of the Right Strophoid

3.3 Analytical and Physical Properties of the Right Strophoid

Using the parametric representation of the Right Strophoid given in Equation 3-5, the following paragraphs delineate the relevant properties of the Right Strophoid.

3.3.1 Derivatives of the Right Strophoid

- $\dot{x} = 4a\sin t\cos t$
- $\ddot{x} = 4a\left(\cos^2 t - \sin^2 t\right)$
- $\dot{y} = a\left(\sec^2 t + 4\sin^2 t - 2\right)$
- $\ddot{y} = 2a\left(\tan t\sec^2 t + 4\sin t\cos t\right)$
- $y' = \dfrac{\sec^2 t + 4\sin^2 t - 2}{4\sin t\cos t}$
- $y'' = \dfrac{3\tan^2 t + 1}{16a\sin^3 t\cos^3 t}$

3.3.2 Metric Properties of the Right Strophoid

If A is the area of the Right Strophoid's loop, then

$$A = a^2\left(\frac{4-\pi}{2}\right).$$

This result is easily obtained using the Cartesian form of the Right Strophoid which yields the following integral for the area under consideration

$$A = \int_0^a y\,dx = \int_0^a x\sqrt{\frac{a-x}{a+x}}\,dx.$$

This integral can best be evaluated by making the substitution

$$\tan^2\theta = \frac{a-x}{a+x}.$$

Under this substitution and much algebraic manipulation, the area integral becomes

$$A = 4a^2\int_0^{\pi/4}\sin^2\theta\,d\theta - 8a^2\int_0^{\pi/4}\sin^4\theta\,d\theta.$$

Using the identity $\sin^2\theta = ½ - ½\cos 2\theta$, the first integral becomes

$$\int_0^{\pi/4}\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta=\frac{\pi-2}{4}.$$

Using the same identity, the second integral becomes

$$\int_0^{\pi/4}\left(\frac{1}{4}-\frac{1}{2}\cos 2\theta+\frac{1}{4}\cos^2 2\theta\right)d\theta=\frac{3\pi-8}{32}.$$

Hence, the area of the loop above the *x*-axis is

$$A=4a^2\left(\frac{\pi-2}{8}\right)-8a^2\left(\frac{3\pi-8}{32}\right)=a^2\left(\frac{4-\pi}{4}\right).$$

The total loop area is therefore, by symmetry, twice this value or as is indicated above,

$$A=a^2\left(\frac{4-\pi}{2}\right).$$

In a similar manner, the area between the curve and its asymptote can also be calculated. This calculation is not shown; however, it is done using the same substitution as was used for the area of the loop. When all integrals have been evaluated, the area between the Right Strophoid and its asymptote will turn out to be

$$A=a^2\left(\frac{4+\pi}{2}\right).$$

Hence, totaling this area with that of the loop we get the very beautiful result that the total area of the Right Strophoid between the asymptote (at $x = + a$) and its tangent (at $x = - a$) is $4a^2$.

Just as the area of the Right Strophoid's loop was calculated above, one can also calculate the volume of the solid of revolution that is formed when that loop area is revolved about the *x*-axis. The total volume of the revolved loop will be

$$V=\pi\int_0^a y^2dx=\pi\int_0^a\frac{x^2(a-x)}{a+x}dx=\pi\int_0^a\frac{ax^2-x^3}{a+x}dx.$$

This integral is most easily evaluated by simply performing the indicated division, that is, dividing the numerator of the integrand by $a + x$, i.e.,

$$V=\pi\int_0^a\left(2ax-x^2-2a^2+\frac{2a^3}{a+x}\right)dx=2\pi a\int_0^a xdx-\pi\int_0^a x^2dx-2\pi a^2\int_0^a dx+2\pi a^3\int_0^a\frac{dx}{a+x}.$$

Hence,

$$V = 2\pi a\left[\frac{1}{2}x^2\right]_0^a - \frac{\pi}{3}\left[x^3\right]_0^a - 2\pi a^2\left[x\right]_0^a + 2\pi a^3\left[\ln(a+x)\right]_0^a .$$

Therefore,

$$V = \frac{2\pi a^3}{3}(3\ln 2 - 2).$$

If r is the distance from the origin to the curve, then

$$r = a(1-2\cos^2 t)\sec t .$$

If p is the distance from the origin to the tangent, then

$$p = \frac{-a(1-2\cos^2 t)^2}{\sqrt{1+4\cos^2 t - 4\cos^4 t}} .$$

3.3.3 Curvature of the Right Strophoid

If ρ represents the radius of curvature of the Right Strophoid, then

$$\rho = \frac{a(1+4\cos^2 t - 4\cos^4 t)^{3/2}}{4\cos^4 t(1+2\sin^2 t)} .$$

If (α, β) are the coordinates of the center of curvature, then

$$\alpha = \frac{a(8\sin^6 t - 12\sin^4 t + 6\sin^2 t - 3)}{4\cos^4 t(1+2\sin^2 t)} \quad \text{and} \quad \beta = \frac{4a\sin^3 t}{\cos t(1+2\sin^2 t)}.$$

3.3.4 Angles for the Right Strophoid

If ψ is the angle between the tangent and the radius vector at the point of tangency, then

$$\tan\psi = \cot t \cdot \frac{1-2\cos^2 t}{1+2\cos^2 t}.$$

If ϕ denotes the tangential angle, i.e., the angle between the tangent to the Right Strophoid and the horizontal, then

$$\tan\phi = \frac{\sec^2 t + 4\sin^2 t - 2}{4\sin t\cos t} .$$

If θ denotes the radial angle, i.e., the angle between the radius vector to the Right Strophoid and the horizontal, then

$$\theta = t.$$

3.4 Geometric Properties of the Right Strophoid

- Intercepts: $(-a, 0)$ and $(0, 0)$.
- Extrema: x-minimum at $(-a, 0)$; x-maximum at $(a, \pm\infty)$
    - y-minimum at $(a, -\infty)$; y-maximum at $(a, +\infty)$.
- Extent: $-a \le x < a$, $-\infty < y < +\infty$.
- Symmetry: symmetric about the x-axis.
- Asymptote: $x = a$.
- Loop: $-a \le x < 0$.

3.5 Dynamic Geometry of the Strophoid

Dynamic geometry applications, such as GSP, can be used to generate the Right Strophoid in a variety of entertaining ways. In fact, six different constructions for the Right Strophoid and two constructions for the general Strophoid follow.

3.5.1 A Construction for the General Strophoid

As alluded to in Chapter 2, if, in the construction of a Cissoid generated by a circle and a straight line, we let the pole point be anywhere on the circumference of the circle and we require the straight line to pass through the center of the circle, the resulting curve is the Strophoid. This is illustrated by the GSP construction delineated below in Table 3-1.

Table 3-1: The General Strophoid

1. Draw circle AB centered at A and passing through point B	8. Draw line segment DE
2. Let C be a random point <u>not</u> on circle AB	9. Construct circle C_2 centered at point F with radius = DE
3. Draw line AC.	10. Drag point D until circle C_2 surrounds circle AB*
4. Let D be a random point on the circumference of circle AB	11. Let point G be the intersection of circle C_2 and line DE*
5. Let E be a 2nd random point on the circumference of circle AB	12. Trace point G and change its color
6. Draw line DE	13. Animate point E around circle AB
7. Let point F be the intersection of lines AC and DE	

*See discussion below

The reason for step 10 is simply to ensure that when executing step 11, the correct intersection point is selected; there are two of them and one will trace the Strophoid and the other will not. The one that should be selected is the intersection point such that point D will lie between points E and G. Note also that each different position of point D on circle AB will cause the point G to trace a different member of the Strophoid family. When point D is at the position on circle AB that is the maximum distance from line AC (i.e., on a perpendicular to AC through the point A), then the Strophoid is a Right Strophoid.

3.5.2 The General Strophoid as the Pedal of a Parabola

Delineated below, in Table 3-2, is an alternate construction for the general Strophoid. The general Strophoid can be generated as the pedal curve of an ordinary parabola when the pedal point is selected as any random point on the directrix of the parabola.

Table 3-2: The General Strophoid as the Pedal of a Parabola

1. Draw horizontal line AB	9. Construct $P_1 \perp$ to line AB through point G.
2. Let C be a random point above line AB	10. Let I be the midpoint of line segment CG
3. Draw circle DE (below line AB) with center at point D	11. Construct $P_2 \perp$ to line segment CG through point I
4. Let F be a random point on the circumference of circle DE	12. Construct $P_3 \perp$ to P_2 through point H
5. Draw line DF	13. Let point J be the intersection of perpendiculars P_2 and P_3
6. Let point G be the intersection of lines AB and DF	14. Trace point J and change its color
7. Let H be a random point on line AB	15. Animate point F around circle DE
8. Draw line segment CG	

Note that if the construction for the general Strophoid shown above is continued as follows,

16. Construct point K, the intersection perpendiculars P_1 and P_2.
17. Trace point K and change its color.

then, as point F travels around circle DE, point K traces the parabola whose pedal curve was constructed. Its focus is at point C and it is tangent to the Strophoid. Additionally, if we continue the construction further,

18. Construct line segment KH.
19. Construct point L, the midpoint of line segment KH.
20. Draw line segment JL.
21. Construct perpendicular P_4 to line segment JL through point J.

Perpendicular P_4 that is drawn in step 21 is tangent to the Strophoid, and remains tangent as it moves with the animation of point F around circle DE. Quite spectacular! Also note that if the pedal point is on the directrix directly below the vertex of the parabola, then the Strophoid is a Right Strophoid.

3.5.3 A Construction Based on the Definition of a Right Strophoid

The Right Strophoid can be defined in the following way (refer to Figure 3-4): Let AB and AC be the sides of an angle of arbitrary, fixed measure, i.e., angle BAC. Let D be a random point lying on line AB, but on the opposite side of A from B. Let *L* be a straight line passing through point D intersecting line AC in point E. We now locate on *L* two points, P_1 and P_2, symmetrical to point E such that the lengths of the segments $EP_1 = EP_2 = EA$. When line *L* rotates around the point D, the points P_1 and P_2 describe a Right Strophoid. A GSP construction based on this definition is delineated below in Table 3-3.

Table 3-3: The Right Strophoid by Definition

1. Draw horizontal line AB	7. Let point G be the intersection of lines AC and DF
2. Draw line AC, where C is any point not on line AB	8. Draw circle GA centered at G and passing trough A
3. Let D be a point on line AB on the opposite side of A from B	9. Let points H & I be the intersections of circle GA and line DF
4. Draw circle DE centered at D with radius > AD	10. Trace points H and I and change their color
5. Let F be a random point on the circumference of circle DE	11. Animate point F around circle DE.
6. Draw line DF	

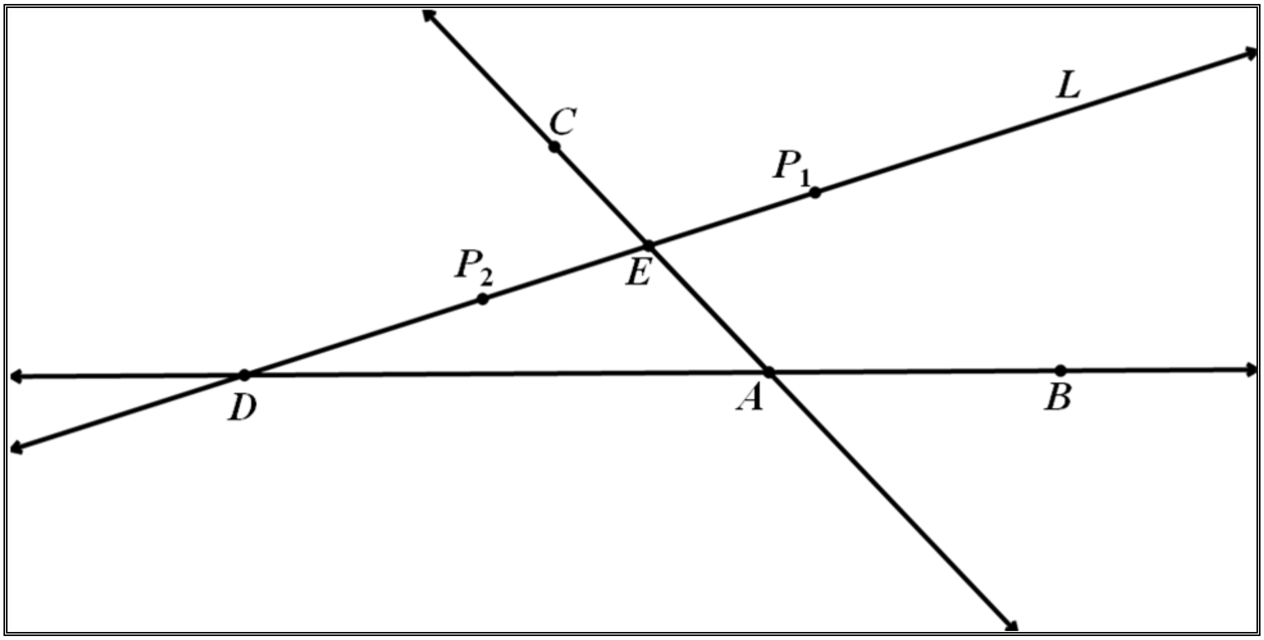

Figure 3-4: Defining the Right Strophoid

3.5.4 Newton's Carpenter Square Construction of the Right Strophoid

As alluded to in Chapter 2, Newton's Carpenter Square method of constructing a Cissoid can also be used to generate a Right Strophoid. That GSP construction is repeated here (Table 3-4) with the appropriate modifications required to produce the Right Strophoid.

Table 3-4: Newton's Construction of the Right Strophoid

1. Draw horizontal line segment AB with A to the right of B	7. Construct circle C_1 centered at F with radius = segment AB
2. Construct $P_1 \perp$ to segment AB through point A	8. Draw line segment AF
3. Draw circle CD to the right of segment AB	9. Construct circle C_2 centered at B with radius = segment AF
4. Let E be a random point on the circumference of circle CD	10. Let points G and H be the intersections of circles C_1 and C_2
5. Draw line EC	11. Trace points G and H and change their color
6. Let point F be the intersection of line EC and P_1	12. Animate point E around circle CD

Again note that if GSP is being used, the trace of points G and H not only produce the Right Strophoid, but also produce a straight line that is tangent to the Right Strophoid. This straight line is not part of the Right Strophoid, but is a result of the way in which GSP switches points G and H at the point of tangency.

3.5.5 The Right Strophoid as an Envelope of Circles

A particularly beautiful construction for the Right Strophoid is given in Table 3-5. In fact, this construction is really a slight variation of the construction for the Strophoid given in section 3.5.2 where the general Strophoid was generated as the pedal curve of a Parabola with the pedal point as an arbitrary point on the Parabola's directrix. In this case, point J traces the Parabola and is the center of the circles whose envelope defines the Right Strophoid. Execute this construction, perform the animation, and watch as the Right Strophoid unfolds. It is a thing of beauty!

Table 3-5: The Right Strophoid as an Envelope of Circles

1. Construct vertical line AB	10. Construct P_2 $\perp$ to line AB through point G
2. Let C be a random point not on line AB	11. Draw line segment CG
3. Let D be a random pt. on the opposite side of line AB from C	12. Let I be the midpoint of line segment CG
4. Draw circle DE centered at D and passing through point E	13. Construct P_3 $\perp$ to line segment CG through point I
5. Construct P_1 $\perp$ to line AB through point C	14. Let point J be the intersection of P_2 and P_3
6. Let F be a random point on the circumference of circle DE	15. Draw circle JH centered at J and passing through point H
7. Draw line DF	16. Trace circle JH and change its color
8. Let point G be the intersection of lines DF and AB	17. Animate point F around circle DE
9. Let point H be the intersection of line AB and P_1	

3.5.6 The Right Strophoid as the Inverse of a Hyperbola

If the point of inversion is taken as the vertex of a Rectangular Hyperbola, then the Hyperbola inverts to a Right Strophoid, as seen in the construction of Table 3-6.

Table 3-6: The Right Strophoid as the Inverse of a Hyperbola

1. Draw circle AB centered at A and passing through point B	13. Let G′ be the image when G is translated by distance EG*
2. Let C be a random point on the circumference of circle AB	14. Draw line FG′
3. Draw horizontal line AD such that AD > AB	15. Draw circle G'H centered at point G' and of any radius = G'H
4. Draw line segment CD	16. Measure distance G'F
5. Let E be the midpoint of line segment CD	17. Measure distance G'H
6. Construct P_1 $\perp$ to line segment CD through point E	18. Calculate $(G'H)^2 / G'F$
7. Draw line AC	19. Let G" be the image of translating G' by result of step 18*
8. Let point F be the intersection of line AC and P_1	20. Draw circle G'G" centered at G' and passing through point G"
9. Trace point F and change its color	21. Let point I be one intersection of circle G'G" and line G'F
10. Draw line segment AD	22. Trace point I and change its color
11. Let G be the midpoint of line segment AD	23. Animate point C around circle AB
12. Measure distance EG	

*The translations in both steps 13 and 19 should be done along line AD, i.e., at 0°

Steps 1 – 9 are the construction of the Rectangular Hyperbola. Steps 10 – 13 are to locate the vertex of the Rectangular Hyperbola, which is then labeled G'. Finally, steps 14 – 21 are the construction of the inverse of the Hyperbola. Note that step 15 is simply the creation of the inversion circle. As we have learned it can be any size, affecting the scale of the inverted curve, but not the nature of the curve; that is why you can draw it with an arbitrary size radius. In step 21, either intersection of the circle and line may be chosen; both will ultimately yield a Right Strophoid, however, they will open in opposite directions. For that matter, either vertex can also be chosen, again yielding Right Strophoids that open in opposite directions.

3.5.7 The Right Strophoid as an Inversion of Itself

Finally, if an inversion circle is centered at the point where the Right Strophoid crosses the *x*-axis and has radius the distance of that point to the origin, then the Right Strophoid is invariant under inversion in that circle. When a curve inverts into itself, it is called anallagmatic with respect to the given point of inversion. The following construction, delineated in Table 3-7, illustrates this invariant concept. In the construction below, steps 18 – 25 are the inversion of the Right Strophoid which is constructed in steps 1 – 17. This is quite a beautiful construction; note that the three perpendiculars of the construction are all concurrent.

Table 3-7: The Right Strophoid as an Inversion of Itself

1. Draw horizontal line AB	14. Draw line EL
2. Draw line CD such that line CD intersects line AB	15. Construct $P_3 \perp$ to line EL through point I
3. Let E be a random point on neither lines AB nor CD	16. Let point M be the intersection of line EL and P_3
4. Construct $P_1 \perp$ to line CD through point E	17. Trace point M and change its color
5. Draw circle FG (any radius) below line AB	18. Draw circle EJ centered at E and passing through point J
6. Let H be a random point on the circumference of circle FG	19. Measure distance EM
7. Draw line FH	20. Measure distance EJ
8. Let point I be the intersection of lines FH and CD	21. Calculate $(EJ)^2$ / EM
9. Let point J be the intersection of line CD and P_1	22. Let E′ be the image when E is translated by $(EJ)^2$ / EM
10. Draw line segment EI	23. Draw circle EE′ centered at E and passing through point E′
11. Let K be the midpoint of line segment EI	24. Let N be one of the intersections of circle EE′ and line LE
12. Construct $P_2 \perp$ to line segment EI through point K	25. Trace point N and change its color
13. Let point L be the intersection of line CD and P_2	26. Animate point H around circle FG

3.5.8 The Right Strophoid and Its Tangent

Table 3-8 contains a construction for the curve and its tangent.

Table 3-8: The Right Strophoid and Tangent

1. Create *x*-*y* axes with origin as point A and unit point as B	12. Let F′ be the image when F is translated by vector A → C
2. Let C be a random point on the negative *x*-axis.	13. Construct the locus of point F′ as point D traverses circle AC
3. Draw circle AC centered at A and passing through point C	14. Let point E_1 be the reflection of point E across line AD
4. Construct $P_1 \perp$ to the *x*-axis through point C	15. Let E_2 be the image when E_1 is translated by vector F → E_1
5. Let D be a random point on the circumference of circle AC	16. Let point G be the intersection of the *x*-axis and P_2
6. Draw line AD	17. Let E_3 be the image when E_1 is translated by vector G → E_1
7. Let point D′ be the reflection of point D across the *x*-axis	18. Let E_4 be the image when E_2 is translated by vector E_3 → E_2
8. Draw line AD′	19. Draw line segment AE_4
9. Let point E be the intersection of line AD′ and P_1	20. Construct $P_3 \perp$ to line segment AE_4 through point F
10. Construct $P_2 \perp$ to line AD through point E	21. Construct line L_1 parallel to P_3 through point F′
11. Let point F be the intersection of line AD and P_2	22. Animate point D around circle AC

Make the locus of step 13 and line L_1 of step 21 thick and colored and run the animation. It's a sight to behold!

3.5.9 The Strophoid and Its Osculating Circle

Finally, as the last construction for this chapter, Table 3-9 contains a construction for the Strophoid's osculating circle.

Table 3-9: The Strophoid's Osculating Circle

1. Draw horizontal line AB	19. Let K be a random point on line AB
2. Draw circle AB with center at A and passing through point B	20. Construct $P_7 \perp$ to P_2 through point K
3. Construct $P_1 \perp$ to line AB through point B	21. Let point L be the intersection of perpendiculars P_2 and P_7
4. Let C be a random point on the circumference of circle AB	22. Construct the locus of point L as point C traverses circle AB
5. Draw line AC	23. Draw line segment JK
6. Let point D be the intersection of line AC and P_1	24. Let M be the midpoint of line segment JK
7. Let E be diametrically opposite point B	25. Draw line segment GK
8. Draw line segment DE	26. Let N be the midpoint of line segment GK
9. Let F be the midpoint of line segment DE	27. Draw line LN
10. Construct $P_2 \perp$ to line segment DE through point F	28. Let point O be the intersection of P_4 and line LN
11. Construct $P_3 \perp$ to P_1 through point D	29. Construct $P_8 \perp$ to line LN through point O
12. Let point G be the intersection of perpendiculars P_2 and P_3	30. Let point P be the intersection of perpendiculars P_7 and P_8
13. Construct $P_4 \perp$ to P_2 through point G	31. Draw line MP
14. Let point H be the intersection of P_4 and line AB	32. Let point Q be the intersection of line MP and line LN
15. Construct $P_5 \perp$ to P_4 through point H	33. Draw circle QL
16. Let point I be the intersection of perpendiculars P_3 and P_5	34. Make circle QL thick and change its color
17. Construct $P_6 \perp$ to P_3 through point I	35. Animate point C around circle AB
18. Let point J be the intersection of perpendiculars P_4 and P_6	

First of all, note that the locus of point L should be a Strophoid. If in your construction it is not, drag point K along line AB until the Strophoid forms. Second, note that there are some hidden goodies in this construction. If you construct the locus of point G as point C travels on circle AB you will find that the locus is a parabola. Further, if you construct circle JG (i.e., the circle centered at point J and passing through point G) and then rerun the animation, you will find that circle JG is the osculating circle of the parabola. Also note that the Strophoid is constructed as the parabola’s pedal curve, as we discussed in section 3.5.2.

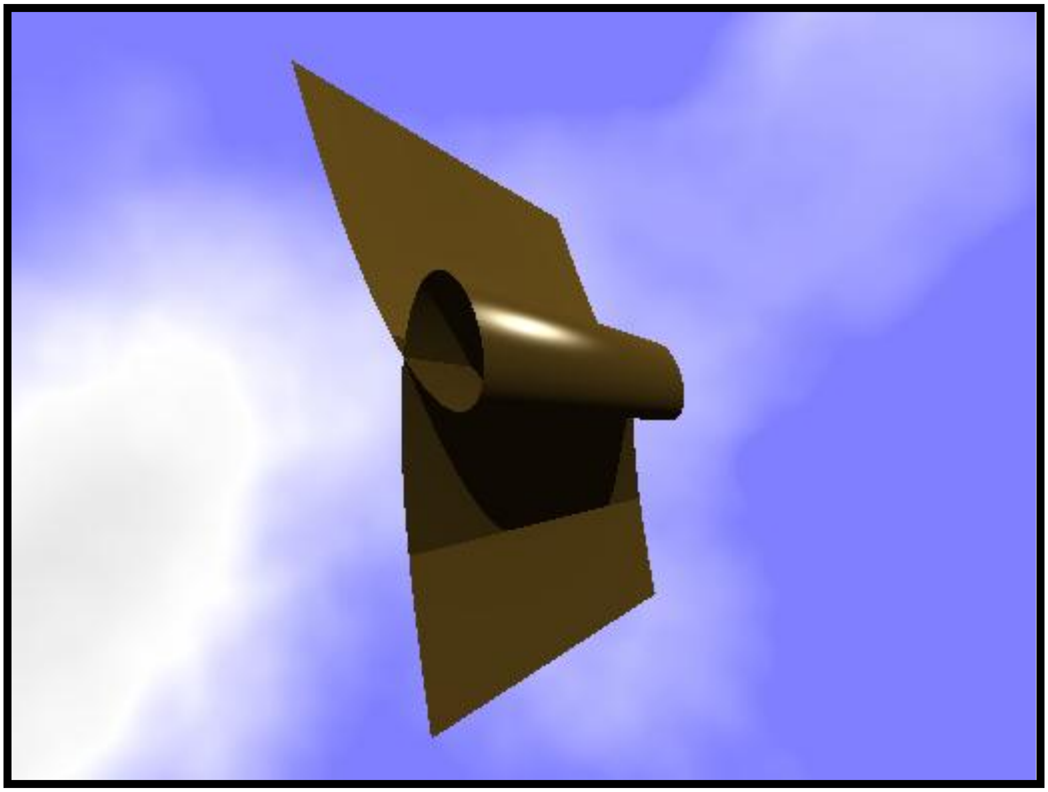

Figure 3-5: The Right Strophoid in Three Dimensions

To see the Right Strophoid, you look along the left edge of this figure. In other words, the cross-section of this figure is a Right Strophoid. To create it, the Right Strophoid was extruded into this third dimension and then the curve was truncated along its asymptote. The surface of the resulting figure was then given a sienna colored finish. The figure was then placed as though floating in an azure-blue sky and a light source was situated so as to cast the shadow caused by the extruded loop portion of the curve. The light source itself can be seen reflecting off the surface of the loop.

Chapter 4 – The Witch of Agnesi

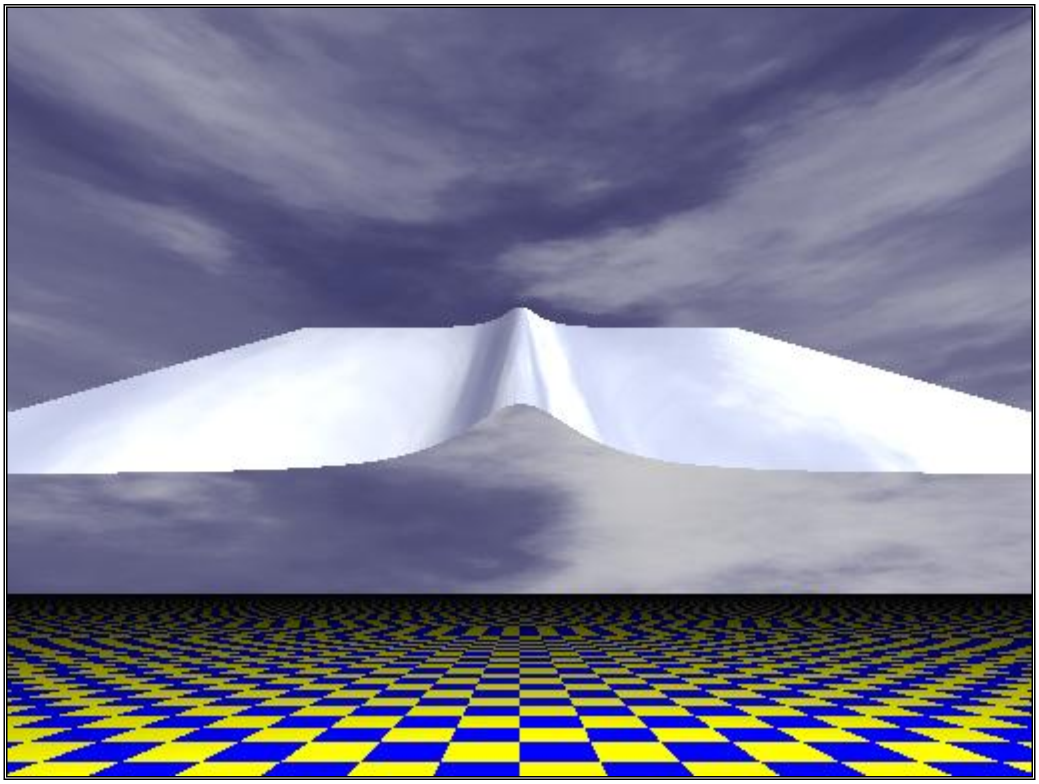

Figure 4-1: The Witch of Agnesi in Three Dimensions

The "*Witch of Agnesi*" *curve can be seen along the leading edge of the three-dimensional figure above. To construct this figure, however, the curve was extruded into the third dimension, truncated along its asymptote, and then given a semi-reflective silver finish that reflects, to some extent, the clouds on the horizon. The infinite blue and yellow checkered plane was then added to complete the figure.*

4.1 Introduction

Maria Agnesi (1718 to 1799) was the author of a famous two-volume work on the methods of Calculus circa 1748. This work by Agnesi is the first surviving mathematical work by a woman. The book includes a discussion of the curve now known as the "Witch of Agnesi." It is unfortunate that this curve has come down to us through the years with this name, for it is certainly not the name that Ms. Agnesi intended, or for that matter, the name which anybody else intended. The curve was first discussed by Fermat and a construction for the curve was given by Grandi[5] in 1703. In 1718 Grandi gave the curve the Latin name *versoria* which means turning curve, so named because of its shape. Grandi also gave the Italian *versiera* for the Latin *versoria* and indeed Agnesi quite correctly states in her book that the curve was called *la versiera*. However, an Englishman by the name of John Colson translated Agnesi's book from Italian into English and Colson mistook *la versiera* for *laversiera* which means ungodly woman or she-devil. Hence, today we know the curve as the Witch. (Colson now has the distinction of being the first mathematical male chauvinist; however, in Colson's favor is the fact that a chapter entitled "Turning Curve" is not anywhere near as romantic sounding as one entitled "The Witch of Agnesi.")

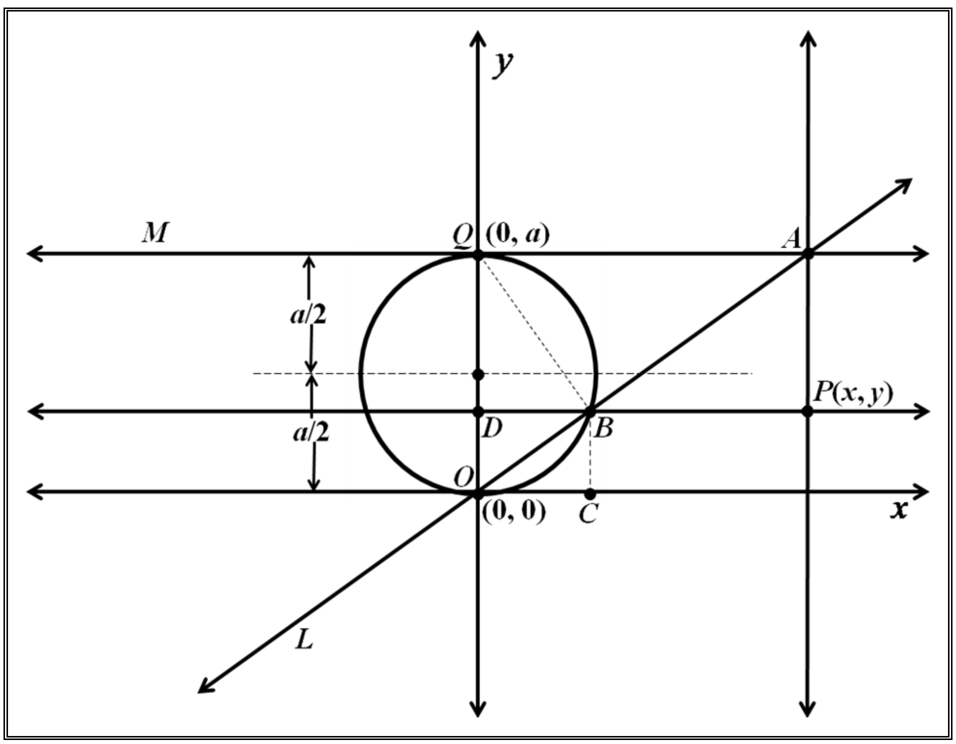

Figure 4-2: Definition of the Witch of Agnesi

Refer to Figure 4-2, which depicts an origin O, a circle of diameter a tangent to the x-axis at the origin that passes through the point Q(0, a), and a line M which is

[5] Guido Grandi (1671 to 1742) was the author of a number of works on geometry. In 1703 he studied the curve that is today known as the Witch of Agnesi; in fact, his work of 1703 was important in introducing Leibniz's calculus into Italy.

parallel to the *x*-axis and also passes through Q. Let any line, *L*, passing through the origin intersect the circle in point B and intersect the line *M* in point A. Let the projection of point B on the *x*- and *y*-axis be points C and D, respectively. Finally, let point P be the intersection of two perpendiculars, the first through point A and perpendicular to line *M* and the second through point B and perpendicular to the *y*-axis. The Witch of Agnesi is defined as the locus of point P for all possible lines *L*.

4.2 Equations and Graph of the Witch of Agnesi

It is relatively straightforward to derive the Cartesian equation for the Witch from the geometric relationships depicted in Figure 4-2. First note that ΔBPA is similar to ΔAQO. Thus, AQ / BP = QO / PA. However, AQ = x, QO = a, and PA = $a - y$. Therefore,

$$\frac{x}{\mathrm{BP}} = \frac{a}{a-y} \quad \text{or} \quad \mathrm{BP} = \frac{x(a-y)}{a}.$$

Now, in ΔOBC, we have $\mathrm{OB}^2 = \mathrm{OC}^2 + \mathrm{BC}^2 = (x - \mathrm{BP})^2 + y^2$. Substituting the value of BP from above, we have

$$\mathrm{OB}^2 = \frac{(x^2 + a^2)y^2}{a^2}.$$

Further, in ΔQDB, we have $\mathrm{BQ}^2 = \mathrm{QD}^2 + \mathrm{DB}^2 = (a - y)^2 + (x - \mathrm{BP})^2$. Again, substituting the value of BP, we have

$$\mathrm{BQ}^2 = \frac{a^2(a-y)^2 + x^2y^2}{a^2}.$$

Finally, in ΔQOB, which incidentally is a right triangle because it is inscribed in a semi-circle, we have, $a^2 = \mathrm{OB}^2 + \mathrm{BQ}^2$. Hence, adding the two previous results, equating it to a^2, and simplifying, we get the Cartesian form of the Witch of Agnesi, i.e.

$$y = \frac{a^3}{x^2 + a^2} \qquad \textbf{Equation 4-1}$$

There are at least two different, useful parametric representations of the Witch which offer a convenient form. First let $x = at$. Then,

$$y = \frac{a^3}{a^2t^2 + a^2} = \frac{a}{1+t^2}.$$

Hence, the first parametric representation is

$$(x, y) = a\left(t, \frac{1}{1+t^2}\right), \quad -\infty < t < +\infty \qquad \textbf{Equation 4-2}$$

For the second representation, let $x = a \cdot \tan t$. Then,

$$y = \frac{a^3}{a^2 + a^2\tan^2 t} = \frac{a}{1+\tan^2 t} = \frac{a}{\sec^2 t} = a\cos^2 t.$$

Therefore, the second parametric representation is

$$(x, y) = a\left(\tan t, \cos^2 t\right), \quad -\tfrac{\pi}{2} < t < \tfrac{\pi}{2} \qquad \textbf{Equation 4-3}$$

Of course, substitution of the usual polar coordinate transformations of $x = r\cos\theta$ and $y = r\sin\theta$ gives the Witch's polar form as

$$r^3\sin\theta - r\sin\theta\left(r^2 + a^2\right) + a^3 = 0 \qquad \textbf{Equation 4-4}$$

Finally, the equation of the tangent line at the point $t = q$ is

$$y + 2\sin q\cos^3 q \cdot x = a\cos^2 q\left(1 + 2\sin^2 q\right) \qquad \textbf{Equation 4-5}$$

The graph of the Witch of Agnesi is depicted in Figure 4-3.

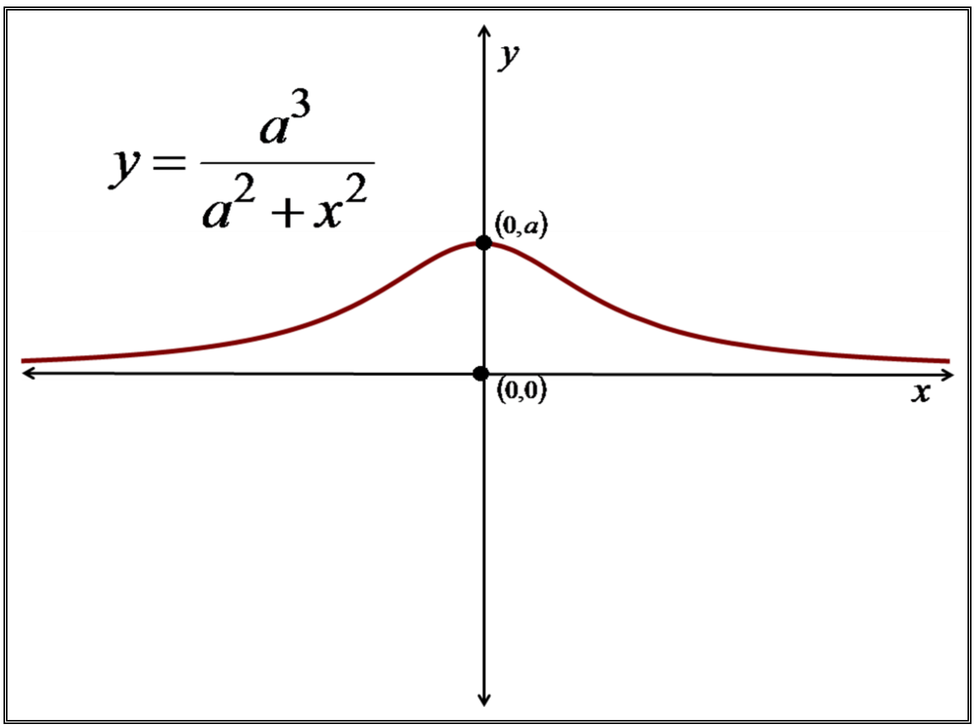

Figure 4-3: Graph of the Witch of Agnesi

4.3 Analytical and Physical Properties of the Witch of Agnesi

Using the parametric representation given in Equation 4-3 (i.e., $x = a\tan t$ and $y = a\cos^2 t$), the following paragraphs delineate the relevant properties of the Witch of Agnesi.

4.3.1 Derivatives of the Witch of Agnesi

- $\dot{x} = a\sec^2 t$
- $\ddot{x} = 2a\sin t\sec^3 t$
- $\dot{y} = -a\sin 2t$
- $\ddot{y} = -2a\cos 2t$
- $y' = -2\sin t\cos^3 t$
- $y'' = \dfrac{2}{a}\cos^4 t\left(3 - 4\cos^2 t\right)$

4.3.2 Metric Properties of the Witch of Agnesi

If A is the area between the Witch and its asymptote, then

$$A = \int_{-\infty}^{+\infty} y \cdot dx = a^3 \int_{-\infty}^{+\infty} \frac{dx}{a^2 + x^2}\,.$$

This integral is recognizable as the inverse tangent form, i.e.,

$$a^3 \int_{-\infty}^{+\infty} \frac{dx}{a^2 + x^2} = a^3 \left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]_{-\infty}^{+\infty} = a^2\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \pi a^2.$$

In other words, the area between the Witch of Agnesi and the x-axis is four times the area of the initial, defining circle pictured in Figure 4-2.

If V is the volume of the solid of revolution that is formed when the Witch is rotated about the x-axis, then

$$V = \pi \int_{-\infty}^{+\infty} y^2 \cdot dx = \pi a^6 \int_{-\infty}^{+\infty} \frac{dx}{\left(a^2 + x^2\right)^2}.$$

This integral is most easily evaluated by making the substitution $x = a\cdot\tan\theta$. Under this substitution, the integral is transformed to

$$V = \pi a^6 \int_{-\pi/2}^{\pi/2} \frac{a\sec^2\theta \cdot d\theta}{a^4 \sec^4\theta} = \pi a^3 \int_{-\pi/2}^{\pi/2} \cos^2\theta \cdot d\theta.$$

Using the identity $\cos^2\theta = ½ + ½\cdot\cos 2\theta$, we have

$$V = \pi a^3 \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta \right) d\theta = \frac{\pi^2 a^3}{2}.$$

If r represents the radial distance from the origin to the curve, then

$$r = a\sqrt{\tan^2 t + \cos^4 t}\ .$$

If p is the distance from the origin to the tangent of the Witch, then

$$p = \frac{a\left(1 + 2\sin^2 t\right)}{\sqrt{\sec^4 t + 4\sin^2 t \cos^2 t}}\ .$$

4.3.3 Curvature of the Witch of Agnesi

If ρ represents the radius of curvature of the Witch of Agnesi, then

$$\rho = \frac{a\left(1 + 4\cos^6 t \sin^2 t\right)^{3/2}}{2\left(4\sin^2 t - 1\right)\cos^4 t}\,.$$

If (α, β) are the coordinates of the center of curvature of the Witch, then

$$\alpha = \frac{4a\left(1 + \cos^6 t\right)\sin^3 t}{\cos t\left(4\sin^2 t - 1\right)} \quad \text{and} \quad \beta = \frac{a\left(1 + 10\cos^6 t - 12\cos^8 t\right)}{2\cos^4 t\left(4\sin^2 t - 1\right)}\,.$$

4.3.4 Angles for the Witch of Agnesi

If ψ is the angle between the tangent and the radius vector at the point of tangency, then

$$\tan\psi = \frac{\left(2\sin^2 t + 1\right)\cos^3 t}{\left(2\cos^6 t - 1\right)\sin t}\,.$$

If ϕ denotes the tangential angle, then

$$\tan\phi = -2\sin t \cos^3 t.$$

If θ denotes the radial angle, then

$$\tan\theta = \frac{\cos^3 t}{\sin t}\,.$$

4.4 Geometric Properties of the Witch of Agnesi

- x-intercept: $x = 0$.
- y-intercept: $y = a$.

- The minimum value of the curve occurs at ($\pm\infty$, 0).
- The maximum value of the curve occurs at (0, a).
- Points of inflection occur at $\left(\pm\dfrac{\sqrt{3}a}{6}, \dfrac{3a}{8}\right)$.
- Extent: $-\infty < x < +\infty$; $0 \le y \le a$.
- Symmetry: The curve is symmetric about the y-axis.
- Asymptote: The curve is asymptotic to the x-axis.

4.5 Dynamic Geometry of the Witch of Agnesi

Five GSP dynamic geometry constructions involving the Witch of Agnesi follow. The first is based on the definition of the Witch, the second is for the Tangent line to the Witch, the third shows how to construct the pedal curve of the Witch, the fourth is an alternate construction of the Witch's tangent, while the fifth is for the Osculating Circle.

4.5.1 The Witch of Agnesi Based on the Definition

See Figure 4-2 and the accompanying write-up to understand how the Witch of Agnesi is defined as a locus of points. The GSP construction shown in Table 4-1 follows from that definition.

Table 4-1: The Witch of Agnesi from the Definition

1. Draw vertical line segment AB with point A below point B	8. Let point E be the intersection of lines AD and L_1
2. Construct $P_1 \perp$ to line segment AB through point A.	9. Construct $P_2 \perp$ to L_1 through point E
3. Construct line L_1 parallel to P_1 through point B.	10. Construct $P_3 \perp$ to P_2 through point D
4. Let C be the midpoint of line segment AB	11. Let point F be the intersection of perpendiculars P_2 and P_3
5. Draw circle CA centered at C and passing through point A	12. Trace point F and change its color
6. Let D be a random point on the circumference of circle CA	13. Animate point D around circle CA
7. Draw line AD	

4.5.2 The Tangent to the Witch of Agnesi

As the next construction we will show how to construct the tangent line to the Witch of Agnesi. Of course, in order to construct the tangent to the Witch, one must first construct the Witch itself (herself?). That is done below in Table 4-2 as it was done in section 4.5.1 with a few slight changes.

Table 4-2: The Witch of Agnesi and Tangent Line

1. Draw vertical line Segment AB with point A below point B	11. Let point F be the intersection of P_2 and P_3
2. Construct $P_1 \perp$ to line segment AB through point A	12. Construct $P_4 \perp$ to P_1 through point E.
3. Construct $P_2 \perp$ to line segment AB through point B	13. Draw line AF
4. Let C be the midpoint of line segment AB	14. Construct $P_5 \perp$ to P_4 through point D
5. Draw circle CA centered at C and passing through point A	15. Let point G be the intersection of circle CA and line AF
6. Let D be a random point on the circumference of circle CA	16. Let point H be the intersection of P_4 and P_5
7. Draw line AD	17. Draw line GH
8. Draw line segment CD	18. Change the color and thickness of line GH
9. Construct $P_3 \perp$ line segment CD through point D	19. Construct the locus of point H as point D traverses circle CA
10. Let point E be the intersection of line AD and P_2	20. Animate point D around circle CA

The thickened line (line GH) is, of course, the tangent. Note how almost all of the construction lines come together (i.e., merge) at point A and then again at point B as the animation is run.

4.5.3 The Pedal Curves of the Witch of Agnesi

We learned in Chapter 1 that a Pedal curve is simply the locus of the intersection point of a given curve's tangent and the perpendicular to that tangent from a given point, called the pedal point or pole. Since we now know how to construct the tangent to the Witch (see previous construction), it is a simple matter to also construct the Witch's Pedal Curves. Table 4-3 below depicts such a construction.

Table 4-3: The Witch of Agnesi's Pedal Curves

1. Draw vertical line segment AB with point A below point B	13. Draw line AF
2. Construct $P_1 \perp$ to line segment AB through point A	14. Construct $P_5 \perp$ to P_4 through point D
3. Construct $P_2 \perp$ to line segment AB through point B	15. Let point G be the intersection of circle CA and line AF
4. Let C be the midpoint of line segment AB	16. Let point H be the intersection of P_4 and P_5
5. Draw circle CA centered at C and passing through point A	17. Construct the locus of point H as D traverses circle CA
6. Let D be a random point on the circumference of circle CA	18. Draw line GH
7. Draw line AD	19. Change the color and thickness of line GH
8. Draw line segment CD	20. Let I be a random point anywhere in the plane
9. Construct $P_3 \perp$ to line segment CD through point D	21. Construct $P_6 \perp$ to line GH through point I
10. Let point E be the intersection of line AD and P_2	22. Let point J be the intersection of line GH and P_6
11. Let point F be the intersection of P_2 and P_3	23. Trace point J and change its color
12. Construct $P_4 \perp$ to P_1 through point E	24. Animate point D around circle CA

No, that's not a typo or grammatical error in using the plural in "Pedal Curves." Although the above construction will only display one of the Witch's Pedal curves, you can drag the pedal point (point I) to another location in your GSP construction and rerun the animation. For each different location of point I, one will get a different pedal curve. It is particularly interesting to drag point I onto line segment AB (or its extension) and rerun the animation. A pedal curve symmetric about segment AB is obtained.

4.5.4 An Alternate Construction for the Tangent to the Witch of Agnesi

Although the tangent line to a curve at a given point on the curve is unique, the method of constructing such a line is not. Table 4-4 gives an alternate construction for the Witch's tangent which is rather interesting (the construction for the curve is slightly different also).

Table 4-4: The Witch's Tangent (Alternate Construction)

1. Create *x*-*y* axes with A as origin and B as the point (1, 0)	13. Construct $P_5 \perp$ to the *x*-axis through point G
2. Draw circle AB centered at A and passing through point B	14. Let point H be the intersection of P_3 and P_5
3. Let C be a random point on the circumference of circle AB	15. Construct the locus of point H as Point C traverses circle AB
4. Draw line AC	16. Construct $P_6 \perp$ to P_1 through point G
5. Construct $P_1 \perp$ to line AC through point A	17. Let point I be the intersection of P_3 and the *y*-axis
6. Let D be the intersection of circle AB with the positive *y*-axis	18. Let F′ be the image when F is translated by vector I → F
7. Let E be the intersection of circle AB with the negative *y*-axis	19. Construct $P_7 \perp$ to the *x*-axis through point F′
8. Construct $P_2 \perp$ to P_1 through point D	20. Let point J be the intersection of P_6 and P_7
9. Let point F be the intersection of P_1 and P_2	21. Draw line AJ
10. Construct $P_3 \perp$ to the *y*-axis through point F	22. Construct $P_8 \perp$ to line AJ through point H
11. Construct $P_4 \perp$ to the *y*-axis through point E	23. Animate point C around circle AB
12. Let point G be the intersection of P_1 and P_4	

Of course, perpendicular P_8, constructed in step 22, is the tangent line. Note how the tangent line coincides with the *x*-axis when the point that traces the curve (point H) approaches infinity—the curve is asymptotic to the *x*-axis.

4.5.5 The Osculating Circle for the Witch of Agnesi

Generally, the constructions associated with the Witch of Agnesi are relatively simple. However, as Table 4-5 portrays, the construction for the Witch's Osculating Circle is rather complex.

Table 4-5: The Osculating Circle for the Witch of Agnesi

1. Draw circle AB with center at A and passing through point B	23. Construct $P_{10} \perp$ to P_1 through point B"
2. Draw line AB	24. Let point K be the intersection of perpendiculars P_9 and P_{10}
3. Let C be a random point on the circumference of circle AB	25. Draw line segment AK
4. Draw line AC	26. Construct $P_{11} \perp$ to line AB through point B'
5. Construct $P_1 \perp$ to line AB through point A	27. Let point L be the intersection of line AB and P_{11}
6. Construct $P_2 \perp$ to line AC through point A	28. Let L' be the image when L is translated by vector A → L
7. Let D and E be the two intersections of circle AB with P_1	29. Construct $P_{12} \perp$ to line AB through point L'
8. Construct $P_3 \perp$ to P_2 through point D	30. Let point M be the intersection of line AC with P_9
9. Let F be the intersection of perpendiculars P_2 and P_3	31. Let M' be the image when M is translated by vector J → M
10. Construct $P_4 \perp$ to P_1 through point F	32. Construct $P_{13} \perp$ to P_1 through point M'
11. Construct $P_5 \perp$ to P_1 through point E	33. Let point N be the intersection of perpendiculars P_{12} and P_{13}
12. Let point G be the intersection of perpendiculars P_2 and P_5	34. Let point N' be the image when N is rotated about A by – 90°
13. Construct $P_6 \perp$ to line AB through point G	35. Construct $P_{14} \perp$ to segment AK through point A
14. Let point H be the intersection of perpendiculars P_4 and P_6	36. Construct $P_{15} \perp$ to P_{14} through point N'
15. Construct the locus of point H as point C traverses circle AB	37. Let point O be the intersection of perpendiculars P_{14} and P_{15}
16. Construct $P_7 \perp$ to line AB through point B	38. Draw line segment KO
17. Let point I be the intersection of line AC and P_7	39. Construct $P_{16} \perp$ to line segment KO through point K
18. Construct $P_8 \perp$ to line AC through point I	40. Let point P be the intersection of perpendiculars P_{14} and P_{16}
19. Let point J be the intersection of line AB and P_8	41. Let H' be the image when H is translated by vector P → A
20. Construct $P_9 \perp$ to line AB through point J	42. Draw circle H'H centered at point H' and passing through H
21. Let B' be the image when B is reflected across line AC	43. Make circle H'H thick and change its color
22. Let B" be the image when point B' is reflected across line AB	44. Animate point C around circle AB

No less than 16 perpendiculars are required for this fantastic construction. By the way, when the center of the osculating circle, i.e., the center of curvature, crosses the curve (in this case the Witch of Agnesi), the point or points where the crossing takes place is (are) the point(s) of inflection of the curve. In other words, the intersection point(s) of the curve and its evolute is (are) the point(s) of inflection.

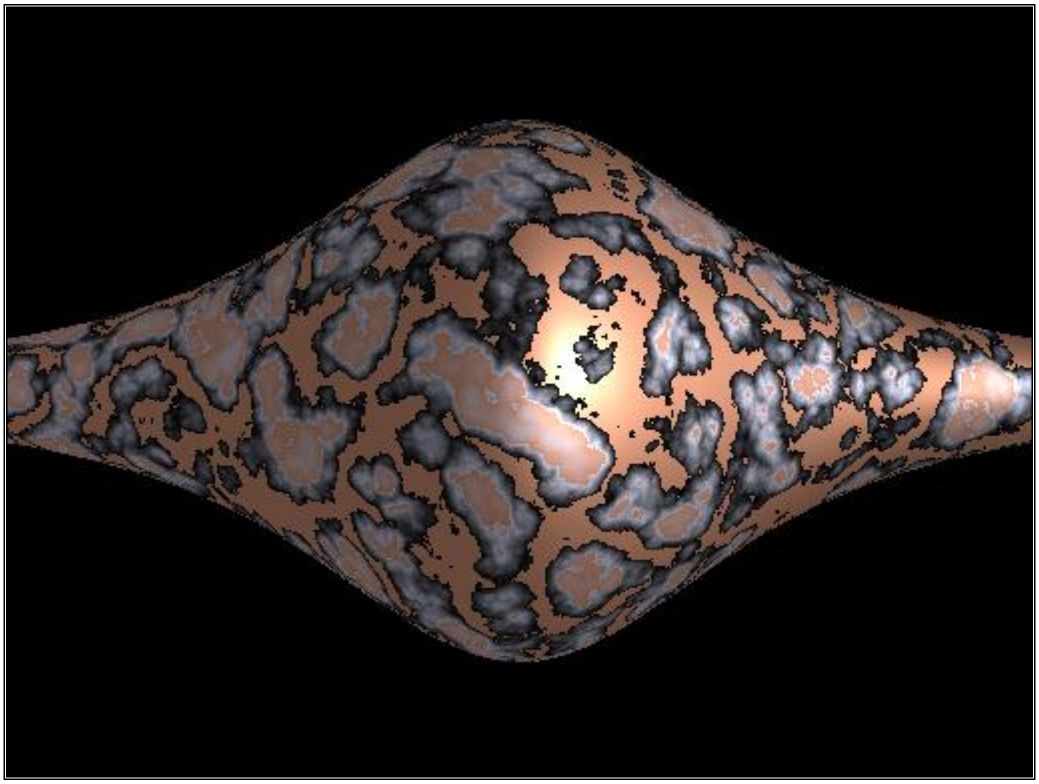

Figure 4-4: The Solid of Revolution Formed by the Witch of Agnesi

This figure shows the solid formed when the Witch of Agnesi is revolved about the x-axis. The background has been made solid black to give the appearance that the solid is floating in space. The surface of the solid has been given a weathered brass finish and the light source has been placed so as to illuminate the upper right portion of the solid and to partially shadow the rest of the object.

Chapter 5 – The Conchoid of Nicomedes

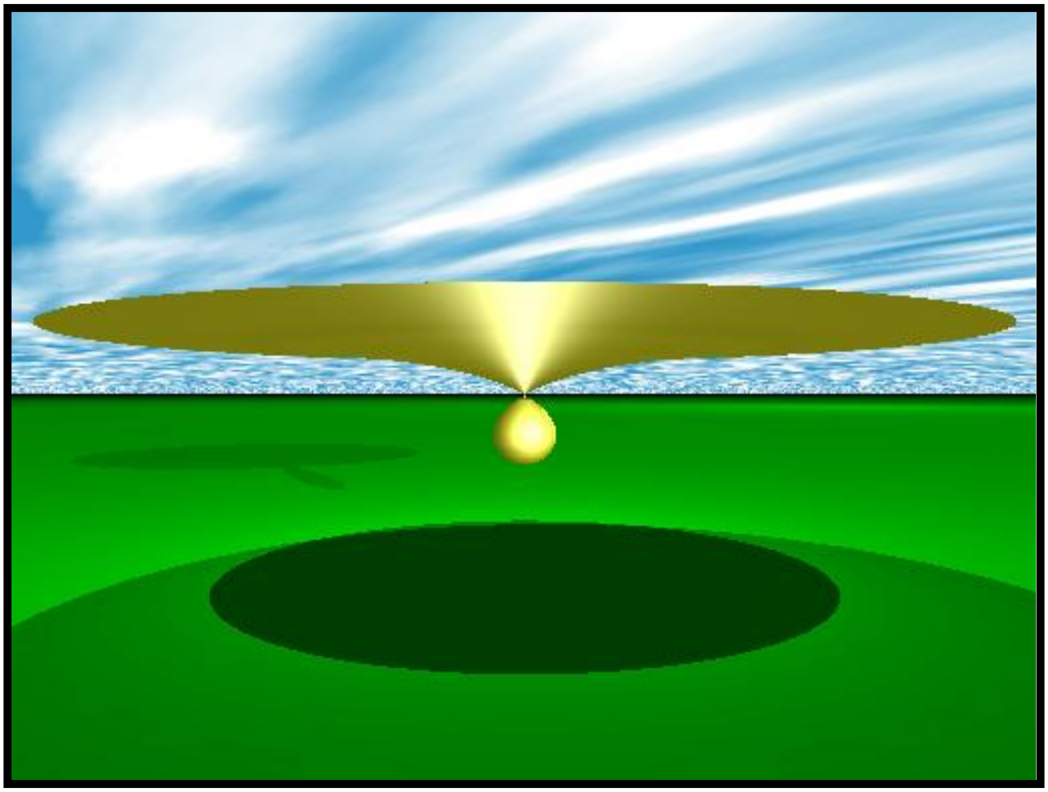

Figure 5-1: The Solid of Revolution Formed from the Conchoid of Nicomedes

The Conchoid of Nicomedes is a curve with two branches. One branch has a loop in it for certain values of the curve's parameters. Figure 5-1 portrays the loop branch of the Conchoid of Nicomedes after it was truncated along its directrix and then rotated about the y-axis. The surface of the solid so formed has been given a bright gold finish and then placed over an infinite green plane that meets a cloud-bedecked sky at the horizon. Two light sources have been configured so as to form the shadows seen on the green plane; one in the background and one directly under the solid of revolution.

5.1 Introduction

In Chapter 1, we learned about the concept of a Conchoid, namely, one method of deriving a new curve from a given curve. That is, given a curve C and a fixed point *O*, points P_1 and P_2 are taken on a variable line through *O* at a distance ± *a* from the intersection of the line and the curve C. Then, the locus of P_1 and P_2 is called the Conchoid of the given curve C with respect to the point *O*. When the given curve C is itself a straight line, the Conchoid is called the Conchoid of Nicomedes.

Nicomedes was born in Greece about 280 BC and died approximately 210 BC. Very little is known of Nicomedes' life; even the birth and death dates are approximations. However, Nicomedes is famous for his treatise *On Conchoid Lines* which contains his discovery of the curve that is today referred to as the Conchoid of Nicomedes. According to modern accounts, the Conchoid of Nicomedes was first conceived by Nicomedes to solve the angle trisection problem (we will address this problem later in the chapter). The name conchoid is derived from Greek and it means "shell," as in the word "conch"; the curve is also sometimes known as a cochloid. In actuality, the Conchoid of Nicomedes describes a whole family of curves, a different curve for each value of the parameter *a*; i.e., given a line *L*, a point *O* that is not on *L*, and a specified distance *a*, the Conchoid of Nicomedes is defined as follows (refer to Figure 5-2). Draw a line *K* passing through point *O* and intersecting line *L* in point *P*. Locate points P_1 and P_2 on line *K* such that the distance $PP_1 = PP_2 = a$. Then, the locus of points P_1 and P_2 for each point *P* on *L* gives the Conchoid of Nicomedes.

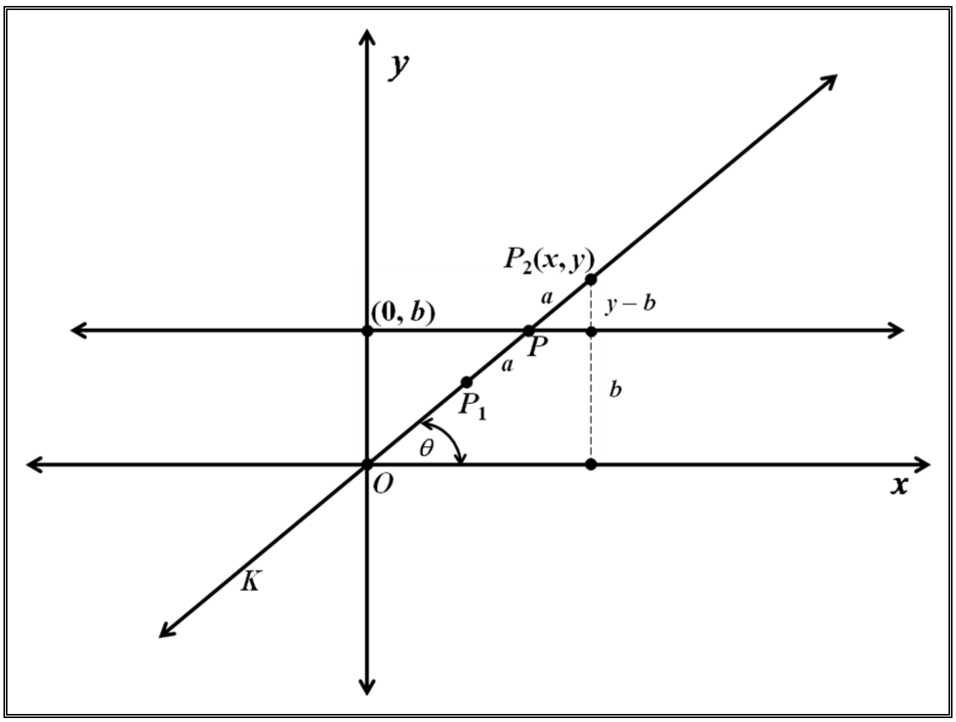

Figure 5-2: The Definition of the Conchoid of Nicomedes

The point O is called the pole point and the given line, L, is called the conchoid's directrix and is an asymptote to the curve.

5.2 Equations and Graph of the Conchoid of Nicomedes

To arrive at a parametric representation for the Conchoid of Nicomedes, we could simply plug a parametric representation for a straight line into Equation 1-15 (given in Chapter 1) and be done with it. However, it is more informative to derive the equations directly from the definition. As a matter of fact, let's do both, since there is something to be learned from performing both exercises. To derive the Cartesian equation for this curve directly, first consider the point P_2 to have coordinates x and y. From Figure 5-2, one can see that

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}.$$

Similarly,

$$\sin\theta = \frac{y-b}{a}.$$

Eliminating θ by squaring and adding, we have

$$\frac{x^2}{x^2+y^2} + \frac{(y-b)^2}{a^2} = 1.$$

Now, clearing the fractions and rearranging, we see that the Cartesian form of the Conchoid of Nicomedes is

$$(y-b)^2\left(x^2+y^2\right) - a^2y^2 = 0 \quad \textbf{Equation 5-1}$$

Note that we could just as easily have considered the point P_1 to have coordinates x and y. In that case,

$$\sin\theta = \frac{b-y}{a.},$$

and when θ is eliminated due to squaring and adding, the same final equation is derived.

For the polar form, substitute $y = r\sin\theta$ and $x = r\cos\theta$. Making these substitutions yields

$$(r\sin\theta - b)^2\left(r^2\right) - a^2r^2\sin^2\theta = 0.$$

Expanding and simplifying, the polar form of the Conchoid of Nicomedes is

$$(r-a)\sin\theta = b \quad \textbf{Equation 5-2}$$

A convenient parametric form can be obtained by letting $y = x \cdot \tan t$, substituting this into Equation 5-1 and solving for x. Thus,

$$(x\tan t - b)^2 \left(x^2 + x^2 \tan^2 t\right) - a^2 x^2 \tan^2 t = 0.$$

Simplifying and rearranging, we get

$$x = \frac{b}{\tan t} + a\cos t.$$

Similarly, solving for y, we obtain

$$y = b^2 + a\sin t.$$

Hence, a parametric representation for the Conchoid of Nicomedes is,

$$(x, y) = \left(\frac{b}{\tan t} + a\cos t, b^2 + a\sin t\right), \quad -\tfrac{\pi}{2} < t < \tfrac{\pi}{2} \qquad \textbf{Equation 5-3}$$

Now let us do the derivation by plugging the parametric equation for the line $y = b$ into Equation 1-15. Doing that yields

$$x = t \pm \frac{at}{\sqrt{t^2 + b^2}} \quad \text{and} \quad y = b \pm \frac{ab}{\sqrt{t^2 + b^2}}.$$

If we now substitute $t = b \tan\phi$, we obtain

$$x = \tan\phi(b + a\cos\phi) \quad \text{and} \quad y = b + a\cos\phi.$$

Since ϕ is just a dummy parameter, we can call it t (our usual notation) and we have an alternate parametric representation, that is,

$$(x, y) = (b + a\cos t)(\tan t, 1), \quad -\tfrac{\pi}{2} < t < \tfrac{3\pi}{2} \qquad \textbf{Equation 5-4}$$

Nevertheless, continuing with our second derivation, if we now square both x and y and add the results together, we have

$$x^2 + y^2 = (b + a\cos\phi)^2 \sec^2\phi.$$

However, $b + a\cos\phi = y$ and $\sec\phi = a / (y - b)$. Therefore,

$$x^2 + y^2 = y^2 \left(\frac{a}{y - b}\right)^2.$$

Simplifying, we get Equation 5-1, as promised. The value of this exercise was that we were able to obtain a convenient alternate parametric representation along the way, namely Equation 5-4.

If we graph the Conchoid of Nicomedes, we get the two-branched curve shown in Figure 5-3.

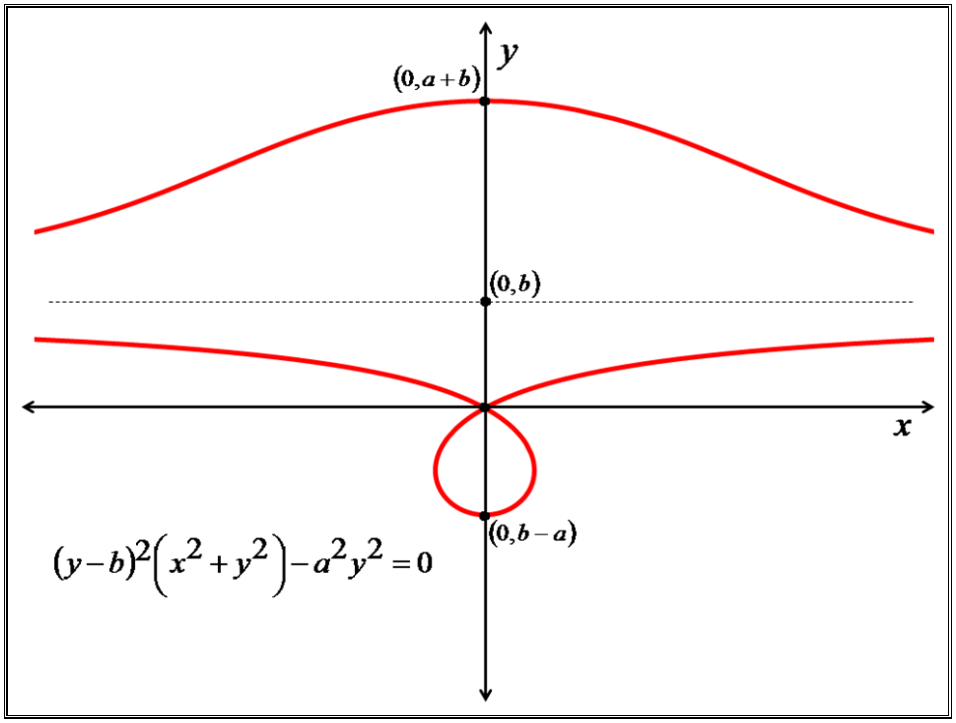

Figure 5-3: Graph of the Conchoid of Nicomedes

Note that the graph of the curve shown in Figure 5-3 shows what the curve looks like if the parameter $b < a$. If b = a, the loop becomes a cusp and if $b > a$, the bottom branch is smooth like the top branch shown in the figure.

The equation of the tangent line at the point $t = q$ is

$$\left(b + a\cos^3 q\right)\cdot y + a\sin q\cos^2 q\cdot x = \left(b + a\cos q\right)^2. \quad \textbf{Equation 5-5}$$

5.3 Analytical and Physical Properties of the Conchoid of Nicomedes

Using the parametric representation of the Conchoid of Nicomedes given in Equation 5-4, i.e., $x = (b + a\cos t)\cdot\tan t$ and $y = b + a\cos t$, the following are the relevant properties of the Conchoid of Nicomedes:

5.3.1 Derivatives of the Conchoid of Nicomedes

- $\dot{x} = b\sec^2 t + a\cos t$
- $\ddot{x} = 2b\sin t\sec^3 t - a\sin t$

- $\dot{y} = -a\sin t$
- $\ddot{y} = -a\cos t$
- $y' = \dfrac{-a\sin t\cos^2 t}{b + a\cos^3 t}$
- $y'' = \dfrac{a\cos^3 t\left(2b - 3b\cos^2 t - a\cos^3 t\right)}{\left(b + a\cos^3 t\right)^3}$

5.3.2 Metric Properties of the Conchoid of Nicomedes

As can be seen from the graph (Figure 5-3), the Conchoid of Nicomedes has the line $y = b$ as an asymptote; however, unlike curves in previous chapters, the area between the curve (that is, either branch of the curve) and the asymptote is infinite. Nevertheless, the area of the loop can be calculated. Consider an incremental portion of the loop area of width $2x$ and height dy. The area of this incremental portion is simply $dA = 2x \cdot dy$. Hence, by integrating between $a - b$ and 0, we can obtain the loop area, i.e.,

$$A = 2\int_{b-a}^{0} x \cdot dy = 2\int_{b-a}^{0} \frac{y}{y-b}\sqrt{a^2 - (y-b)^2}\,dy\,.$$

This integral is most easily evaluated by making the substitution $u = y - b$. Under this substitution, we get

$$A = 2\int_{-a}^{-b}\sqrt{a^2 - u^2}\,du + 2b\int_{-a}^{-b}\frac{\sqrt{a^2 - u^2}}{u}\,du.$$

The first integral has the value

$$\left[u\sqrt{a^2 - u^2} + a^2\sin^{-1}\left(\frac{u}{a}\right)\right]_{-a}^{-b} = a^2\cos^{-1}\left(\frac{b}{a}\right) - b\sqrt{a^2 - b^2}.$$

The second integral has the value

$$\left[2b\sqrt{a^2 - u^2} - 2ab\log\left(\frac{a + \sqrt{a^2 - u^2}}{u}\right)\right]_{-a}^{-b} = 2b\sqrt{a^2 - b^2} - 2ab\log\left(\frac{a + \sqrt{a^2 - b^2}}{b}\right).$$

Hence, adding these two values together gives the loop area of the Conchoid of Nicomedes, that is,

$$A = b\sqrt{a^2 - b^2} - 2ab\log\left(\frac{a + \sqrt{a^2 - b^2}}{b}\right) + a^2\cos^{-1}\left(\frac{b}{a}\right).$$

The volume of the solid of revolution that is formed when the loop of the Conchoid of Nicomedes is rotated about the *y*-axis can also be calculated. Consider an incremental disk of width dy. Its volume is simply $dV = \pi x^2 dy$. Therefore, by integrating from $b - a$ to 0, we can calculate the total volume. That is,

$$V = \pi\int_{b-a}^{0} y^2\left[\frac{a^2 - (y-b)^2}{(y-b)^2}\right]dy.$$

Making the substitution $y - b = u$ yields the following readily integrable form

$$V = \pi\int_{-a}^{-b}\frac{(b+u)^2(a^2 - u^2)}{u^2}du = \pi\int_{-a}^{-b}\frac{a^2b^2 + 2a^2bu + a^2u^2 - b^2u^2 - 2bu^3 - u^4}{u^2}du.$$

This now breaks into six different integrals, each of which is integrable as either a positive or negative power of u. Performing the indicated integrations, evaluating the results between the two limits of integration, and then collecting like terms, we obtain a final form for the desired volume.

$$V = \pi ab(a - 2b) + 2\pi a^2 b\log\left(\frac{b}{a}\right) + \frac{\pi}{3}(4a^3 - b^3).$$

If r is the radial distance from the origin to the curve, then

$$r = \frac{b + a\cos t}{\cos t}.$$

If p is the distance from the origin to the tangent line, then

$$p = \frac{(b + a\cos t)^2}{\sqrt{b^2 + 2ab\cos^3 t + a^2\cos^4 t}}.$$

5.3.3 Curvature of the Conchoid of Nicomedes

If ρ denotes the radius of curvature of the Conchoid of Nicomedes, then,

$$\rho = \frac{(b^2 + 2ab\cos^3 t + a^2\cos^4 t)^{3/2}}{\cos^5 t\left[ab(2\tan^2 t - 1) - a^2\cos t\right]}.$$

If (α, β) denote the coordinates of the center of curvature for the Conchoid of Nicomedes, then

$$\alpha = \frac{\sin^2 t \tan t\left(3b^2 + 2ab\cos t\right)}{2b - 3b\cos^2 t - a\cos^3 t} \quad \text{and} \quad \beta = \frac{b^3 + ab^2\cos^3 t\left(5 - 3\cos^2 t\right) + a^2 b\cos^4 t\left(3 - 2\cos^2 t\right)}{a\cos^3 t\left(2b - 3b\cos^2 t - a\cos^3 t\right)}$$

5.3.4 Angles for the Conchoid of Nicomedes

If ψ is the angle between the tangent and the radius vector (i.e., the tangential-radial angle), then

$$\tan\psi = -\frac{\cos t\left(b + a\cos t\right)}{b\sin t}.$$

If θ denotes the radial angle, then

$$\theta = \pi / 2 - t.$$

If ϕ denotes the tangential angle, then

$$\tan\phi = \frac{-a\sin t\cos^2 t}{b + a\cos^3 t}.$$

5.4 Geometric Properties of the Conchoid of Nicomedes

- Intercepts: $(0, b - a)$; $(0, 0)$; and $(0, a + b)$.
- Maximum: $(0, a + b)$.
- Extent: $-\infty < x < +\infty$; $(b - a) \le y \le a + b$.
- Symmetry: The curve is symmetric about the *y*-axis.
- Asymptote: The curve is asymptotic to the line $y = b$.

5.5 Trisecting the Angle

As alluded to in the introductory section of this chapter, the Conchoid of Nicomedes can be used to solve the Greek angle trisection problem. Given an acute ∠AOB, we must construct an angle that is ⅓ of ∠AOB. (If the given angle is obtuse, then one simply performs the construction on its supplement.) Refer to Figure 5-4.

1. Draw a line *J* that is perpendicular to segment AO of ∠AOB.
2. Let point C be the intersection point of segment AO and line *J*.
3. Let point D be the intersection point of line *J* and segment BO of ∠AOB.
4. Let a Conchoid of Nicomedes be constructed with pole at point O, directrix of line J, and distance of 2·OD.
5. Draw line *K* through point D and perpendicular to line *J*.
6. Let E be the intersection of the curve (on the opposite side of the pole) and line *K*.

7. Draw line segment OE.
8. Let point F be the intersection of line J and segment OE.
9. Let point G be the midpoint of segment FE.
10. Draw segment DG.

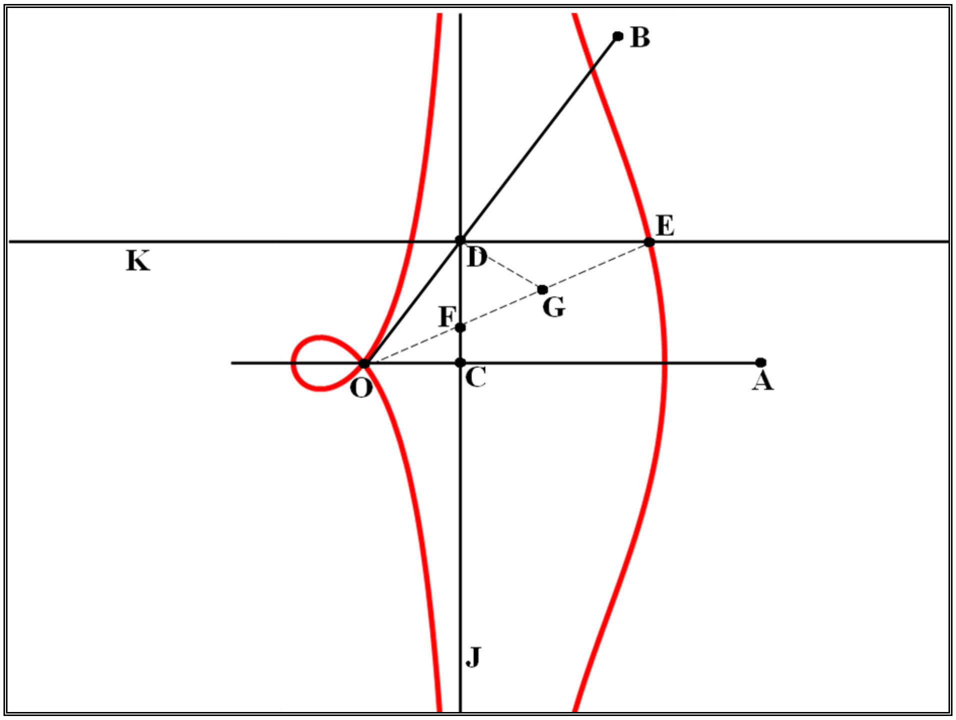

Figure 5-4: The Trisection of an Angle Using the Conchoid of Nicomedes

Since triangles DEG, OGD, and GFD are all isosceles and since triangle FDE is a right triangle, it can easily be shown that ∠AOE = ∠OED = ⅓∠AOB. The essential element that makes the trisection possible is the construction of the point E on line *K* such that the segment FE is equal to twice segment OD. A slight modification of this construction can actually be used to generate the Conchoid of Nicomedes (i.e., essentially reversing the steps of the trisection process) and is presented as one of the dynamic geometry constructions in section 5.6.

5.6 Dynamic Geometry Considerations

Dynamic geometry programs such as GSP can be used to generate the Conchoid of Nicomedes and demonstrate other properties of the curve. A few such constructions follow.

5.6.1 The Conchoid of Nicomedes Based on the Definition

The construction presented below in Table 5-1 basically follows from the definition of the Conchoid of Nicomedes given in section 5.1. Note that in this construction, by moving point I into different locations in the plane relative to points G and H, one can cause one branch of the curve to have a cusp, or a loop, or to be smooth (as briefly addressed in section 5.2).

Table 5-1: The Conchoid of Nicomedes Based on the Definition

1. Draw horizontal line AB	8. Draw circle HI with center at H and passing through point I
2. Let C and D be two random points anywhere below line AB	9. Let C_1 be the image as circle HI is translated by vector H → F
3. Draw circle CD with center at C and passing through point D	10. Draw line GF
4. Let E be a random point on the circumference of circle CD	11. Let J and K be the intersections of line GF and circle C_1
5. Draw line CE	12. Trace points J and K and change their color
6. Let point F be the intersection of lines AB and CE	13. Animate point E around circle CD
7. Let G, H, and I be random points anywhere above line AB	

5.6.2 The Trisection Construction

A rather complex but beautiful construction can be based on the trisection problem as discussed earlier. It is presented below in Table 5-2.

Table 5-2: The Trisection Construction

1. Draw horizontal line segment AB with B to the left of point A	16. Let point I be the intersection of line AB and P_1
2. Let C be a random point <u>above</u> line segment AB	17. Draw circle IB with center at I and passing through point B
3. Draw line segment BC	18. Let point J be the intersection of ray BC' and P_1
4. Draw a dashed line through points A and B	19. Construct $P_2 \perp$ to P_1 through point I
5. Draw a dashed line through points B and C	20. Draw line HB
6. Let m_1 be the measure of ∠CBA	21. Let m_3 be the measure of distance BI
7. Let $m_2 = ⅓ \cdot m_1$	22. Let $m_4 = 2 \cdot m_3$
8. Let C' be the image when C is rotated about point B by $\angle m_2$	23. Let point K be the intersection of circle IB and ray BC'
9. Draw ray BC' from point B through point C'	24. Draw line segment KI
10. Let D be a random point on line segment BC	25. Let H' be the image when point H is translated by distance m_4
11. Construct $P_1 \perp$ to line CB through point D	26. Draw circle HH' centered at H and passing through point H'
12. Draw circle EF centered at point E where EF is any radius	27. Let L and M be the intersections of circle HH' and line HB
13. Let G be a random point on the circumference of circle EF	28. Trace points L and M and change their color
14. Draw line EG	29. Animate point G around circle EF
15. Let point H be the intersection of line EG and P_1	

In step 8 of the above construction, make sure that the angle units are set for directed degrees (i.e., select directed degrees in the object preferences box found under the preference entry of GSP's display menu).

5.6.3 The Generalized Conchoid

We learned in Chapter 1 and again at the beginning of this chapter that the Conchoid of Nicomedes can be thought of as a special case of a more general type of curve, namely, something that we call the generalized Conchoid or just Conchoid. As an example of this more general concept, consider the dynamic geometry construction of the Conchoid of a circle with respect to a given pole and distance *a*.

Table 5-3: The Generalized Conchoid

1. Draw line segment AB	6. Construct circle C_2 centered at point E with radius = AB
2. Draw circle CD with center at point C and CD = to any radius	7. Let G and H be the intersections of circle C_2 and line FE
3. Let E be a random point on the circumference of circle CD	8. Trace points G and H and change their color
4. Let F be a random point <u>not</u> on circle CD	9. Animate point E around circle CD
5. Draw line FE	

In step 1, the length of line segment AB represents the distance *a*. In step 2, circle CD represents the circle for which we desire to construct the Conchoid. And, in step 4, point F represents the pole point. Note that by dragging point A (or point B) in order to change the length of segment AB and thereby change the radius of circle C_2, different members of the family for the Conchoid of a circle can be generated. Dragging point F will also have this same effect.

5.6.4 The Tangent Lines of the Conchoid of Nicomedes

Below, in Table 5-4, is a construction of the tangent lines to the Conchoid of Nicomedes, one tangent line for each branch.

Table 5-4: Tangent Lines to the Conchoid of Nicomedes

1. Draw horizontal line AB	12. Let J and K be the two intersections of line GF and circle C_2
2. Let C and D be two random points <u>below</u> line AB	13. Construct $P_2 \perp$ to line GF through point G
3. Draw circle CD with center at C and passing through point D	14. Construct the locus of point J while E traverses circle CD
4. Let E be a random point on the circumference of circle CD	15. Construct the locus of point K while E traverses circle CD
5. Draw line CE	16. Let point L be the intersection of perpendiculars P_1 and P_2
6. Let point F be the intersection of lines AB and CE	17. Draw lines JL and KL
7. Let G, H, and I be three random points <u>above</u> line AB	18. Construct $P_3 \perp$ to line JL through point J
8. Draw circle HI with center at H and passing through point I	19. Construct $P_4 \perp$ to line KL through point K
9. Let C_2 be the translation of circle HI by the vector H $\rightarrow$ F	20. Change the color of the two loci
10. Draw line GF	21. Change the color and thickness of perpendiculars P_3 and P_4
11. Construct $P_1 \perp$ to line AB through point F	22. Animate point E around circle CD

5.6.5 The Pedal Curves of the Conchoid of Nicomedes

As we learned earlier, the pedal of a given curve is defined to be the locus of the intersection point of the tangent to the given curve and the perpendicular to that tangent from the pole or pedal point. In the last section (section 5.6.4), we constructed the tangents to both branches (one tangent per branch) of the Conchoid of Nicomedes. It should therefore be "duck soup" to construct the pedal curves to either branch of the Conchoid of Nicomedes. Let's do it! The pedal curve construction is found below in Table 5-5.

Table 5-5: Pedal Curves of the Conchoid of Nicomedes

1. Draw horizontal line AB	15. Let point L be the intersection of perpendiculars P_1 and P_2
2. Let C and D be two random points <u>below</u> line AB	16. Draw lines JL and KL
3. Draw circle CD with center at C and passing through point D	17. Construct $P_3 \perp$ to line JL through point J
4. Let E be a random point on the circumference of circle CD	18. Construct $P_4 \perp$ to line KL through point K
5. Draw line CE	19. Let M be a random point anywhere in the plane
6. Let point F be the intersection of lines AB and CE	20. Construct $P_5 \perp$ to P_3 through point M
7. Let G, H, and I be three random points <u>above</u> line AB	21. Let the intersection of P_3 and P_5 be point N
8. Draw circle HI with center at H and passing through point I	22. Trace point N and change its color
9. Let circle C_2 be the translation of circle HI by vector H $\rightarrow$ F	23. Let point O be another random point anywhere in the plane
10. Draw line GF	24. Construct $P_6 \perp$ to P_4 through point O
11. Construct $P_1 \perp$ to line AB through point F	25. Let point P be the intersection of perpendiculars P_4 and P_6
12. Let J and K be the intersections of line GF and circle C_2	26. Trace point P and change its color
13. Trace points J and K and change their color	27. Animate point E around circle CD
14. Construct $P_2 \perp$ to GF through point G	

Obviously, points M and O serve as the pole points in this construction. Either point M or point O, or both, may be dragged to different positions and the animation then rerun. Each different position of the pole point(s) yields a member of the pedal curve family. Some very weird curves can be generated by playing around with this construction. Have fun!

5.6.6 The Conchoid as the Cissoid of a Line and Circle

In Chapter 1, we learned of the concept of a Cissoid as a means of deriving a new curve from two given curves. That is, given two curves C_1 and C_2, a fixed point O (called the pole point), and a line L that intersects the two curves in Q_1 and Q_2; if we now locate a point P on L such that $OP = Q_1Q_2$, then the locus of P for all lines L is called the Cissoid of C_1 and C_2 with respect to the point O. Well guess what? If the two curves are a circle and a straight line and if the pole point is the center of the circle, it turns out that the locus will be a Conchoid of Nicomedes. You don't believe it? Well, take a look below in Table 5-6 where just such a construction should make a believer out of you.

Table 5-6: The Conchoid of a Circle and a Straight Line

1. Draw circle AB with center at A and passing through point B	6. Let G be the unlabeled intersection of circle AB and line AE
2. Draw a random line CD anywhere in the plane	7. Let A' be the translation of point A by vector F → G
3. Let E be a random point on the circumference of circle AB	8. Trace point A′ and change its color
4. Draw line AE	9. Animate point E around circle AB
5. Let point F be the intersection of lines CD and AE	

It probably doesn't need to be said, but here goes: circle AB is curve C_1, line CD is curve C_2, and point A is the pole point.

5.6.7 One Tangent Line for Both Branches

In section 5.6.4 we geometrically constructed the tangent to the Conchoid of Nicomedes, one such tangent line for each branch of the curve. We will now see a construction that uses only a single tangent line but moves during the animation so as to be tangent to both branches, one after the other. Also note that the construction of the Conchoid of Nicomedes itself is different than what has been used previously. Refer to Table 5-7.

Table 5-7: One Tangent Line for Both Branches

1. Draw circle AB with center at A and passing through point B	11. Let E' be the image when E is translated by vector G → E
2. Let C be a random point on the circumference of circle AB	12. Construct the locus of point E' while C traverses circle AB
3. Draw line AC	13. Construct $P_2 \perp$ to line AC through point D
4. Draw line AB	14. Let point H be the intersection of line AB andP_2
5. Construct $P_1 \perp$ to line AB through point B	15. Construct line L_1 parallel to line AC through point H
6. Let point D be the intersection of P_1 and line AC	16. Construct $P_3 \perp$ to line L_1 through point E'
7. Draw line segment AD	17. Let point I be the intersection of perpendicular P_3 and line L_1
8. Let E be the midpoint of line segment AD	18. Draw line AI
9. Draw circle AF centered at A such that radius AF > radius AB	19. Construct $P_4 \perp$ to line AI through point E'
10. Let point G be either intersection of circle AF and line AC	20. Animate point C around circle AB

Perpendicular P_4 is, of course, the tangent. Note that if the other intersection of line AC with circle AF is used, point E is translated to the other branch. So, in either case, the locus of E' gives the same curve.

5.6.8 The Osculating Circle of the Conchoid of Nicomedes

In the construction of Table 5-8 below, the osculating circle of the Conchoid of Nicomedes is presented. This construction shares the osculating circle with its two branches just as the previous construction shared the tangent line. This construction is rather complex; however, geometric constructions of the center of curvature of most curves tend to be complex. Execute this construction and watch with wonder as the osculating circle grows to infinite radius and shrinks to fit inside the loop of the curve as the center of curvature traces the curve's evolute.

Table 5-8: The Osculating Circle of the Conchoid of Nicomedes

1. Draw circle AB centered at A and passing through point B	22. Construct $P_7 \perp$ to line AB through point H
2. Draw line AB	23. Let point J be the intersection of P_7 and line AC
3. Let C be a random point on the circumference of circle AB	24. Let J′ be the image when point J is translated by vector E $\rightarrow$ J
4. Draw line AC	25. Draw line segment AJ′
5. Draw circle AD centered at A with radius AD > AB	26. Let K be the midpoint of line segment AJ′
6. Construct $P_1 \perp$ to line AB through point B	27. Let K′ be the image when K is translated by vector G′ $\rightarrow$ K
7. Let point E be the intersection of P_1 and line AC	28. Construct $P_8 \perp$ to line AC through point K′
8. Let F be one of the intersections of circle AD and line AC	29. Let E′ be the image when E is translated by vector H $\rightarrow$ E
9. Draw line segment EF	30. Let E″ be the image when E′ is translated by vector E $\rightarrow$ E′
10. Let G be the midpoint of line segment EF	31. Construct $P_9 \perp$ to P_3 through point E″
11. Let G′ be the image when G is translated by vector A $\rightarrow$ G	32. Let point L be the intersection of P_8 and P_9
12. Construct the locus of point G′ as point C traverses circle AB	33. Construct $P_{10} \perp$ to line AI through point L
13. Construct $P_2 \perp$ to line AB through point A	34. Let point M be the intersection of P_{10} and line AI
14. Construct $P_3 \perp$ to line AC through point E	35. Draw line segment I′M
15. Construct $P_4 \perp$ to line AC through point G′	36. Construct $P_{11} \perp$ to line segment I′M through point I′
16. Let point H be the intersection of P_3 and line AB	37. Let point N be the intersection of line AI and P_{11}
17. Construct $P_5 \perp$ to P_3 through point H	38. Let G″ be the image when G′ is translated by vector N $\rightarrow$ A
18. Let point I be the intersection of P_4 and P_5	39. Draw circle G″G′ centered at G″ and passing through G′
19. Let I′ be the image when point I is rotated about A by 90°	40. Make circle G″G′ thick and a different color
20. Draw line AI	41. Animate point C around circle AB
21. Construct $P_6 \perp$ to line AI through point G′	

Figure 5-5: The Loop of the Conchoid of Nicomedes in Three Dimensions

To create this figure, the loop branch of the Conchoid of Nicomedes was truncated along its asymptote and then extruded into the third dimension. The resulting figure was then given a brown-agate finish and super-imposed over the arc of a rainbow. Light sources were placed so as to cast shadows on the inside of the loop.

Chapter 6 – The Cardioid

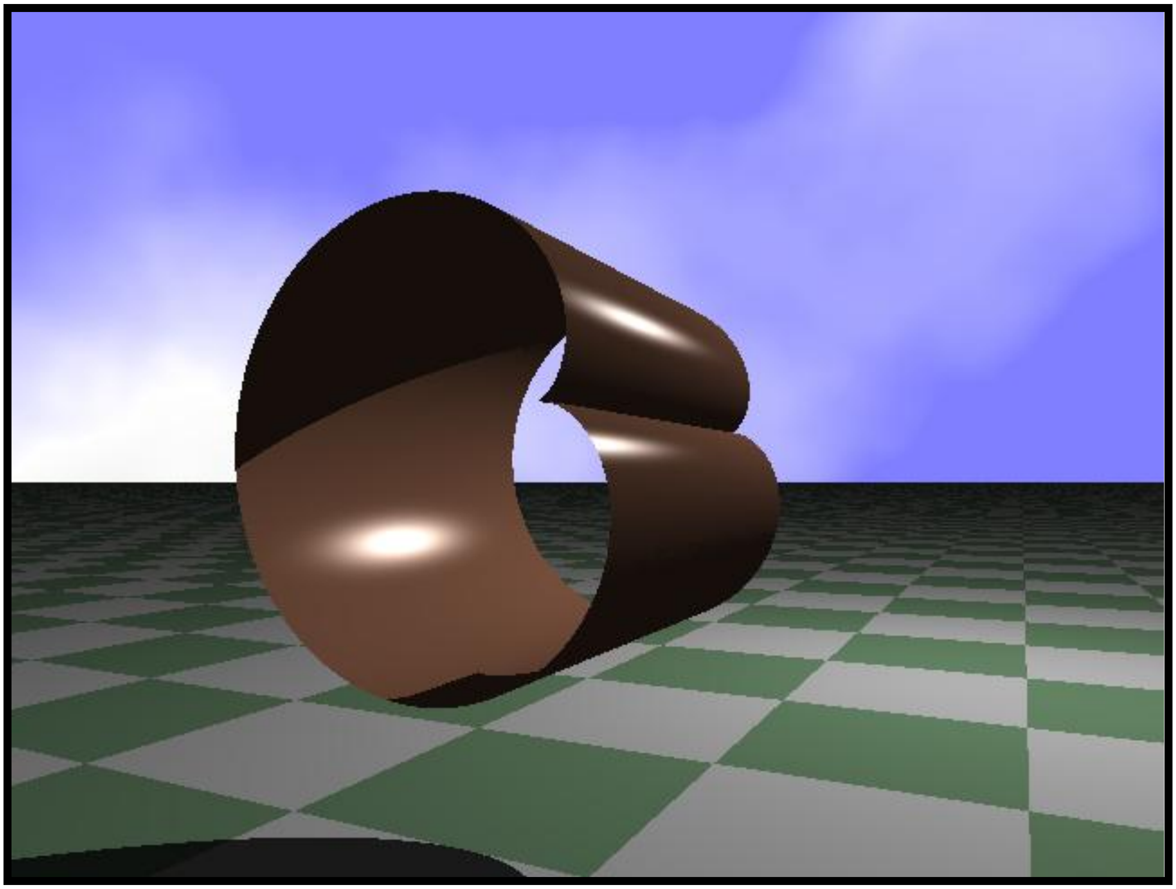

Figure 6-1: The Cardioid in Three Dimensions

The cross-section of this pseudo-cylinder is the curve known as the Cardioid. To create the object above, the Cardioid was extruded into the third dimension, given a lustrous tan finish, and placed above an infinite checkered plane which meets the wispy, clouded sky at the horizon. Light sources were placed so as to cast the object's shadow on the plane and to partially shade the inner surface of the object.

6.1 Introduction

The word cardioid is from the Greek root *cardi*, meaning heart; hence cardioid means heart-shaped. We learned in Chapter 1 that a Roulette is the curve resulting as the trace of a fixed point on a curve C_1 that rolls without slipping along another fixed curve, C_2. A special name is given to the Roulette when both C_1 and C_2 are circles and when the fixed point is on the circumference of the rolling circle; that name is Epicycloid and the Cardioid is a special instance of an Epicycloid. Before we define the Cardioid, let us more precisely define an Epicycloid. An Epicycloid is defined as the path of a point *P* fixed on the circumference of a circle of radius *b*, as it rolls at a uniform speed along the circumference and outside of a second circle of radius *a*. Let the fixed circle be centered at the origin of the *x*-*y* plane. Suppose the moving circle is rolling along the fixed one in such a way that its center has rotated about the origin to an angle *t* at time *t* (see Figure 6-2).

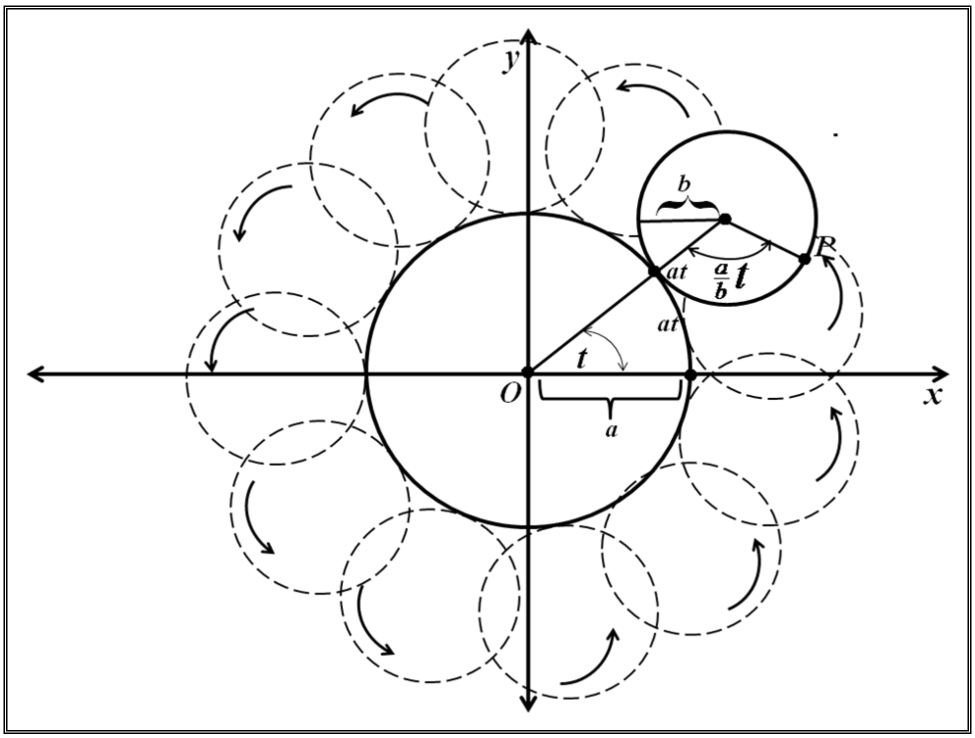

Figure 6-2: The Concept of an Epicycloid

We then find for the position at the time *t* of the point P = [*x* (*t*), *y* (*t*)], which at the time *t* = 0 is the point of contact (*a*, 0), the parametric equations

$$x = (a+b)\cos t - b\cos\left(\frac{a+b}{b}t\right)$$
$$y = (a+b)\sin t - b\sin\left(\frac{a+b}{b}t\right)$$

Equation 6-1

This then defines an Epicycloid. As alluded to above, a Cardioid is a special case of an Epicycloid; namely, when $a = b$ (i.e., when the radius of the fixed circle is the same as the radius of the rolling circle), the curve traced is called a Cardioid.

6.2 Equations and Graph of the Cardioid

Obtaining a parametric equation for the Cardioid is simply a matter of letting $b = a$ in Equation 6-1. Thus,

$$\begin{aligned} x &= 2a\cos t - a\cos 2t \\ y &= 2a\sin t - a\sin 2t \end{aligned} \qquad -\pi < t < \pi \qquad \textbf{Equation 6-2}$$

By eliminating the parameter t between the two components of Equation 6-2, one can derive the Cartesian equation, which is

$$\left(x^2 + y^2 - 2ax\right)^2 = 4a^2\left(x^2 + y^2\right) \qquad \textbf{Equation 6-3}$$

Similarly, one can derive the polar equation of the Cardioid by making the familiar substitutions of $x = r{\cdot}\cos\theta$, $y = r{\cdot}\sin\theta$, and $r^2 = x^2 + y^2$. That is,

$$r = 2a(1 + \cos\theta) \qquad \textbf{Equation 6-4}$$

With the origin taken at the center of the fixed circle, the pedal equation of the Cardioid is

$$9\left(r^2 - a^2\right) = 8p^2 \qquad \textbf{Equation 6-5}$$

Further, the Whewell equation is

$$s = 8a\cos\frac{\varphi}{3} \qquad \textbf{Equation 6-6}$$

And the Cesáro equation is

$$s^2 + 9\rho^2 = 64a^2 \qquad \textbf{Equation 6-7}$$

Finally, the equation of the tangent line to the Cardioid at the point $t = q$ is

$$y = \frac{(1-\cos q)(1+2\cos q)}{\sin q(2\cos q - 1)} \cdot x + \frac{3a(\cos q - 1)}{\sin q(2\cos q - 1)} \qquad \textbf{Equation 6-8}$$

The graph of the Cardioid is shown in Figure 6-3. Note that the vertex of the Cardioid is defined to be the point opposite the Cardioid's cusp, and the diameter of the Cardioid is the segment between the cusp and the vertex.

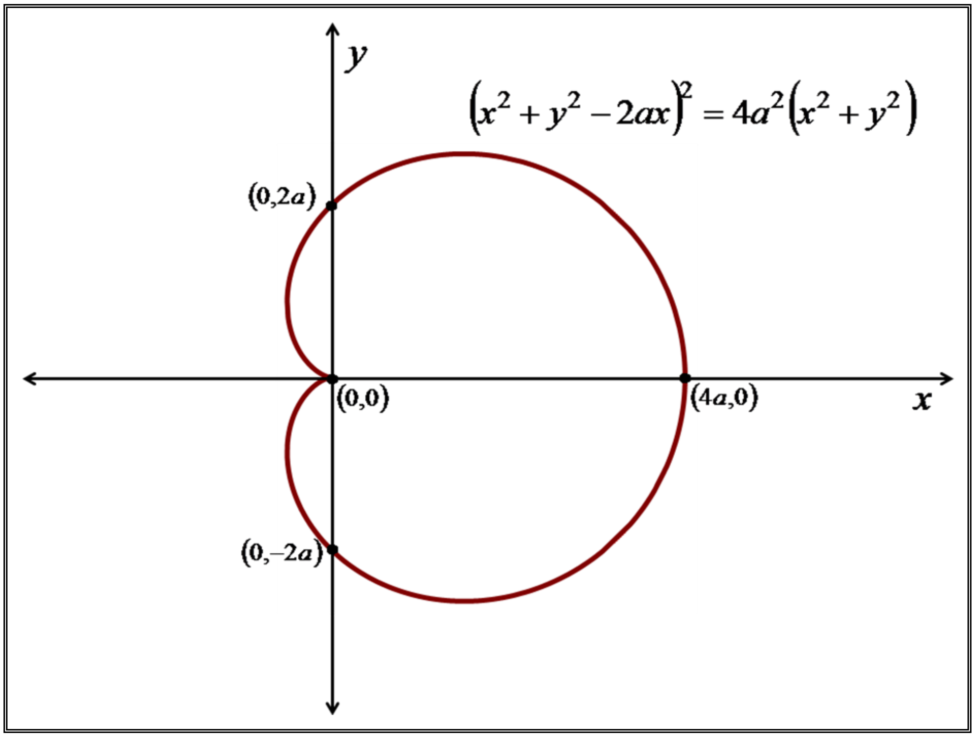

Figure 6-3: Graph of the Cardioid

6.3 Analytical and Physical Properties of the Cardioid

Using the parametric representation of the Cardioid given in Equation 6-2, i.e., $x = 2a\cos t - a\cos 2t$ and $y = 2a\sin t - a\sin 2t$, the following subparagraphs delineate further properties and characteristics of the Cardioid:

6.3.1 Derivatives of the Cardioid

- $\dot{x} = 2a\sin t(2\cos t - 1)$.
- $\ddot{x} = 2a\left(4\cos^2 t - \cos t - 2\right)$.
- $\dot{y} = 2a(1-\cos t)(1+2\cos t)$.
- $\ddot{y} = 2a\sin t(4\cos t - 1)$.
- $y' = \dfrac{(1-\cos t)(1+2\cos t)}{\sin t(2\cos t - 1)}$.
- $y'' = \dfrac{3(1-\cos t)}{2a\sin^3 t(2\cos t - 1)^3}$.

6.3.2 Metric Properties of the Cardioid

In Chapter 1, we addressed the arc length of a curve in polar coordinates as

$$s=\int_{\theta_1}^{\theta_2}\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}\,d\theta\,.$$

Therefore, ½ the length of the Cardioid would be its arc length between $\theta = 0$ and $\theta = \pi$. That is,

$$s=\int_0^{\pi}\sqrt{4a^2\sin^2\theta+4a^2(1+\cos\theta)^2}\,d\theta=2a\sqrt{2}\int_0^{\pi}\sqrt{1+\cos\theta}\,d\theta.$$

Now, using the identity

$$\sqrt{1+\cos\theta}=\sqrt{2}\cos\frac{\theta}{2}\,,$$

we have,

$$s=8a\int_0^{\pi}d\left(\sin\frac{\theta}{2}\right)=8a\left[\sin\frac{\theta}{2}\right]_0^{\pi}=8a.$$

Since the Cardioid is symmetric about the *x*-axis, the total length of the Cardioid will be twice this value or $s = 16a$.

Chapter 1 also expresses the area in polar coordinates by considering the area of a small circular sector of incremental angle $d\theta$. The area of this incremental sector is simply $dA = ½\, r^2\cdot d\theta$. Therefore, the area of the Cardioid above the *x*-axis is

$$A=\frac{1}{2}\int_0^{\pi}4a^2(1+\cos\theta)^2\,d\theta=2a^2\int_0^{\pi}d\theta+4a^2\int_0^{\pi}\cos\theta\cdot d\theta+2a^2\int_0^{\pi}\cos^2\theta\cdot d\theta$$

The contribution from the first integral is $2\pi a^2$, the contribution from the second integral is zero, and the contribution from the third integral is πa^2. Hence, the area above the *x*-axis is $3\pi a^2$ and the total area, by symmetry, is $6\pi a^2$.

The surface area of the solid of revolution that is formed when the Cardioid is rotated about the *x*-axis can be calculated using the parametric form addressed in Chapter 1. That is,

$$S=2\pi\int_{t_1}^{t_2}y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt\,,$$

where *x* and *y* are the components from the parametric representation in Equation 6-2 and where t_1 and t_2 are 0 and π, respectively. We know from section 6.3.1 that

$$\frac{dx}{dt}=2a\sin 2t-2a\sin t \qquad \text{and} \qquad \frac{dy}{dt}=2a\cos t-2a\cos 2t\,.$$

Therefore,

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=2a\sqrt{2}\cdot\sqrt{1-\cos t}\,,\ \text{and}$$

$$S = 2\pi\int_0^{\pi}(2a\sin t - a\sin 2t)\left(2a\sqrt{2}\cdot\sqrt{1-\cos t}\right)dt.$$

Breaking this up into two integrals, we have

$$S = 8\pi a^2\sqrt{2}\int_0^{\pi}\sin t\sqrt{1-\cos t}\,dt - 4\pi a^2\sqrt{2}\int_0^{\pi}\sin 2t\sqrt{1-\cos t}\,dt.$$

Both integrals can be evaluated by making the substitution $u = 1 - \cos t$. Under this substitution, $du = \sin t \cdot dt$ and when $t = 0$, $u = 0$ and when $t = \pi$, $u = 2$. Therefore, the two integrals are transformed to the following

$$S = 8\pi a^2\sqrt{2}\int_0^{2}u^{1/2}du - 4\pi a^2\sqrt{2}\int_0^{2}2(1-u)u^{1/2}du\ .$$

The first integral has the value $64\pi a^2/3$ and the second integral has the value $-64\pi a^2/15$. Taking the difference of these two quantities therefore yields S, the total surface area of the surface of revolution formed when the Cardioid is rotated about the x-axis:

$$S = \frac{64\pi a^2}{3} - \left(-\frac{64\pi a^2}{15}\right) = \frac{128\pi a^2}{5}\ .$$

If p is the distance from the origin to the tangent of the Cardioid, then

$$p = -\frac{3a\sqrt{2}}{2}(1-\cos t)^{1/2}\ .$$

If r is the radial distance from the origin to the Cardioid, then

$$r = a\sqrt{5-4\cos t}\ .$$

6.3.3 Curvature of the Cardioid

If ρ represents the radius of curvature of the Cardioid, then

$$\rho = \frac{4a\sqrt{2}}{3}(1-\cos t)^{1/2}\ .$$

If (α, β) are the coordinates of the center of curvature of the Cardioid, then

$$\alpha = \frac{a}{3}\left(2\cos^2 t + 2\cos t - 1\right) \quad \text{and} \quad \beta = \frac{2a}{3}\sin t(1+\cos t).$$

6.3.4 Angles for the Cardioid

If ψ is the angle between the tangent and the radius vector at the point of tangency to the Cardioid, then

$$\tan\psi = \frac{3(1-\cos t)}{\sin t}.$$

If θ denotes the radial angle of the Cardioid, then

$$\tan\theta = \frac{2\sin t(1-\cos t)}{2\cos t - \cos 2t}.$$

If ϕ denotes the tangential angle of the Cardioid, then

$$\tan\phi = \frac{(1-\cos t)(1+2\cos t)}{\sin t(2\cos t - 1)}.$$

6.4 Geometric Properties of the Cardioid

- Intercepts: (0, ± 2*a*); (4*a*, 0).
- *x*-maximum: (4*a*, 0).
- *x*-minima: $\left(-a, \pm\frac{1}{2}\sqrt{3}a\right)$.
- *y*-maximum: $\left(\frac{3a}{2}, \frac{3\sqrt{3}}{2}a\right)$.
- *y*-minimum: $\left(\frac{3a}{2}, -\frac{3\sqrt{3}}{2}a\right)$.
- Extent: $-a \le x \le 4a$ and $-\frac{3\sqrt{3}}{2}a \le y \le \frac{3\sqrt{3}}{2}a$.
- Symmetry: The Cardioid is symmetric about the *x*-axis.
- Cusp: (0, 0).
- Loop: The entire Cardioid is one loop.

6.5 Dynamic Geometry of the Cardioid

The Cardioid has many interesting properties and can be generated in a variety of different ways. The following subsections present a few of these properties as well as an assortment of the methods that can be used to generate the curve itself.

6.5.1 The Cardioid as an Epicycloid

As alluded to in section 6.1, an Epicycloid is defined as the path of a fixed point, P, on the circumference of a circle of radius *b*, as it rolls at a uniform speed without slipping around the circumference and outside of a second stationary circle of radius *a*. The Cardioid occurs when the two radii are equal, i.e., when $a = b$. The GSP construction delineated below in Table 6-1 is based on this relationship.

Table 6-1: The Cardioid as an Epicycloid

1. Draw horizontal line AB	9. Construct line L_1 parallel to line AB through point A'
2. Let C be a random point on line AB	10. Let F be the intersection of L_1 and translated circle
3. Draw circle AC centered at A and passing through point C	11. Let m_1 be the measure of ∠CAD
4. Let D be a random point on the circumference of circle AC	12. Let $m_2 = 2m_1$
5. Draw line AD	13. Let F' be the image when F is rotated about point A' by $\angle m_2$
6. Let point E be diametrically opposite to point D	14. Trace point F' and change its color
7. Translate circle AC by vector E → D	15. Animate point D around circle AC
8. Let A' be the image when A is translated by vector E → D	

In the above construction, the circle created at step 3 (circle AC) plays the role of the fixed circle. Steps 4 through 8 are then merely the methodology used to obtain another circle of equal radius tangent to the fixed circle. Let us call this circle A'D because it is centered at point A' and passes through point D. Ostensibly, circle A'D is supposed to be the rolling circle although this is somewhat of a misnomer. When point D is animated around the fixed circle (in step 15), it looks like circle A'D is rolling around the fixed circle, but in actuality, it is not – it is sliding around the fixed circle, not rolling. If you need proof of this fact, simply trace any random point on the circumference of circle A'D and you will obtain the trace of another circle. If circle A'D were truly rolling (without slipping), any random point on its circumference would trace a Cardioid. Steps 9 through 13 are then simply an artifice to construct a point that emulates that of a point on a rolling circle. Note that we have created point F' by rotating point F about point A' by twice angle CAD. Since GSP always maintains that relationship as points are moved, point F' moves around circle A'D as though it were truly on a rolling circle.

An interesting fact to note is that the diameter of the Cardioid generated in this manner is 4 times the radius of the fixed circle.

6.5.2 The Cardioid as an Epicycloid Revisited

The previous construction is based upon a fixed circle and a circle of equal radius rotating without slipping around the outside of the fixed circle. The construction shown below in Table 6-2 is similar except that the rotating circle is twice the radius of the fixed circle and it rotates in such a way that the fixed circle is inside of the rotating circle. That a Cardioid may also be generated this way was discovered by Daniel Bernoulli in 1725 and is known as the Double Generation theorem.

Table 6-2: The Cardioid as an Epicycloid Revisited

1. Draw horizontal line AB	6. Let A" be the image when A is translated by vector A' → C'
2. Draw circle AB with center at A and passing through point B	7. Draw circle A"C' centered at A" and passing through point C'
3. Let C be a random point on the circumference of circle AB	8. Trace point C' and change its color
4. Let A' be the image when A is rotated about point C by 180º	9. Animate point C around circle AB
5. Let C' be the image when C is rotated about A' by ∠BAC	

6.5.3 Orthogonal Tangents to the Cardioid

The following simple, but elegant, construction demonstrates an interesting property of Cardioids. That is, given a tangent to the Cardioid, one can always find another tangent that is perpendicular to the given tangent. After completing this construction, note that segments CC_2 and C_1C_4 are normals of the Cardioid. Refer to Table 6-3.

Table 6-3: Orthogonal Tangents to the Cardioid

1. Draw circle AB centered at A and passing through point B	9. Let C_3 be the image when C_2 is rotated about point A by 180º
2. Let C be a random point on the circumference of circle AB	10. Draw line segment CC_2
3. Let B' be the image when B is dilated about A by a factor of 3	11. Let C_4 be the image when C_3 is rotated about A" by 180º
4. Draw circle AB' with center at A and passing through point B'	12. Construct P_1 ⊥ to line segment CC_2 through point C_2
5. Let point A' be the image when A is rotated about C by 180º	13. Draw line segment C_1C_4
6. Let C_1 be the image when C is rotated about point A by 180º	14. Construct P_2 ⊥ to line segment C_1C_4 through point C_4
7. Let C_2 be the image when C is rotated about A' by ∠BAC	15. Construct the locus of point C_2 while C traverses circle AB
8. Let A" be the image when A' is rotated about point A by 180º	16. Animate point C around circle AB

There are at least two items of interest with this construction and Cardioid property. First, note that even though the main thrust here is to demonstrate this orthogonal tangent property, the construction contains a general method of constructing a tangent, something that was pointed out in previous chapters. Second, we have two tangents meeting at a constant angle as the animation is executed; in Chapter 1 we learned that the locus of the intersection point of two tangents with that property is called an isoptic and, further, when the constant angle is 90°, the locus is called an orthoptic. In this case, the locus is a circle, specifically circle AB′. Therefore the orthoptic produced by the Cardioid's orthogonal tangents is a circle.

6.5.4 The Cardioid as the Conchoid of a Circle

In Chapter 1, we discussed the concept of a generalized Conchoid. To review, let O be a fixed point (called the pole point) and let L be a line through O that intersects a curve C at a point Q. The locus of points P_1 and P_2 on L such that $P_1Q = P_2Q = a$, where a is a constant, is a conchoid of the curve C with respect to point O. Now, consider a circle of radius r. The Conchoid of this circle with respect to a fixed point on the circumference of the circle where the constant $a = 2r$ is a Cardioid. See Table 6-4 below for the GSP construction of this Cardioid.

Table 6-4: The Cardioid as the Conchoid of a Circle

1. Draw circle AB centered at A and passing through point B	6. Construct circle C_2 centered at C and radius = to segment BD
2. Draw line AB	7. Draw line BC
3. Let C be a random point on the circumference of circle AB	8. Let E and F be the two intersections of line BC and circle C_2
4. Let D be the point diametrically opposed to point B	9. Trace points E and F and change their color
5. Draw line segment BD	10. Animate point C around circle AB

This construction, as can be seen, also appears to be a rolling circle of twice the radius of the fixed circle, and rotating in such a way that the fixed circle is inside of the rolling circle. However, note that both points E and F need to be traced in order to generate the entire Cardioid. Tracing only one of the points generates only a fraction of the curve. This is due to the way GSP is designed. (Other dynamic geometry programs do not necessarily have this limitation.)

6.5.5 A Cardioid Sliding on Mutually Orthogonal Lines

This construction produces a Cardioid that slides on two mutually perpendicular lines. This very beautiful construction is delineated below in Table 6-5.

Table 6-5: A Cardioid on Mutually Orthogonal Lines

1. Draw horizontal line AB	14. Let H be the midpoint of line segment DF
2. Let C be a random point on line AB	15. Draw line segment GH
3. Draw circle AC centered at A and passing through point C	16. Let I be the midpoint of line segment GH
4. Let D be a random point on the circumference of circle AC	17. Draw circle ID centered at I and passing through point D
5. Construct $P_1 \perp$ to line AB through point A	18. Let point G_1 be the translation of point G by vector I $\rightarrow$ G
6. Construct $P_2 \perp$ to line AB through point D	19. Draw circle G_1G centered at G_1 and passing through point G
7. Construct $P_3 \perp$ to P_2 through point D	20. Let J be a random point on the circumference of circle ID
8. Let point E be the intersection of perpendiculars P_1 and P_3	21. Let G_2 be the image when G_1 is rotated about point I by $\angle$GIJ
9. Let F be the intersection of line AB and perpendicular P_2	22. Let E_1 be the image when E is rotated about point I by $\angle$GIJ
10. Draw line segment EF	23. Let E_2 be the image when E_1 is rotated about G_2 by $\angle$GIJ
11. Draw line segment ED	24. Construct the locus of E_2 while point J traverses circle ID
12. Let G be the midpoint of line segment ED	25. Animate point D around circle AC
13. Draw line segment DF	

Note the following about this fascinating construction. Lines AB and AE are the two mutually orthogonal lines upon which the Cardioid slides. While running the animation, observe that point E performs simple harmonic motion along line AE while, simultaneously, point F performs simple harmonic motion along line AB and both of these points are points on the Cardioid. Further, observe that the cusp of the Cardioid is confined to two of the constructs, namely, the circumference of circle ID and line segment EF.

If one constructs the two intersection points of line segment EF with circle ID and traces those two points, one obtains a curve called the Astroid, a curve addressed in Chapter 11. In this case, the Astroid is produced with an inscribed circle. However, both points must be traced to obtain the Astroid with inscribed circle; tracing only one of the points gives only half of the Astroid and half of the circle. If one traces point E_2, one obtains a curve called the Nephroid, also addressed in Chapter 7. If one traces point E_1, one obtains a curve called the Limacon of Pascal. Finally, if one traces point G_1 and/or circle G_1G, one obtains an ellipse. This construction is just full of goodies!

6.5.6 The Cardioid as the Caustic of a Circle

In Chapter 1, we briefly addressed the concept of a caustic. To review, the caustic of a given curve C is the envelope of light rays emitted from a point source after reflection or refraction at the curve C. When the envelope is due to reflection, the caustic is referred to as a catacaustic, and if the envelope is due to refraction, the caustic is referred to as a diacaustic. It turns out that the catacaustic of a circle when the light source is on the circumference of the circle is a Cardioid. The Cardioid produced in this manner is inside the circle and its vertex coincides with the light source. Further, the Cardioid has a diameter that is 2/3 of the circle's diameter. See Table 6-6 below for this construction.

Table 6-6: The Cardioid as the Caustic of a Circle

1. Draw circle AB centered at A and passing through point B	5. Reflect line segment BC across line segment AC
2. Let C be a random point on the circumference of circle AB	6. Let the reflected line segment intersect circle AB in point D
3. Draw line segment AC	7. Trace line segment CD and change its color
4. Draw line segment BC	8. Animate point C around circle AB

As can be seen from Figure 6-4, which is a snapshot of the executing animation, this construction makes for quite a spectacular looking trace. For best effects, have the animation do only one revolution and do it at the highest animation speed, i.e., quickly.

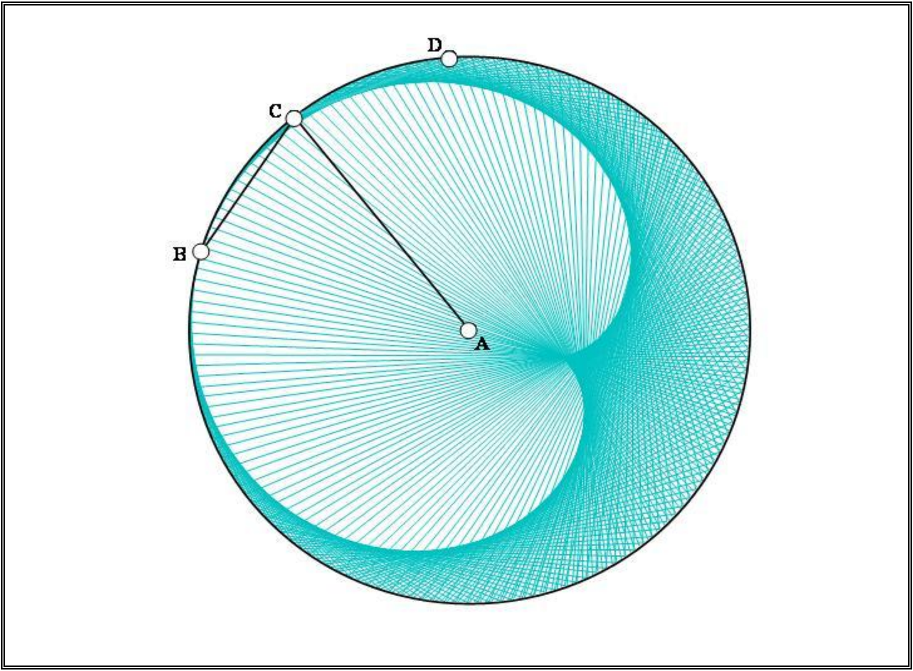

Figure 6-4: The Cardioid as the Caustic of a Circle

6.5.7 The Cardioid as the Pedal Curve of a Circle

It just so happens that the pedal curve of a circle when the pedal point is on the circumference of the circle is a Cardioid, as can be seen from the construction below in Table 6-7. In this case, the pedal point (point B) forms the cusp of the Cardioid and the diameter of the circle is the same as the diameter of the Cardioid.

Table 6-7: The Cardioid as the Pedal of a Circle

1. Draw circle AB with center at A and passing through point B	5. Construct $P_2 \perp$ to P_1 through point B
2. Let C be a random point on the circumference of circle AB	6. Let point D be the intersection of perpendiculars P_1 and P_2
3. Draw line segment AC	7. Trace point D and change its color
4. Construct $P_1 \perp$ to line segment AC through point C	8. Animate point C around circle AB

6.5.8 The Cardioid and Simple Harmonic Motion

Who would have thought that a rotating Cardioid could produce simple harmonic motion? But it is a remarkable fact that it can, as can be seen from the construction delineated below in Table 6-8.

Table 6-8: The Cardioid and Simple Harmonic Motion

1. Draw circle AB with center at A and passing through point B	11. Let C_3 be the image when C is rotated about C_1 by $\angle BAC$
2. Let A_1 be the image when A is dilated about B by a factor of 2	12. Construct $P_1 \perp$ to line segment BA_2 through point C_2
3. Dilate circle AB about point B be a factor of 2	13. Let D_2 be the image when D is rotated about D_1 by $\angle BAD$
4. Let A_2 be the image when A_1 is dilated about B by the factor 2	14. Draw line segment C_2A_1
5. Let C and D be two random points on circle AB	15. Let E be the intersection of line segment BA_2 and P_1
6. Draw line segment BA_2	16. Let D_3 be the image when D_2 is rotated about B by $\angle C_3BE$
7. Let C_1 be the image when C is dilated about A by a factor of 2	17. Draw line segment EC_3
8. Let C_2 be the image when C is dilated about B by a factor of 2	18. Draw line segment EA_2
9. Let D_1 be the image when D is dilated about A by a factor of 2	19. Construct the locus of point D_3 as point D traverses circle AB
10. Let B_1 be the image when B is rotated about A_1 by $\angle BA_1C$	20. Animate point C around circle AB

When running this animation, hide the following elements for aesthetic purposes and visual clarity: points A, C, D, C_1, D_1, B_1, C_3, D_2, D_3, circle AB, segment BA_2, segment EA_2, and the perpendicular P_1. Additionally, trace point E. Then it is very easy to see that as the Cardioid rotates about its cusp, point E which is a point on the Cardioid oscillates between points B and A_2 tracing a straight line – simple harmonic motion. In other words, if the Cardioid is pivoted at its cusp and rotated with a constant angular velocity, a pin constrained to a fixed straight line and bearing on the Cardioid will move with simple harmonic motion. Recall, from Equation 6-4, that the polar equation of the Cardioid is,

$$r = 2a(1+\cos\theta).$$

Thus,

$$\frac{dr}{dt} = -2a\sin\theta \cdot \frac{d\theta}{dt} \quad \text{and}$$

$$\frac{d^2r}{dt^2} = -2a\cos\theta \cdot \left(\frac{d\theta}{dt}\right)^2 - 2a\sin\theta \cdot \frac{d^2\theta}{dt^2}.$$

Now, if $d\theta / dt = k$, a constant, we have

$$d^2r \big/ dt^2 = -k^2(2a\cos\theta) = -k^2(r-2a).$$

Slightly rewriting this, we have the differential equation governing the motion of any point of the pin as,

$$\frac{d^2(r-a)}{dt^2} = -k^2(r-a). \quad \textbf{Equation 6-9}$$

6.5.9 The Cardioid as an Envelope of Circles

Similar to the pedal, an envelope can be thought of as a way of deriving a new curve based on a set of curves. The envelope of a set of curves is a new curve C such that C is tangent to every member of the set. It so happens that the Cardioid is the envelope of a specific set of circles as can be seen in the simple, but elegant, construction found in Table 6-9.

Table 6-9: The Cardioid as an Envelope of Circles

1. Draw circle AB with center at A and passing through point B	4. Trace circle CB and change its color
2. Let C be a random point on the circumference of circle AB	5. Animate point C around circle AB
3. Draw circle CB with center at C and passing through point B	

The Cardioid generated in this manner has a diameter of twice that of circle AB, as can be seen from the dotted circle shown in Figure 6-5 (the dotted circle is not part of the construction but has been added to make the diameter relationship clear).

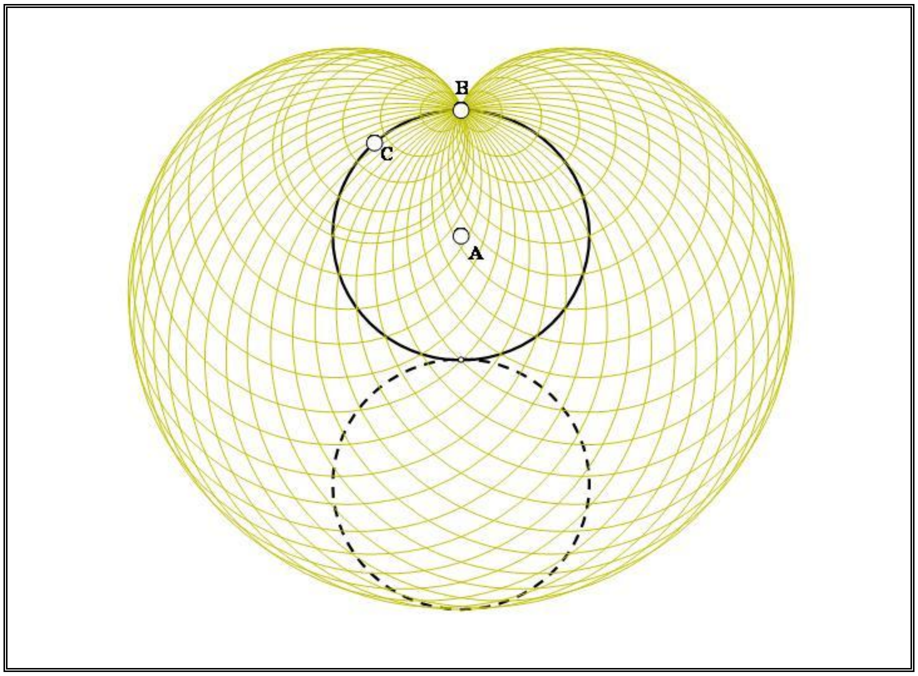

Figure 6-5: The Cardioid as an Envelope of Circles

6.5.10 The Cuspidal Chords of the Cardioid

A chord of the Cardioid is merely any line segment whose endpoints lie on the circumference of the Cardioid. A cuspidal chord is a chord that passes through the cusp. It is interesting to note that all cuspidal chords of a given Cardioid are equal and therefore equal the Cardioid's diameter (since the diameter is a cuspidal chord). This fact can be demonstrated with the construction found in Table 6-10.

As can be seen from this construction, the cuspidal chord, which is segment C_2C_4, is constant in length and equals 4 times the radius of the initial circle. It is also interesting to note that the midpoint of the cuspidal chord, point D, always lies on the initial circle (this can be verified by tracing point D). Finally, if one traces the cuspidal chord itself, it will color in the Cardioid making quite a striking picture.

Table 6-10: The Cuspidal Chords of the Cardioid

1. Draw circle AB with center at A and passing through point B	10. Construct the locus of point C_2 as point C traverses circle AB
2. Let C be a random point on the circumference of circle AB	11. Let C_3 be the image when C_2 is rotated about point A by 180º
3. Let circle O_2 be the image as circle AB is dilated about A by 3	12. Let C_4 be the image when C_3 is rotated about A_2 by 180º
4. Let A_1 be the image when A is rotated about point C by 180º	13. Draw line segment C_2C_4 and change its color
5. Let O_3 be the image as circle AB is rotated about C by 180º	14. Measure the distance from point C_2 to point C_4
6. Let C_1 be the image when C is rotated about point A by 180º	15. Let point D be the midpoint of line segment C_2C_4
7. Let C_2 be the image when C is rotated about A_1 by $\angle$BAC	16. Calculate 4 times the radius of circle AB (, i.e., $4 \cdot AD$)
8. Let O_4 be the image when circle O_3 is rotated about A by 180º	17. Animate point C around circle AB
9. Let A_2 be the image when A_1 is rotated about point A by 180º	

6.5.11 The Osculating Circle of the Cardioid

In previous chapters, we have at times, given constructions for the osculating circle of the curve under consideration. Remember that the center of the osculating circle is the center of curvature for the curve and as such, its trace draws the curve's evolute. This chapter is no exception; however, in this case, the evolute of the Cardioid is another Cardioid. Hence, our point here is not so much to demonstrate a construction for the Cardioid's osculating circle but to show that the Cardioid's evolute is, indeed, another Cardioid. Table 6-11 contains this construction.

Table 6-11: The Osculating Circle of the Cardioid

1. Draw horizontal line AB	17. Draw line A'F and line EG
2. Draw circle AB centered at A and passing through point B	18. Let point H be the intersection of lines A'F and EG
3. Draw circle BA centered at B and passing through point A	19. Construct the locus of point H as point F traverses circle DA
4. Let C be the point of circle BA diametrically opposite point A	20. Let point I be the intersection of circle DE and line DF
5. Draw circle CB centered at C and passing through point B	21. Draw line HI
6. Let D be the point of circle CB diametrically opposite point B	22. Let point J be the intersection of lines A'D and HI
7. Draw circle DC centered at D and passing through point C	23. Let E' be the image when E is dilated about point D by ⅓
8. Let E be the point of circle DC diametrically opposite point C	24. Draw circle DE' centered at D and passing through point E'
9. Hide all circles drawn (unnecessary, but it avoids clutter)	25. Let point K be the intersection of circle DE' and line AB
10. Draw circle DA centered at D and passing through point A	26. Draw line segment DJ
11. Draw circle DE centered at D but passing through point E	27. Let L be the intersection of circle DE' and line segment DJ
12. Let F be a random point on the circumference of circle DA	28. Draw line LK
13. Draw line DF	29. Let point M be the intersection of lines HI and LK
14. Let A' be the image when point A is reflected across line DF	30. Trace point M and change its color
15. Draw line A'D	31. Draw circle MH centered at M and passing through point H
16. Let point G be the intersection of circle DE and line A'D	32. Animate point F around circle DA

If all of this complex construction is done correctly, the locus of point M while point F revolves around circle C_1 is the evolute of the locus produced by point H and is another Cardioid ⅓ the size of the original. And, of course, circle MH is the osculating circle of the large Cardioid.

Steps 1 through 9 are simply a method of obtaining three collinear points such that the distance $AD = 3 \cdot DE$. That is why one can hide the circles (there is no longer any need for them). A very bizarre fact is that this construction contains no perpendiculars!

6.5.12 The Cardioid as the Inverse of a Parabola

The concept of inversion of a curve forms the basis for the construction shown below in Table 6-12. The inversion of a parabola is a Cardioid when the center of the inversion circle is taken as the focus of the parabola.

Table 6-12: The Cardioid as the Inversion of a Parabola

1. Construct horizontal line AB	14. Let point J be the intersection of perpendiculars P_2 and P_3
2. Let C be a random point on line AB	15. Construct the locus of point J as point G traverses circle EF
3. Construct $P_1 \perp$ to line AB through point C	16. Draw line DJ
4. Let D be a random point on perpendicular P_1	17. Measure distance DJ
5. Draw line segment CD and then hide perpendicular P_1	18. Draw circle DK with center at D where DK is any radius
6. Draw circle EF centered at point E passing through point F	19. Measure distance DK
7. Let G be a random point on the circumference of circle EF	20. Calculate $(DK)^2$ / DJ
8. Draw line EG	21. Let D' be the image when D is translated by $(DK)^2$ / DJ
9. Let point H be the intersection of line AB and line EG	22. Draw circle DD' with center at D and passing through D'
10. Draw line segment DH	23. Let point L be the intersection of circle DD' and line DJ
11. Let I be the midpoint of line segment DH	24. Trace point L and change its color
12. Construct $P_2 \perp$ to line segment DH through point I	25. Animate point G around circle EF
13. Construct $P_3 \perp$ to line AB through point H	

In this construction, the focus of the parabola and the cusp of the Cardioid coincide (point D in the construction). Circle DK is the inversion circle and distance DD' multiplied by distance DJ will always equal the square of the radius DK. The locus of point L is, of course, the Cardioid. Note that angles are preserved under inversion but with a reversed sense. That is, if two curves intersect with angle α, their inverses will also intersect with angle α but in a counter-direction of sweeping.

6.5.13 The Cardioid by Relative Velocity

This very unusual construction of the Cardioid is based on the idea of the two end points of a line segment traveling around a circle, but one traveling twice as fast as the other. The envelope of the line segment then forms a Cardioid. Refer to Table 6-13 for this construction.

Table 6-13: The Cardioid by Relative Velocity

1. Draw circle AB with center at A and passing through point B	7. Draw ray AE starting at point A and passing through point E
2. Draw line segment AB	8. Let point F be the intersection of ray AE and circle AB
3. Let C be the midpoint of line segment AB	9. Draw line segment FD
4. Draw circle AC with center at A and passing through point C	10. Trace line segment FD and change its color
5. Let D be a random point on the circumference of circle AB	11. Animate point D once around circle AB while
6. Let E be a random point on the circumference of circle AC	simultaneously animating point E around circle AC

Note that this entire construction is simply designed to result in two points on the larger circle with the characteristics that when animated to travel around that circle, one will travel twice as fast as the other. Specifically, point F will travel twice as fast as point D. It is also instructional to trace the midpoint of line segment FD. For best results, both animations should be run as fast as possible.

6.5.14 Three Parallel Tangents to the Cardioid

At any arbitrary point on the circumference of a Cardioid, construct the tangent to the Cardioid. Then, no matter what point was chosen, there are two more tangents to the Cardioid that are parallel to the given tangent. This property can be demonstrated using a dynamic geometry program such as GSP, and the methodology for doing so is delineated below in Table 6-14.

Table 6-14: Three Parallel Tangents to the Cardioid

1. Draw circle AB with center at A and passing through point B	15. Change the color of perpendicular P_1 (say green)
2. Let C be a random point on the circumference of circle AB	16. Draw line segment C_1C_5
3. Let C_1 be the image when C is rotated about point A by 120º	17. Construct $P_2 \perp$ to line segment C_1C_5 through point C_5
4. Let A_1 be the image when A is rotated about point C by 180º	18. Change the color of perpendicular P_2 (say green)
5. Let C_2 be the image when C is rotated about A_1 by $\angle BAC$	19. Draw line segment C_3C_7
6. Let A_2 be the image when A_1 is rotated about point A by 120º	20. Construct $P_3 \perp$ to line segment C_3C_7 through point C_7
7. Let C_3 be the image when C_1 is rotated about point A by 120º	21. Change the color of perpendicular P_3 (say green)
8. Let C_4 be the image when C_2 is rotated about point A by 120º	22. Construct the interior of triangle (polygon) $C_2C_5C_7$
9. Let A_3 be the image when A_2 is rotated about point A by 120º	23. Change color of the polygon interior $C_2C_5C_7$ (say light green)
10. Let C_5 be the image when C_4 is rotated about A_2 by 120º	24. Measure the area of polygon $C_2C_5C_7$
11. Let C_6 be the image when C_5 is rotated about point A by 120º	25. Construct the locus of point C_2 as C traverses the circle AB
12. Let C_7 be the image when C_6 is rotated about A_3 by 120º	26. Change the color of the locus (say yellow)
13. Draw line segment CC_2	27. Animate point C around circle AB
14. Construct $P_1 \perp$ to line segment CC_2 through point C_2	

There are at least three things worthy (and interesting) to note about the Cardioid that can easily be investigated with this GSP construction. First, as point C revolves about circle AB, the area of the triangle (that is, triangle $C_2C_5C_7$) is constant; however, the lengths of the sides of the triangle change. Second, the perpendiculars that were constructed in steps 14, 17, and 20 (e.g., P_1, P_2, and P_3) are all tangent to the Cardioid at points C_2, C_5, and C_7 respectively, and they remain tangent as point C revolves. Further, they are always parallel to one another (this is, of course, what we set out to demonstrate). Finally, if the line segments from the points of tangency to the cusp of the Cardioid are all drawn, the three angles so formed, that is $\angle C_2BC_5$, $\angle C_7BC_2$, and $\angle C_5BC_7$, are always equal to 120°. This last property is depicted in Figure 6-6, but is not included as part of the construction above.

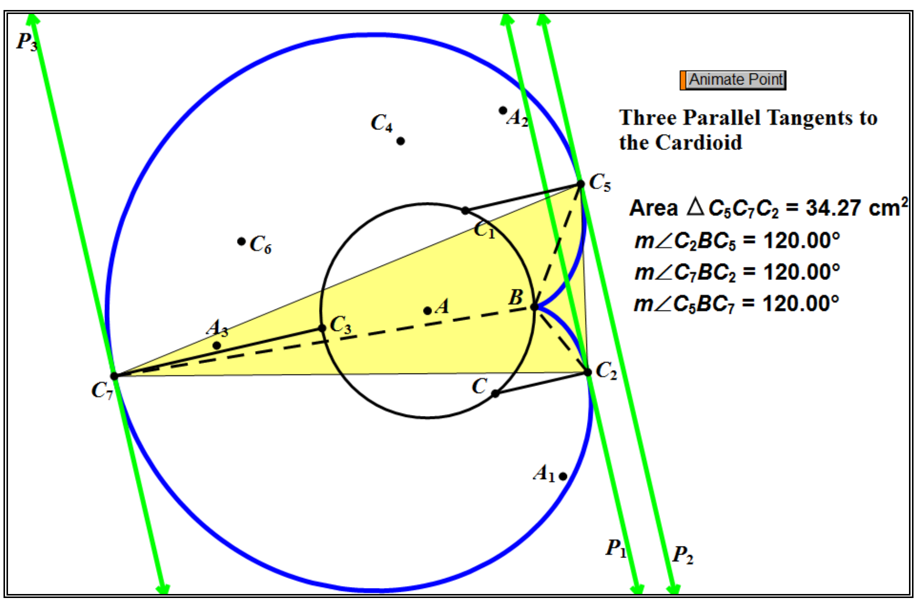

Figure 6-6: Three Parallel Tangents to the Cardioid

6.5.15 A Cardioid Derived from a Compass-Only Construction

GSP has (at least in the version used by your author) a minor flaw in its design. It does not always handle the intersection of two circles correctly when the center of one of the circles is on the circumference of the other circle. If you are using GSP, you can prove this to yourself by performing a little experiment. Draw circle AB, place a point C on its circumference, and then draw circle CB. Let the unlabeled intersection of the two circles be point D. Now drag point C one full revolution around circle AB. If you drag point C counterclockwise, you will find that when point D coincides with point B, point D stops even though point C is still being dragged. Point D will remain coincident with point B until point C is dragged past point B. Then, and only then, will point D continue around circle AB. This is not correct; Point D should not pause in its motion. (If you drag clockwise, point D pauses when it is diametrically opposite point B; this is not correct either.) Because of this flaw, GSP does not handle a compass-only construction correctly. However, there is a way around this dilemma, as the construction in Table 6-15 illustrates.

Table 6-15: A Cardioid Derived from a Compass-Only Construction

1. Draw circle AB with center at A and passing through point B	7. Let D and E be the intersections of circle CB and circle B'C
2. Let C be a random point on the circumference of circle AB	8. Draw circle DE with center at D and passing through point E
3. Draw circle CB with center at C and passing through point B	9. Let point F be the unlabeled intersection of circles DE and CB
4. Draw line segment AC	10. Trace point F and change its color
5. Let B' be the image as B is reflected across line segment AC	11. Animate point C around circle AB
6. Draw circle B'C with center at B' and passing through point C	

Note that except for steps 4 and 5, no straight edge is required for this construction. By making point B' the reflected image of point B across segment AC instead of the intersection of circle AB and circle CB, we eliminate the GSP flaw, but our construction is not totally compass-only (unfortunately). In this regard, any constructions found in the rest of the text that are labeled compass-only will require this type of work-around solution.

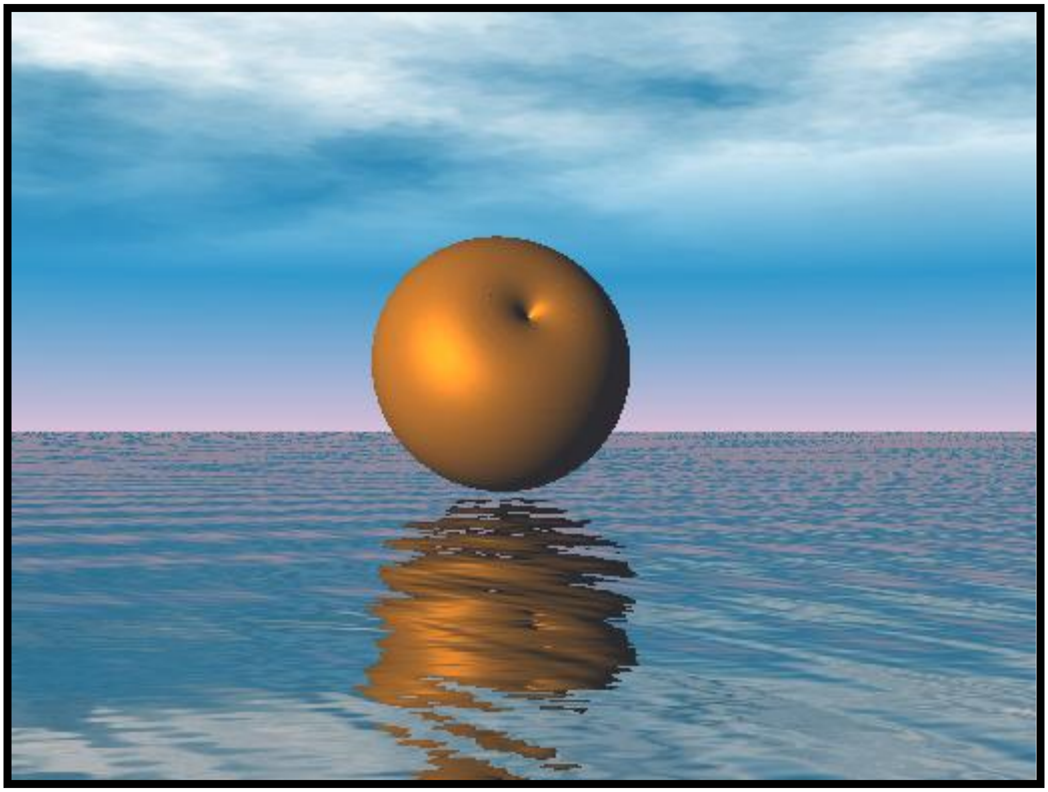

Figure 6-7: A Solid of Revolution Formed from the Cardioid

The Cardioid was rotated about the x-axis thereby forming the solid of revolution seen in the figure above. As can be seen, the cusp of the Cardioid forms an indentation in the solid of revolution. It has been placed over an infinite plane; however, in this case, the infinite plane has been made to resemble water. The object has been given a golden-bronze colored finish which can be seen reflected in the water that it appears to be floating above. The light source has been placed so as to illuminate the left side of the solid and also partially illuminates the indentation formed by the Cardioid's cusp.

Chapter 7 – The Nephroid

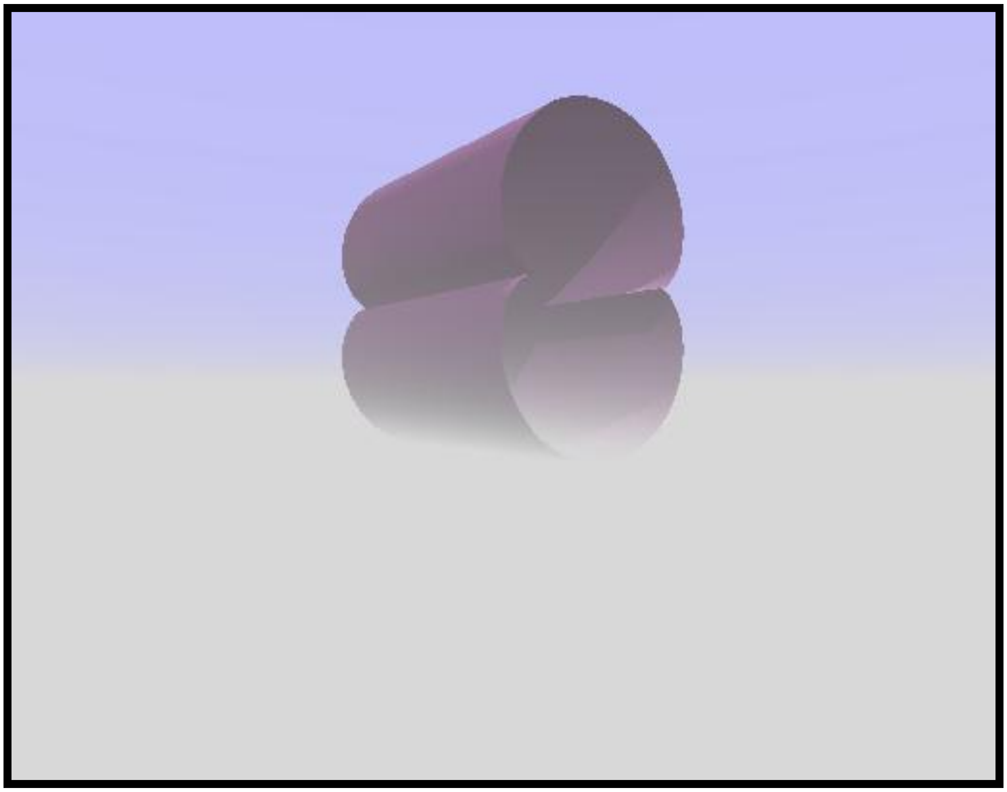

Figure 7-1: The Nephroid in Three Dimensions

The cross-section of the object in the figure above is the curve known as the Nephroid. To create the object, the Nephroid was simply extruded into the third dimension, given an orchid-colored finish and placed over and slightly into a cloud or fogbank. Light sources are positioned so as to cast a shadow on the inner surface of the extruded object.

7.1 Introduction

Chapter 6 introduced the concept of an Epicycloid as the trace of a fixed point on the circumference of a circle rolling around the outside of the circumference of a second, stationary circle. It further stated that when the radius of the rolling circle is equal to the radius of the stationary circle, the curve traced by the fixed point is called a Cardioid. It turns out that if the radius of the rolling circle is ½ of the radius of the stationary circle, the curve so traced is called a Nephroid. The Nephroid was studied extensively by both Christian Huygens and Ehrenfried Tschirnhaus circa 1679 in connection with the theory of caustics. Nephroid means kidney shaped.

7.2 Equations and Graph of the Nephroid

By letting $b = \frac{1}{2}a$ in the parametric equation for the Epicycloid (Equation 6-1), we obtain a parametric representation for the Nephroid, that is,

$$\begin{aligned} x &= \frac{3a}{2}\cos t - \frac{a}{2}\cos 3t \\ y &= \frac{3a}{2}\sin t - \frac{a}{2}\sin 3t \end{aligned} \qquad -\pi < t < \pi \quad \textbf{Equation 7-1}$$

By eliminating the parameter t between the two components of Equation 7-1, one can derive the Cartesian equation, which is

$$4\left(x^2 + y^2 - a^2\right)^3 = 27a^4y^2 \quad \textbf{Equation 7-2}$$

Similarly, one can derive the polar equation of the Nephroid by making the familiar substitutions of $x = r\cdot\cos\theta$, $y = r\cdot\sin\theta$, and $r^2 = x^2 + y^2$. That is, under these substitutions, Equation 7-2 becomes

$$\left(\frac{r}{a}\right)^{2/3} = \left(\sin\frac{\theta}{2}\right)^{2/3} + \left(\cos\frac{\theta}{2}\right)^{2/3} \quad \textbf{Equation 7-3}$$

Likewise, the pedal, Whewell, and Cesáro equations for the Nephroid are, respectively

$$4r^2 - 3p^2 = 4a^2 \quad \textbf{Equation 7-4}$$

$$s = 3a\sin\frac{\varphi}{2} \quad \textbf{Equation 7-5}$$

$$4\rho^2 + s^2 = 9a^2 \quad \textbf{Equation 7-6}$$

Finally, the equation of the Nephroid's tangent line at the point $t = q$ is

$$\cos 2q \cdot y = \sin 2q \cdot x - 2a\sin q. \quad \textbf{Equation 7-7}$$

The graph of the Nephroid is shown in Figure 7-2.

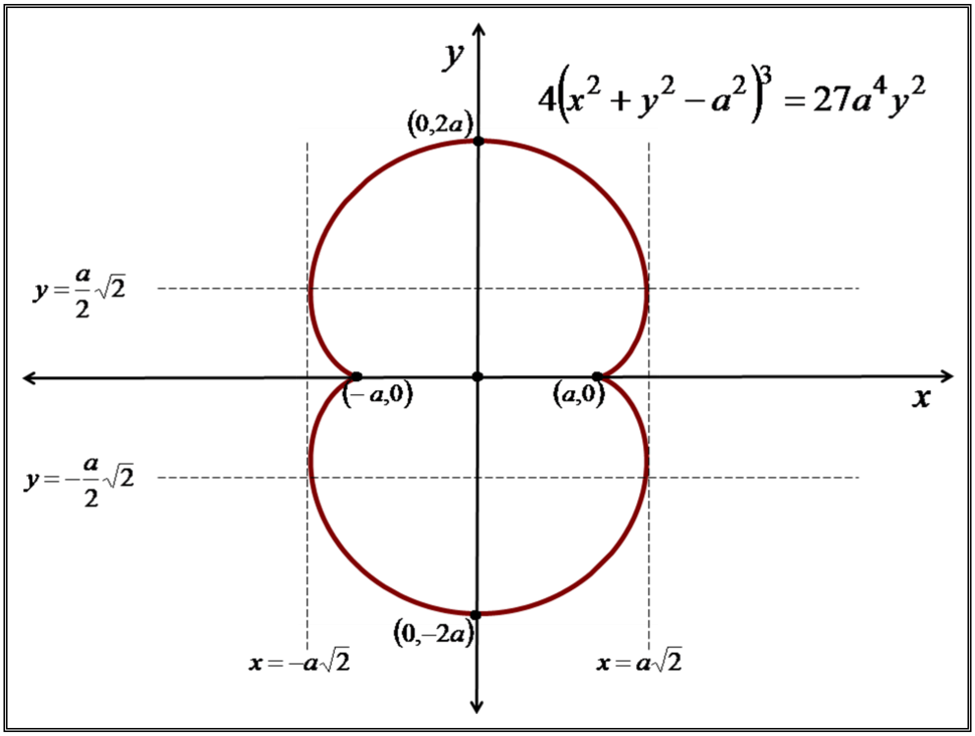

Figure 7-2: Graph of the Nephroid

7.3 Analytical and Physical Properties of the Nephroid

Using the parametric representation of the Nephroid given in Equation 7-1, i.e., $x=\frac{3a}{2}\cos t-\frac{a}{2}\cos 3t,\ y=\frac{3a}{2}\sin t-\frac{a}{2}\sin 3t$, the following subparagraphs delineate further properties of the Nephroid.

7.3.1 Derivatives of the Nephroid

- $\dot{x}=3a\sin t\cos 2t$.
- $\ddot{x}=3a\cos t\cdot\left(6\cos^2 t-5\right)$.
- $\dot{y}=6a\cos t\sin^2 t$.
- $\ddot{y}=6a\sin t\cdot\left(2-3\sin^2 t\right)$.
- $y'=\tan 2t$.
- $y''=\dfrac{2}{3a\sin t\cos^3 2t}$.

7.3.2 Metric Properties of the Nephroid

The length of the Nephroid can be calculated using the formula

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\,.$$

Since the curve is symmetric about the *x*-axis, we may integrate from $t_1 = 0$ to $t_2 = \pi$ and then simply double the result. Hence, we have from Equation 7-1,

$$dx = \frac{3a}{2}(\sin 3t - \sin t)dt \qquad \text{and} \qquad dy = \frac{3a}{2}(\cos t - \cos 3t)dt\,.$$

Therefore,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{9a^2}{2}(1 - \cos t \cos 3t - \sin t \sin 3t) = \frac{9a^2}{2}(1 - \cos 2t)$$

Now, putting this result under the radical sign, we have

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\frac{9a^2}{2}(1 - \cos 2t)} = \frac{3a}{\sqrt{2}}\sqrt{1 - \cos 2t} = 3a \sin t.$$

Hence the desired integral is simply

$$3a\int_0^{\pi} \sin t\, dt = 3a[-\cos t]_0^{\pi} = 6a.$$

The total length of the Nephroid is therefore twice this result or s = 12*a*.

The area of the Nephroid can be calculated using the formula

$$A = \int_{t_0}^{t_1} y \cdot \frac{dx}{dt} \cdot dt,$$

where the limits of integration are from $t_0 = -\pi$ to $t_1 = +\pi$. We therefore have, after performing the indicated operations, the following integrals to evaluate.

$$3a^2\int_{-\pi}^{\pi} \sin t \sin 3t\, dt - \frac{3a^2}{4}\int_{-\pi}^{\pi} \sin^2 3t\, dt - \frac{9a^2}{4}\int_{-\pi}^{\pi} \sin^2 t\, dt\,.$$

In the first integral, write sin3*t* as sin (*t* + 2*t*), expand, and multiply accordingly. After much manipulation the first integral can be written as

$$3a^2\int_{-\pi}^{\pi}\left(3\sin^2 t-4\sin^4 t\right)dt\,.$$

Now, by using the identity $\sin^2 t$ = ½ - ½·cos2t, we can show that this integral evaluates to zero. Using this same identity on the second and third integrals, it can be shown that their values are $-3\pi a^2/4$ and $-9\pi a^2/4$, respectively. We therefore have for the area of the Nephroid, $A = 0 - (-3\pi a^2/4) - (-9\pi a^2/4) = 3\pi a^2$.

The area of the surface of revolution that results when the Nephroid is revolved about the *x*-axis can be calculated by the formula

$$S=2\pi\int_{t_0}^{t_1} y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt\,,$$

where the limits of integration are from $t_0 = -\pi$ to $t_1 = +\pi$. However, we already know that the expression under the radical (from the Nephroid length calculation) is $3a\sin t$. Therefore, we have

$$S=9\pi\cdot a^2\int_{-\pi}^{\pi}\sin^2 t\,dt-3\pi\cdot a^2\int_{-\pi}^{\pi}\sin t\sin 3t\,dt\,.$$

From the Nephroid area calculation we know that this second integral evaluates to zero, so the required area of revolution is simply the value of the first integral, which is upon reduction using the identity $\sin^2 t$ = ½ - ½·cos2t,

$$S=9\pi\cdot a^2\int_{-\pi}^{\pi}\left(\frac{1}{2}-\frac{1}{2}\cos 2t\right)dt=9\pi^2a^2\,.$$

If p is the distance from the origin to the tangent of the Nephroid, then

$$p=-2a\sin t\,.$$

If r is the radial distance from the origin to the curve, then

$$r=a\sqrt{1+3\sin^2 t}\,.$$

7.3.3 Curvature of the Nephroid

If ρ is the radius of curvature of the Nephroid, then

$$\rho=\frac{3a}{2}\sin t\,.$$

If (α, β) are the coordinates of the center of curvature of the Nephroid, then

$$\alpha = \frac{a}{4}(3\cos t + \cos 3t) \quad \text{and} \quad \beta = \frac{a}{4}(3\sin t + \sin 3t).$$

7.3.4 Angles for the Nephroid

If ψ is the angle between the tangent and the radius vector at the point of tangency to the Nephroid, then

$$\tan\psi = 2\tan t\,.$$

If θ denotes the radial angle for the Nephroid, then,

$$\tan\theta = \frac{3\sin t - \sin 3t}{3\cos t - \cos 3t}.$$

If ϕ denotes the tangential angle for the Nephroid, then

$$\phi = 2t.$$

7.4 Geometric Properties of the Nephroid

- Intercepts: (a, 0); (-a, 0); (0, ±2a)
- y-maximum: (0, 2a)
- y-minimum: (0, -2a)
- x-maximum: $\left(a\sqrt{2}, \pm\frac{a}{2}\sqrt{2}\right)$
- x-minimum: $\left(-a\sqrt{2}, \pm\frac{a}{2}\sqrt{2}\right)$
- Extent: Same as maxima and minima.
- Symmetries: The Nephroid is symmetric about both the x and y-axis and about the origin.
- Cusp: (a, 0); (-a, 0)

7.5 Dynamic Geometry of the Nephroid

The next 14 subsections deal with the dynamic geometry of the Nephroid. This includes GSP constructions that generate the Nephroid and GSP constructions that demonstrate selected properties of the Nephroid.

7.5.1 The Nephroid as an Epicycloid

As alluded to in section 7.1, an Epicycloid where the radius of the rolling circle is ½ of the radius of the stationary circle is called a Nephroid. That definition forms the basis for the construction found below in Table 7-1.

Table 7-1: The Nephroid as an Epicycloid

1. Draw circle AB with center at A and passing through point B	5. Let C_2 be the image when C is rotated about C_1 by $\angle BAC$
2. Let C be a random point on the circumference of circle AB	6. Let C_3 be the image when C_2 is rotated about C_1 by $\angle BAC$
3. Let point C_1 be the image when C is dilated about A by 1.5	7. Trace point C_3 and change its color
4. Draw circle C_1C with center at C_1 and passing through point C	8. Animate point C around circle AB

The trace of point C_2 in this construction forms a Limacon, another of the classic curves.

7.5.2 The Nephroid as an Epicycloid Part II

The Double Generation theorem of Daniel Bernoulli also applies to the Nephroid. That is, another way to get a Nephroid with a rolling circle type of construction is to make the radius of the rolling circle equal to 3/2 the radius of the stationary circle and have the stationary circle on the inside of the rolling circle. Such a construction is shown below in Table 7-2.

Table 7-2: The Nephroid as an Epicycloid Part II

1. Draw circle AB with center at A and passing through point B	6. Let A' be the image when A is translated by vector $C_1 \rightarrow C_3$
2. Let C be a random point on the circumference of circle AB	7. Draw circle A'C_3 with center at A' and passing through C_3
3. Let C_1 be the image when C is dilated about A by 1.5	8. Trace point C_3 and change its color
4. Let C_2 be the image when C is rotated about C_1 by $\angle BAC$	9. Animate point C around circle AB
5. Let C_3 be the image when C_2 is rotated about C_1 by $\angle BAC$	

Note how similar this construction is to that of the previous subsection (i.e., section 7.5.1). The only real difference between these two constructions is the radii of the rolling circles; i.e., in one case it is ½ of the radius of the stationary circle and in the other it is 3/2 the radius of the stationary circle.

7.5.3 The Nephroid as the Caustic of a Circle

Christian Huygens showed in 1678 that the Nephroid is the catacaustic of a circle when the light source is at infinity. In other words, the envelope of parallel rays that are reflected from the circumference of a circle creates a Nephroid. See Table 7-3.

Table 7-3: The Nephroid as the Caustic of a Circle

1. Draw horizontal line AB	8. Draw line segment CC_1
2. Draw circle AB with center at A and passing through point B	9. Draw line segment C_1C_3
3. Let C be a random point on the circumference of circle AB	10. Draw line segment AC_3
4. Let C_1 be the image when point C is reflected by line AB	11. Draw line segment CC_3
5. Let C_2 be the image when C is rotated about A by $\angle BAC$	12. Trace line segment CC_3 and change its color
6. Let C_3 be the image when C_2 is rotated about A by $\angle BAC$	13. Animate point C around circle AB
7. Draw line segment CA	

For best results with this construction, when point C is animated around circle AB, do it only once around the circle and at as high a speed as the animation will allow. Enjoy—it's a beautiful construction (as can be seen in Figure 7-3)!

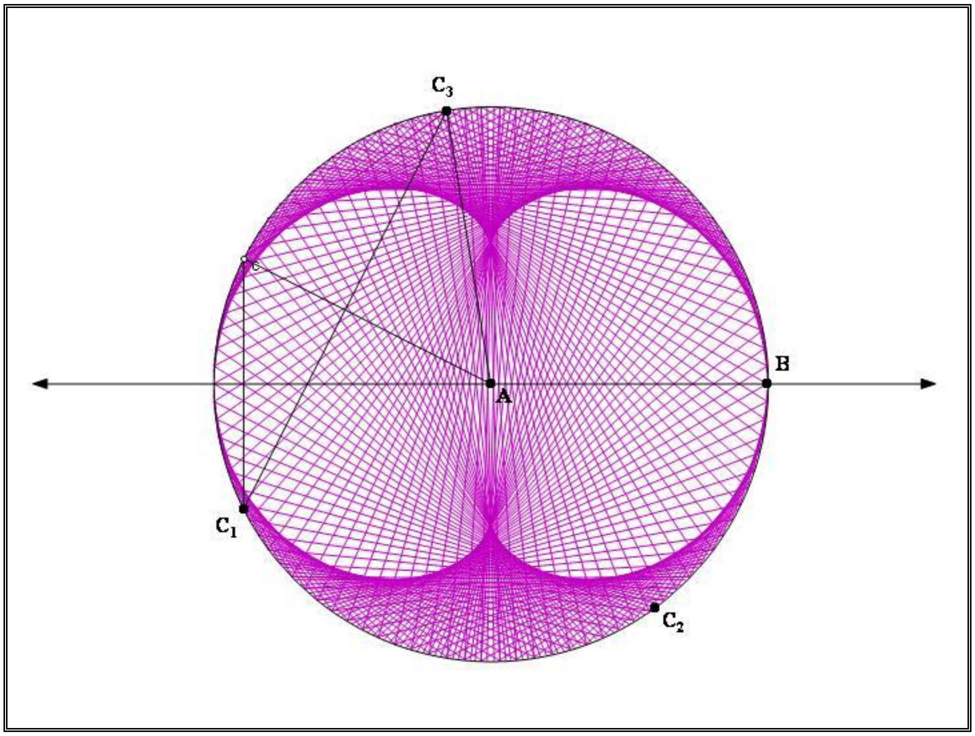

Figure 7-3: The Nephroid as a Caustic of a Circle

7.5.4 The Nephroid as an Envelope of Diameters

The Nephroid can also be generated as the envelope of a diameter of the circle that itself generates the Cardioid as an Epicycloid. In other words, construct a Cardioid as an Epicycloid (section 6.5.1) and then construct one of the diameters of the rolling circle. The trace of that diameter will generate an envelope that is a Nephroid. This is quite a beautiful construction; it is delineated below in Table 7-4 and illustrated in Figure 7-4.

Table 7-4: The Nephroid as an Envelope of Diameters

1. Draw horizontal line AB	6. Draw circle A'C' with center at A' and passing through C'
2. Draw circle AB with center at A and passing through point B	7. Let C" be the image when C' is rotated about point A' by 180º
3. Let C be a random point on the circumference of circle AB	8. Draw line segment C'C"
4. Let A' be the image when A is rotated about point C by 180º	9. Trace line segment C'C" and change its color
5. Let C' be the image when C is rotated about A' by ∠BAC	10. Animate point C around circle AB

7.5.5 The Concurrent Tangents of the Nephroid

One of the very fascinating characteristics of the Nephroid is that for any given tangent to the Nephroid, two other tangents can be found such that all three tangents will intersect in a common point. The construction for this remarkable characteristic is delineated below in Table 7-5. (Of course, if three tangents are concurrent, then the three normals through the three points of tangency are also concurrent.)

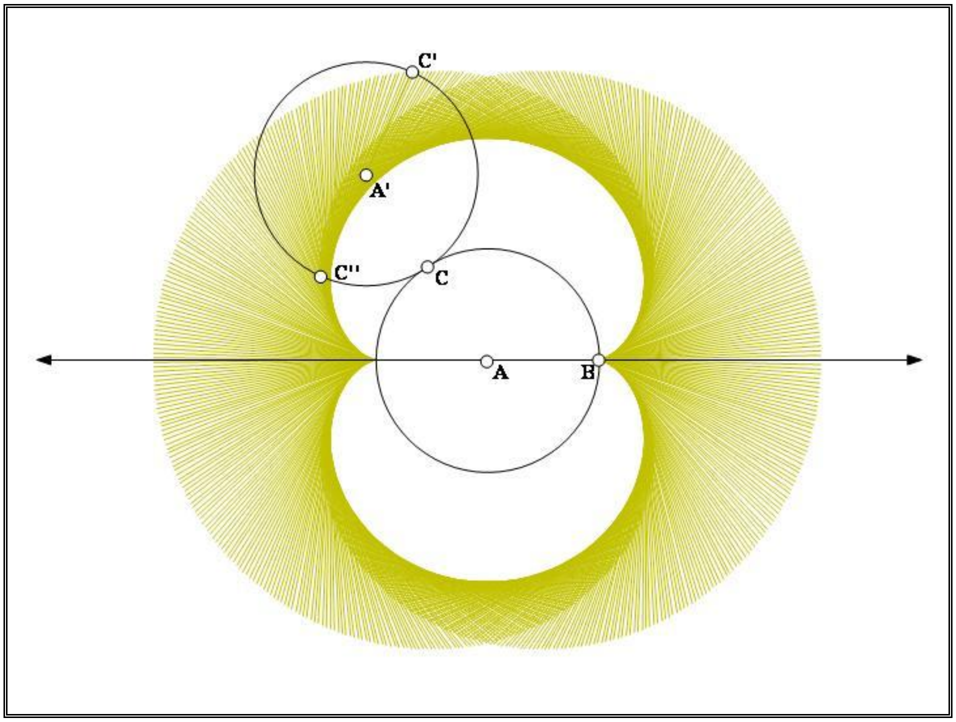

Figure 7-4: The Nephroid as an Envelope of Diameters

The perpendiculars constructed in steps 17, 19, and 21 (Table 7-5) are the three concurrent tangents. Obviously, line segments CC_6, C_2C_9, and C_5C_{11} are the respective normals. If the normals are extended, they too will meet in a common point. Note that circle AA' is circumscribed about the Nephroid and the point of tangent concurrency is confined to the circumference of this circumscribed circle; however, the point of normal concurrency is confined to the circumference of the inner circle, circle AB.

Table 7-5: The Concurrent Tangents of the Nephroid

1. Draw circle AB with center at A and passing through point B	13. Let C_9 be the image when C_8 is rotated about C_4 by –120º
2. Let C be a random point on the circumference of circle AB	14. Let C_{10} be the image when C_9 is rotated about A by 120º
3. Let A' be the image when A is rotated about point B by 180º	15. Let C_{11} be the image when C_{10} is rotated about C_7 by –120º
4. Draw circle AA' with center at A and passing through point A'	16. Draw line segment CC_6
5. Let C_1 be the image when C is dilated about A by 1.5	17. $P_1 \perp$ to line segment CC_6 through point C_6
6. Let C_2 be the image when C is rotated about point A by 120º	18. Draw line segment C_2C_9
7. Let C_3 be the image when C is rotated about C_1 by $\angle$BAC	19. Construct $P_2 \perp$ to line segment C_2C_9 through point C_9
8. Let C_4 be the image when C_1 is rotated about point A by 120º	20. Draw line segment C_5C_{11}
9. Let C_5 be the image when C_2 is rotated about point A by 120º	21. Construct $P_3 \perp$ to line segment C_5C_{11} through point C_{11}
10. Let C_6 be the image when C_3 is rotated about C_1 by $\angle$BAC	22. Construct the locus of C_6 while point C traverses circle AB
11. Let C_7 be the image when C_4 is rotated about point A by 120º	23. Animate point C around circle AB
12. Let C_8 be the image when C_6 is rotated about point A by 120º	

7.5.6 The Nephroid as the Caustic of a Cardioid

In 1692, Jacques Bernoulli showed that the Nephroid is the catacaustic of a Cardioid for a luminous cusp. In other words, if the light source is located at the Cardioid's cusp, the rays reflected from the circumference of the Cardioid form a Nephroid. This construction follows in Table 7-6, and Figure 7-5 is a snapshot of the final construction.

Table 7-6: The Nephroid as the Caustic of a Cardioid

1. Draw circle AB with center at A and passing through point B	7. Let D be the intersection of line segment BC' and circle AB
2. Let C be a random point on the circumference of circle AB	8. Draw line segment C'D
3. Let A' be the image when A is rotated about point C by 180º	9. Reflect line segment C'D in line segment CC'
4. Let C' be the image when C is rotated about A' by ∠BAC	10. Trace the reflected line segment and change its color
5. Draw line segment CC'	11. Animate point C around circle AB
6. Draw line segment BC'	

Point C', of course, is the point on the circumference of the Cardioid and point B is the source of the light rays. Line segment BC' therefore represents the incident light ray and since the angle of incidence must equal the angle of reflection, reflecting DC' in the normal to the Cardioid at the point C' will represent the reflected rays. Obviously, line segment CC' is the normal (normal to the Cardioid, that is).

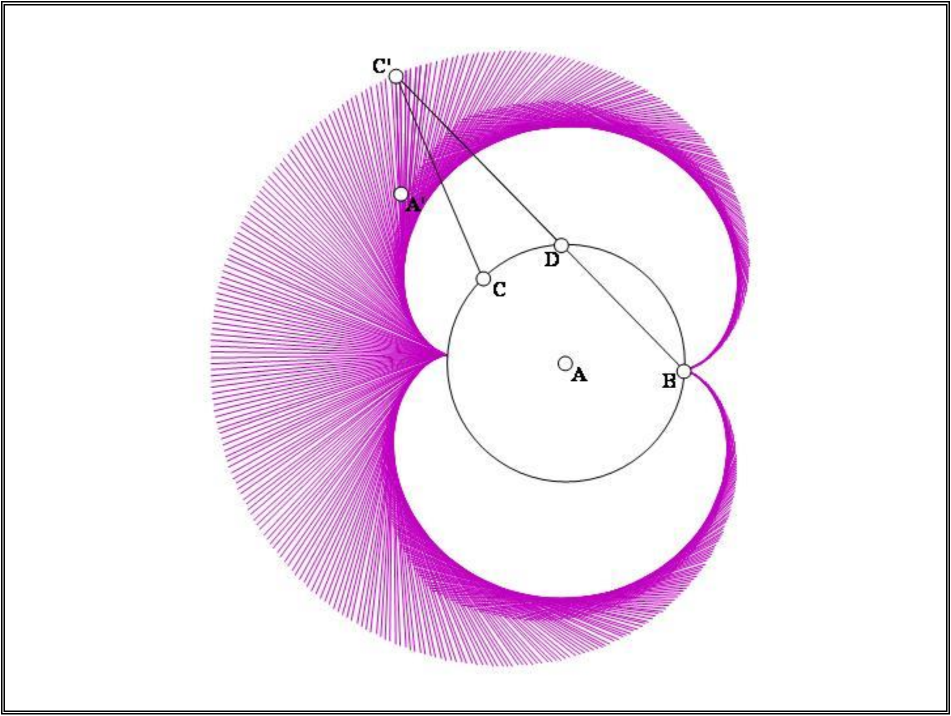

Figure 7-5: The Nephroid as the Caustic of a Cardioid

7.5.7 The Nephroid's Equilateral Triangle

The construction following in Table 7-7 demonstrates the fact that given *any* point on the circumference of a Nephroid, two other circumferential points can be found such that these three points form the vertices of an equilateral triangle.

Note that the locus of point C_5 in this construction is that of the Limacon of Pascal. Also note that the locus of either point C_8 or point C_{10} is another Nephroid that is rotated by 90º from that of the Nephroid traced by point C_7. Further, if one draws line segment C_7C_8 and constructs its midpoint, the locus of that midpoint as point C revolves about circle AB is another Nephroid which is inscribed in circle AB.

Table 7-7: The Equilateral Triangle of the Nephroid

1. Draw circle AB with center at A and passing through point B	10. Let C_8 be the image when C_7 is rotated about point A by 120º
2. Let C be a random point on the circumference of circle AB	11. Let C_9 be the image when C_8 is rotated about C_4 by 240º
3. Let C_1 be the image when C is rotated about point A by 120º	12. Let C_{10} be the image when C_9 is rotated about A by 120º
4. Let C_2 be the image when C is dilated about A by 1.5	13. Let C_{11} be the image when C_{10} is rotated about C_6 by 240º
5. Let C_3 be the image when C_1 is rotated about point A by 120º	14. Construct the locus of point C_7 while C traverses circle AB
6. Let C_4 be the image when C_2 is rotated about point A by 120º	15. Draw line segments C_7C_9, C_9C_{11}, and C_7C_{11}
7. Let C_5 be the image when C is rotated about C_2 by $\angle$BAC	16. Construct the interior of polygon $C_7C_9C_{11}$ and color it
8. Let C_6 be the image when C_4 is rotated about point A by 120º	17. Measure distances C_7C_9, C_9C_{11}, and C_7C_{11}
9. Let C_7 be the image when C_5 is rotated about C_2 by $\angle$BAC	18. Animate point C around circle AB

7.5.8 The Nephroid as an Envelope of Circles

This simple but elegant construction is based on the idea that a Nephroid is the envelope of a set of circles with their centers on a given base circle, such that each member of the set is tangent to a diameter of the base circle. This construction follows in Table 7-8 and an illustration of it is shown in Figure 7-6.

Table 7-8: The Nephroid as an Envelope of Circles

1. Draw horizontal line segment AB	7. Draw circle ED with center at E and passing through point D
2. Let C be the midpoint of line segment AB	8. Trace circle ED and change its color
3. Let D be a random point on line segment AB	9. Draw circle FD with center at F and passing through point D
4. Draw circle CA with center at C and passing through point A	10. Trace circle FD and change its color
5. Construct $P_1 \perp$ to line segment AB through point D	11. Animate point D along line segment AB
6. Let points E and F be the intersections of circle CA with P_1	

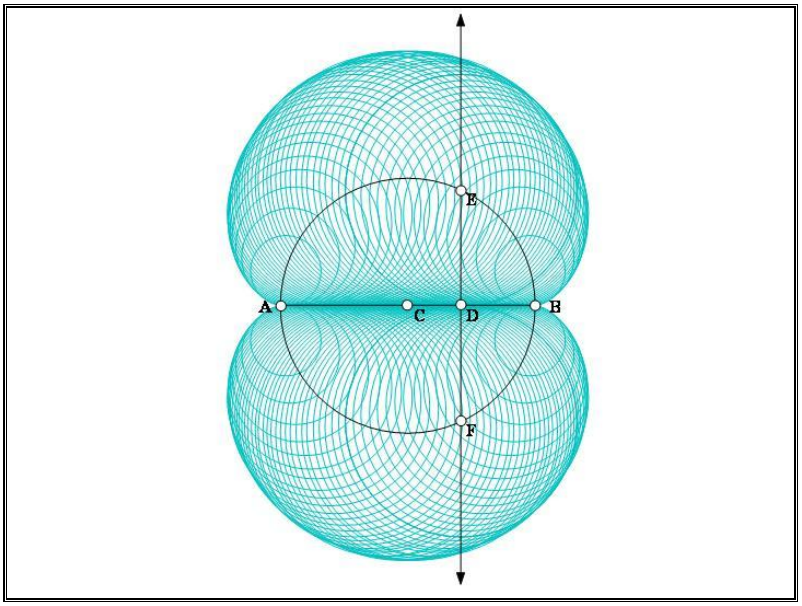

Figure 7-6: The Nephroid as an Envelope of Circles

7.5.9 A Nephroid Moving Around a Cardioid

The following construction is more for fun than to illustrate a specific property of the Nephroid. However, it has some instructive characteristics and is presented below in Table 7-9.

Table 7-9: A Nephroid Moving Around a Cardioid

1. Draw circle AB with center at A and passing through point B	9. Let D be a random point on circle B'C'
2. Let C be a random point on the circumference of circle AB	10. Construct $P_2 \perp$ to line BC' through point D
3. Let A' be the image when A is rotated about point C by 180º	11. Let point E be the intersection of line BC' and P_2
4. Let C' be the image when C is rotated about A' by ∠BAC	12. Let E' be the image when E is rotated about D by ∠C'B'D
5. Draw line BC'	13. Let E" be the image when E' is rotated about D by ∠C'B'D
6. Construct $P_1 \perp$ to line BC' through point A	14. Construct the locus of C' while point C traverses circle AB
7. Let B' be the image when point B is reflected across P_1	15. Construct the locus of point E" while D traverses circle B'C'
8. Draw circle B'C' with center at B' and passing through C'	16. Animate point C around circle AB

Note how the two points at the cusps of the Nephroid describe the Cardioid. It is also interesting to note that any random point placed on the circumference (i.e., the locus) of the Nephroid will trace a tangent line to the Cardioid as the animation is run. Further, the trace of point E' describes an unusual looking double-looped curve.

7.5.10 A Nephroid by Relative Velocity

This unusual construction of the Nephroid is based on the relative velocity of the two end points of a line segment. If the two end points of a line segment travel around a circle, but one end point travels three times faster than the other, the envelope of the line segment forms a Nephroid. The construction is delineated below in Table 7-10 and illustrated in Figure 7-7.

Table 7-10: A Nephroid by Relative Velocity

1. Draw line segment AB (in upper-right portion of the screen)	9. Construct circle C_2 centered at C and radius = segment EA
2. Let C be a random point <u>not</u> on line segment AB	10. Let F be a random point on circle C_1
3. Let D be the midpoint of line segment AB	11. Let G be a random point on circle C_2
4. Draw line segment BD	12. Draw ray FC starting at point F and passing through point C
5. Let E be the midpoint of line segment BD	13. Let point H be the intersection of ray FC and circle C_2
6. Draw line segment ED	14. Draw line segment HG
7. Draw line segment EA	15. Trace line segment HG and change its color
8. Construct circle C_1 centered at C and radius = segment ED	16. Animate F and G around circles C_1 and C_2, respectively

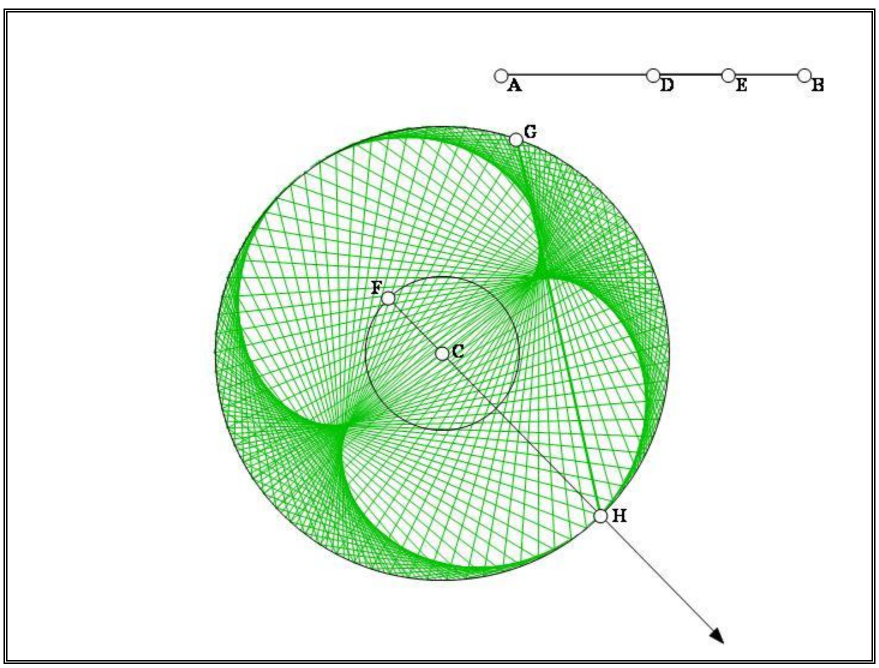

Figure 7-7: A Nephroid By Relative Velocity

Note that in this construction, steps 1 through 7 are merely to construct two line segments, one of which is three times the length of the other, i.e., EA = 3·ED. Now construct two concentric circles with radii in this same ratio (steps 8 and 9). Then, if two points can be made to revolve around the circles (one point around one of the circles and the other point around the other circle), while the point on the larger circle performs one revolution, the point around the smaller circle will perform 3 revolutions. Hence we have the desired relative velocity. For best results, in step 16 have point G make one revolution "quickly."

7.5.11 Orthogonal Nephroids

The following (see Table 7-11) is a construction for two Nephroids which remain orthogonal to each other as the animation is run. What this really means is that the intersection point of two mutually perpendicular lines is a point that is common to two different Nephroids. Further, the perpendicular lines are each normal, respectively, to the Nephroids. As a result, it's as though the two Nephroids were orthogonal.

Table 7-11: Orthogonal Nephroids

1. Draw horizontal line AB	13. Let F be a random point on circle A_2E
2. Draw circle AB with center at A and passing through point B	14. Construct $P_1 \perp$ to line CB through point C
3. Let A_1 be the image when A is dilated about B by –2	15. Let C_2 be the image when C_1 is rotated about A_3 by $\angle BA_1D$
4. Draw circle A_1B with center at A_1 and passing point B	16. Let C_3 be the image when C is rotated about A_2 by $\angle AA_2F$
5. Let C be a random point on the circumference of circle AB	17. Let A_4 be the image when A is rotated about A_2 by $\angle AA_2F$
6. Let A_2 be the image when A_1 is rotated about point A by 180º	18. Let C_4 be the image when C_2 is rotated about A_3 by $\angle BA_1D$
7. Let D be a random point on the circumference of circle A_1B	19. Let C_5 be the image when C_3 is rotated about A_4 by $\angle AA_2F$
8. Let E be the point of circle AB diametrically opposite of B	20. Let C_6 be the image when C_5 is rotated about A_4 by $\angle AA_2F$
9. Draw circle A_2E with center at A_2 and passing through point E	21. Construct the locus of point C_4 while D traverses circle A_1B
10. Draw line CB	22. Construct the locus of point C_6 while F traverses circle A_2E
11. Let C_1 be the image when C is rotated about A_1 by $\angle BA_1D$	23. Animate point C around circle AB
12. Let A_3 be the image when A is rotated about A_1 by $\angle BA_1D$	

7.5.12 A Nephroid from a Compass-Only Construction

Refer to Chapter 6, section 6.5.15 for a discussion of the GSP version of a compass-only construction. Table 7-12 contains the GSP version of a compass-only construction for the Nephroid.

Table 7-12: A Nephroid Derived from a Compass-Only Construction

1. Draw circle AB with center at A and passing through point B	14. Let F be the unlabeled intersection of circles DE and C'C
2. Let C be a random point on the circumference of circle AB	15. Draw circle FC with center at F and passing through point C
3. Draw circle CB with center at C and passing through point B	16. Let points G and H be the intersections of circles FC and CC'
4. Draw line segment AC	17. Draw circle GC with center at G and passing through point C.
5. Let B' be the image as B is reflected across line segment AC	18. Draw circle HC with center at H and passing through point C
6. Draw circle B'C with center at B' and passing through point C	19. Let I be the unlabeled intersection of circle HC and circle GC
7. Draw line segment AB'	20. Draw circle CI with center at C and passing through point I
8. Let C' be the image as C is reflected across line segment AB'	21. Draw circle IC with center at I and passing through point C
9. Hide segment AB'	22. Let J and K be the intersections of circle IC and circle CI
10. Draw circle CC' with center at C and passing through point C'	23. Draw circle JK with center at J and passing through point K
11. Draw circle C'C with center at C' and passing through point C	24. Let L be the unlabeled intersection of circle JK and circle CI
12. Let points D and E be the intersections of circles CC' and C'C	25. Trace point L and change its color
13. Draw circle DE with center at D and passing through point E	26. Animate point C around circle AB

Steps 4, 5, 7, and 8 are, of course, the only non-compass construction steps. The reason for hiding line segment AB' in step 9 is that attempting to carry out step 19, the construction of point I, will result in an ambiguous intersection if line segment AB' is visible. Note how, during the animation, all but one of the circles contract to a single point at one of the Nephroid's cusps (the cusp at point B). The one circle that doesn't collapse to a point is the circle that is the path for the animated point, circle AB. And at the other cusp, again all of the circles collapse with the exception of circles AB, CB, and B'C, the three initial circles. Also, for a construction that one might entitle "The Nephroid as an Envelope of Circles Derived from a Compass-Only Construction," try tracing circle CI and rerun the animation. And last but not least, the Osculating Circle of the Nephroid may be constructed by continuing the construction above. That is,

Table 7-12 (Continued): Osculating Circle Addition to Compass-Only Construction of Nephroid

27. Let M and N be the intersections of circle CI and circle C'C	30. Let O be the unlabeled intersection of circles NC and MC
28. Draw circle MC with center at M and passing point C	31. Draw circle OL with center at O and passing through point L
29. Draw circle NC with center at N and passing through point C	32. Rerun the animation

Circle OL is, of course, the Osculating Circle. In sum, a truly beautiful construction!

7.5.13 The Nephroid as an Envelope of Straight Lines

A quite simple but beautiful construction is the one shown below in Table 7-13.

Table 7-13: The Nephroid as an Envelope of Straight Lines

1. Draw circle AB with center at A and passing through point B	6. Construct $P_1 \perp$ to line AC' through point C
2. Draw line AB	7. Let point D be the intersection of P_1 and line AB
3. Let C be a random point on the circumference of circle AB	8. Draw line C'D
4. Let C' be the image when C is translated by vector $A \rightarrow C$	9. Trace line C'D and change its color
5. Draw line AC'	10. Animate point C around circle AB

7.5.14 The Osculating Circle of the Nephroid

Although a construction for the osculating circle of the Nephroid has already been presented (see the continuation of Table 7-12), here is a different construction of it. Constructions of osculating circles (or centers of curvature) are usually quite complex; however, the construction for the osculating circle of the Nephroid following in Table 7-14 is relatively simple and therefore worth the redundancy.

Table 7-14: The Osculating Circle of the Nephroid

1. Draw circle AB with center at A and passing through point B	8. Draw line CC_3
2. Let C be a random point on the circumference of circle AB	9. Line C_1C_3
3. Let C_1 be the image as point C is dilated about point A by 1.5	10. Let D be the unlabeled intersection of line C_1C_3 and circle C_1C
4. Let C_2 be the image as C is rotated about point C_1 by $\angle BAC$	11. Draw line AD
5. Draw circle C_1C with center at C_1 and passing through C	12. Let E be the intersection of lines CC_3 and AD
6. Let C_3 be the image as C_2 is rotated about C_1 by $\angle BAC$	13. Draw circle EC_3 with center at E and passing through C_3
7. Construct the locus of point C_3 while C traverses circle AB	14. Animate point C around circle AB

Quite a nice little construction for the osculating circle, which is, of course, circle EC_3. Additionally, construct the locus of point E while point C traverses circle AB. This locus gives you the Nephroid's evolute (which is another Nephroid rotated 90° from the original, ⅓ the size of the original, and inscribed between its cusps). Also, draw and trace line segment EC_3, the radius of curvature of the Nephroid. Now, rerun the animation and you will find that the traced radius of curvature fills in the original Nephroid, but envelopes the evolute Nephroid. Quite spectacular!

7.5.15 A Nephroid-Cardioid Waltz

As a final construction for this chapter, consider the Nephroid-Cardioid dance that is delineated below in Table 7-15.

Table 7-15: **A Nephroid-Cardioid Waltz**

1. Draw horizontal line AB	12. Construct $P_1 \perp$ to line segment BC through point C
2. Draw circle AB with center at A and passing through point B	13. Let C_1 be the image when C is rotated about A_3 by $\angle BA_3D$
3. Let A_1 be the image when A is dilated about B by a factor of 2	14. Let A_4 be the image when A is rotated about A_3 by $\angle BA_3D$
4. Let A_2 be the image when A is dilated about B by –2	15. Let C_2 be the image when C is rotated about A_2 by $\angle BA_2E$
5. Let A_3 be the image when A is dilated about B by –1	16. Let A_5 be the image when A is rotated about A_2 by $\angle BA_2E$
6. Draw circle A_2B with center at A_2 and passing through B	17. Let C_3 be the image when C_1 is rotated about A_4 by $\angle BA_3D$
7. Draw circle A_3B with center at A_3 and passing through B	18. Let C_4 be the image when C_2 is rotated about A_5 by $\angle BA_2E$
8. Let C be a random point on the circumference of circle AB	19. Let C_5 be the image when C_4 is rotated about A_5 by $\angle BA_2E$
9. Let D be a random point on the circumference of circle A_3B	20. Construct the locus of C_3 while point D traverses circle A_3B
10. Let E be a random point on the circumference of circle A_2B	21. Construct the locus of C_5 while point E traverses circle A_2B
11. Draw line segment BC	22. Animate point C around circle AB

Note that the Nephroid and Cardioid are tangent at point C and that the cusps of the Nephroid remain on the circumference of the Cardioid as the animation revolves. The dance is most evident if the following elements of the construction are hidden before running the animation: points A, B, A_1, D, E, C_1, A_4, C_2, A_5, C_3, C_4, and C_5, circles AB, A_2B, and A_3B, line AB, and line segment BC. If Roemer (see Chapter 8) had been able to use a dynamic geometry application such as GSP, his study for the best form of gear teeth would probably have been much easier.

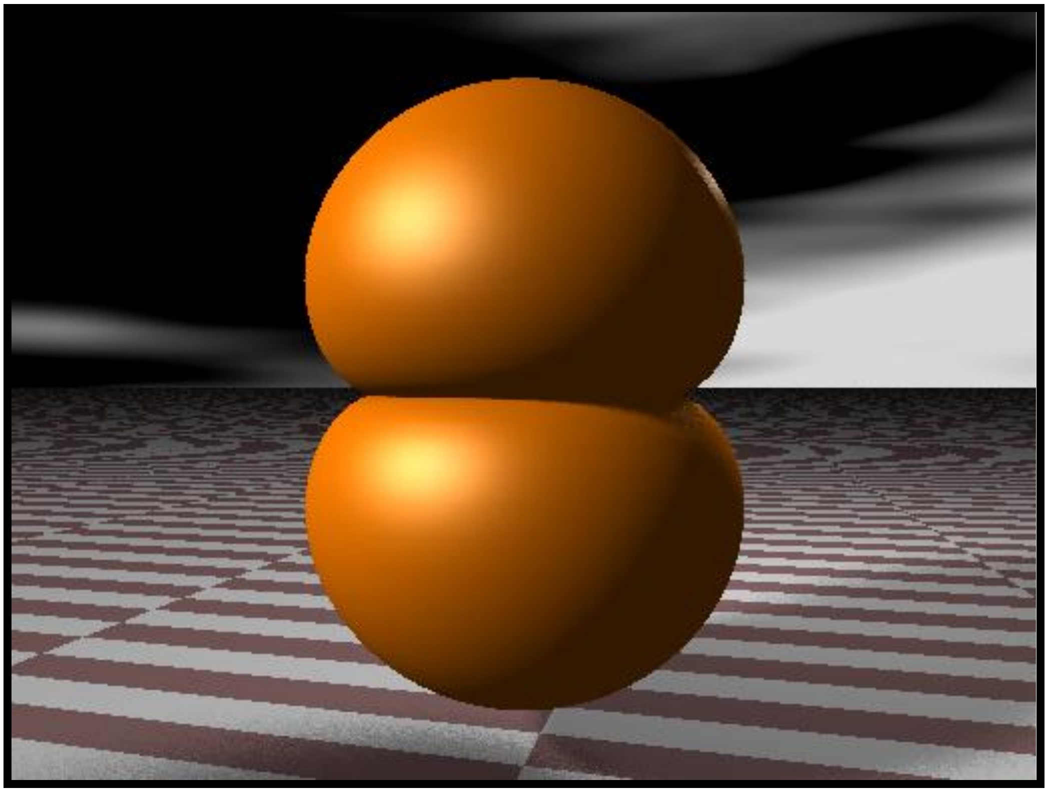

Figure 7-8: The Solid of Revolution Formed by the Nephroid

The Nephroid was revolved about the y-axis to produce the object seen in the figure above. The resulting solid of revolution was then given a coral-colored finish and placed above the grey and rose colored checkered plane which meets a dark and forbidding sky at the horizon. Light sources were placed so as to illuminate the solid of revolution as shown and to cast shadows on the checkered plane.

Chapter 8 – The Epicycloid

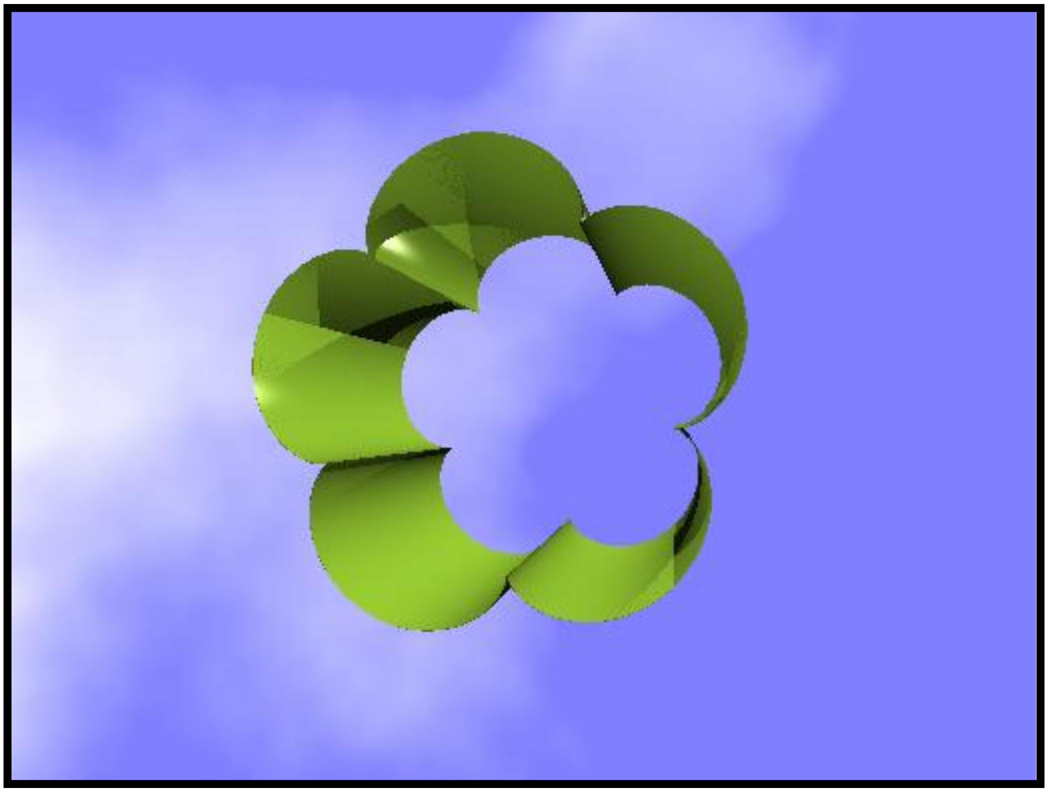

Figure 8-1: A Five-Cusped Epicycloid in Three Dimensions

The cross-section of the object in Figure 8-1 is a Five-Cusped Epicycloid. It was created by taking a normal, two-dimensional Epicycloid with five cusps and simply extruding into the dimension that is normal to the plane of the page. It was then given a shiny, yellow-green finish and configured as though it were floating in a bright, summer sky. Light sources have been placed so as to partially shadow the upper, inner surface.

8.1 Introduction

In Chapter 6 we introduced the concept of the Epicycloid and showed that when the radii of the fixed and rolling circles are the same, a curve called the cardioid results. Then, in Chapter 7 we showed that if the radius of the rolling circle is half the radius of the fixed circle, the resulting curve is a Nephroid. We now take up the Epicycloid in general.

Back to Roemer, briefly (a very interesting character). Cycloidal curves were first conceived by Roemer (a Dane) in 1674 while studying the best form for gear teeth. However, prior to Roemer’s work, in 1599, both Galileo and Mersenne had already discovered the ordinary cycloid. Olaf Roemer (1644-1710) was a mathematician who gave the first good estimate of the speed of light. This was done in 1675 by means of the eclipses of Jupiter’s satellites. Roemer also constructed the fountains at the Versailles castle near Paris. Relevant to this chapter, Roemer deduced from the properties of epicycloids the form of the teeth in toothed-wheels best fitted to secure a uniform motion. As already alluded to, the beautiful Double Generation theorem of these curves was first noticed by Daniel Bernoulli in 1725. Astronomers find forms of the cycloidal curves in various coronas. They also occur as caustics. Rectification was first given by Newton in his *Principia*.

8.2 Equations and Graph of the Epicycloid

We already have shown (in Chapter 6, Equation 6-1) that when the radius of the fixed circle is a, and the radius of the rolling circle is b, the parametric equation of the Epicycloid is

$$x=(a+b)\cos t-b\cos\left(\tfrac{a+b}{b}t\right) \quad \text{and} \quad y=(a+b)\sin t-b\sin\left(\tfrac{a+b}{b}t\right) \qquad \textbf{Equation 8-1}$$

We have relabeled it as Equation 8-1 here. A polar equation can easily be derived by making the following computations

$$x^2=(a+b)^2\cos^2 t-2b(a+b)\cos t\cos\left(\tfrac{a+b}{b}t\right)+b^2\cos^2\left(\tfrac{a+b}{b}t\right) \quad \text{and}$$

$$y^2=(a+b)^2\sin^2 t-2b(a+b)\sin t\sin\left(\tfrac{a+b}{b}t\right)+b^2\sin^2\left(\tfrac{a+b}{b}t\right).$$

By adding these last two expressions, we have,

$$r^2=x^2+y^2=(a+b)^2+b^2-2ab\left[\cos t\cos\left(1+\tfrac{a}{b}\right)t+\sin t\sin\left(1+\tfrac{a}{b}\right)t\right].$$

However, the expression in the square brackets is simply the cosine of the difference of the two arguments of the functions within the brackets. Therefore,

$$r^2=(a+b)^2+b^2-2b(a+b)\cos\tfrac{a}{b}t \qquad \textbf{Equation 8-2}$$

Note, however, that the parameter t in the above expression for r^2 is not the polar angle. The polar angle is

$$\tan\theta = \frac{y}{x} = \frac{(a+b)\sin t - b\sin\left(\frac{a+b}{b}t\right)}{(a+b)\cos t - b\cos\left(\frac{a+b}{b}t\right)} \quad \textbf{Equation 8-3}$$

If the pedal point is located at the center of the Epicycloid, then the pedal equation is

$$p^2 = \frac{(2a+b)^2}{4a(a+b)}\left(r^2 - b^2\right) \quad \textbf{Equation 8-4}$$

Similarly, the Whewell equation is

$$s = b\sin a\varphi,\ a<1 \quad \textbf{Equation 8-5}$$

The Cesáro equation is

$$\rho^2 + a^2s^2 = a^2b^2 \quad \textbf{Equation 8-6}$$

And, finally, the equation of the tangent to the Epicycloid at the point $t = q$ is

$$\left(\sin\tfrac{a+b}{b}q - \sin q\right)\cdot y = \left(\cos q - \cos\tfrac{a+b}{b}q\right)\cdot x + (a+2b)\left(\cos\tfrac{a}{b}q - 1\right) \quad \textbf{Equation 8-7}$$

To obtain n cusps in the epicycloid, let $b = a / n$, because n rotations of the rolling circle bring the point on its circumference back to its starting position. As we have already learned, a one-cusped Epicycloid is called a Cardioid and a two-cusped Epicycloid is called a Nephroid. The only other named Epicycloid is one with five cusps. It is called the Ranunculoid, named after the buttercup genus Ranunculus. Figure 8-2 depicts the graph of four different Epicycloids, namely, three-, four-, five-, and six-cusped Epicycloids in red, green, blue and violet, respectively, while Figure 8-3 portrays a variety of different Epicycloids for various selected values of the two radii of the associated circles (the fixed circle with radius a and the rolling circle with radius b). Note that as long as the ratio of a to n is a rational number, the Epicycloid will be closed; however, in some cases, the rolling circle will make more than one revolution before the tracing point comes back to its starting position, as is illustrated in all but one of the Epicycloids in Figure 8-3.

8.3 Analytical and Physical Properties of the Epicycloid

Using the parametric representation of the Epicycloid given in Equation 8-1 i.e., $x = (a+b)\cos t - b\cos\left(\frac{a+b}{b}t\right)$ and $y = (a+b)\sin t - b\sin\left(\frac{a+b}{b}t\right)$, the following subparagraphs delineate further properties of the Epicycloid.

8.3.1 Derivatives of the Epicycloid

- $\dot{x} = (a+b)\left[\sin\left(\frac{a+b}{b}t\right) - \sin t\right]$
- $\ddot{x} = \dfrac{a+b}{b}\left[(a+b)\cos\left(\frac{a+b}{b}t\right) - b\cos t\right]$
- $\dot{y} = (a+b)\left[\cos t - \cos\left(\frac{a+b}{b}t\right)\right]$

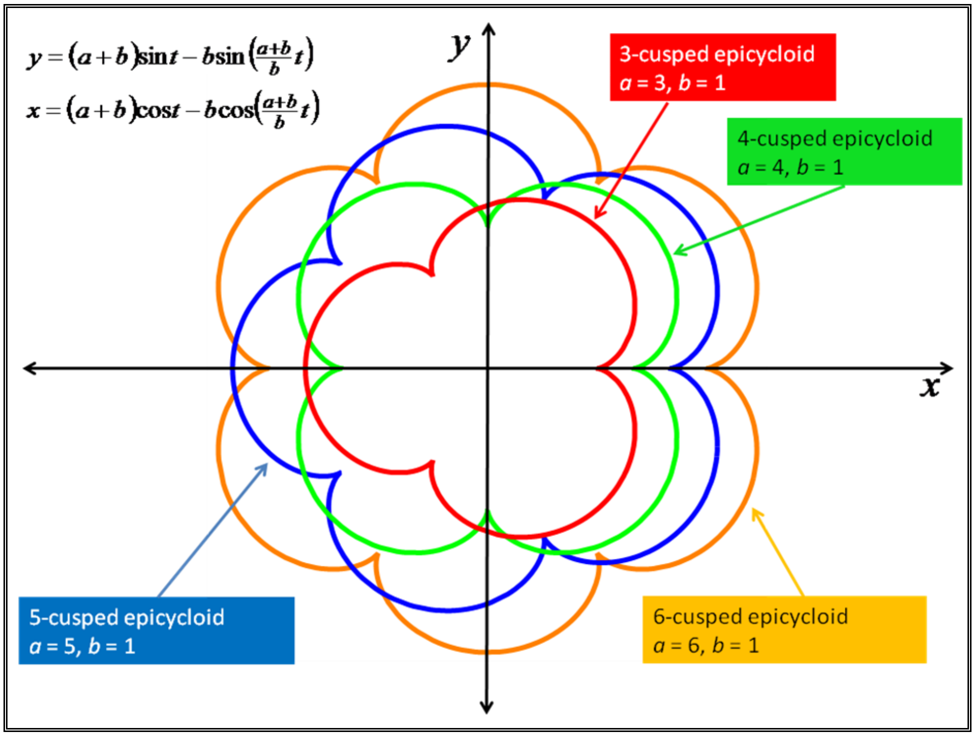

Figure 8-2: Four Different Epicycloids

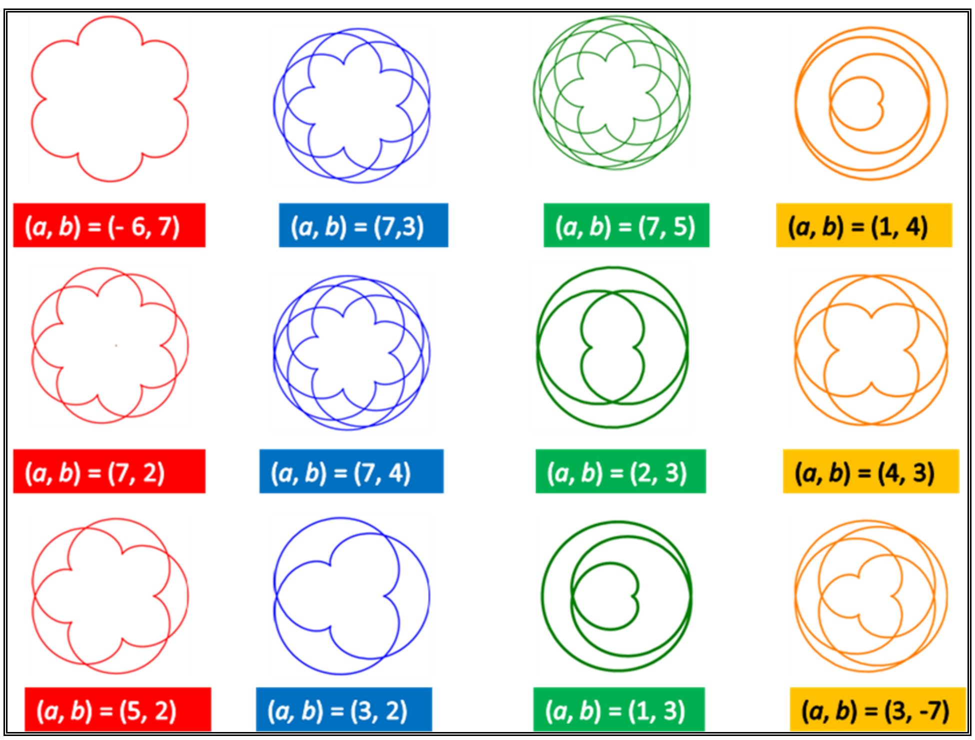

Figure 8-3: A Variety of Epicycloids as a Function of the Parameters *a* and *b*

- $\ddot{y} = \dfrac{a+b}{b}\left[(a+b)\sin\left(\frac{a+b}{b}t\right) - b\sin t\right]$

- $y' = \dfrac{\cos t - \cos\left(\frac{a+b}{b}t\right)}{\sin\left(\frac{a+b}{b}t\right) - \sin t}$

- $y'' = \dfrac{(a+2b)\left(1-\cos\frac{a}{b}t\right)}{b(a+b)\left(\sin\frac{a+b}{b}t - \sin t\right)^3}$.

8.3.2 Metric Properties of the Epicycloid

The following addresses the length and area of the Epicycloid; however, rather than derive the formulas for length and area through the laborious process of integration (as we have done in previous chapters), we will make a departure here and show how one might intuitively arrive at the formulas. Additionally, we present the formulas for the radial distance and the distance to the tangent for the Epicycloid.

Chapters 6 and 7 showed that the length of the Cardioid and Nephroid were 16*a* and 24*a*, respectively, where *a* is the radius of the rolling circle. Table 8-1 tabulates the lengths for the first six epicycloids (the calculations for the three-cusped through the six-cusped are left as an exercise for the reader).

Table 8-1: Epicycloid Lengths

Number of Cusps	Name of the Curve	Calculated Length
1	Cardioid	16*a*
2	Nephroid	24*a*
3	3-Cusped Epicycloid	32a
4	4-Cusped Epicycloid	40*a*
5	Ranunculoid	48*a*
6	6-Cusped Epicycloid	56*a*

One can quite easily intuit from this table that the length as a function of the number of cusps is simply $L = 8a\ (n + 1)$, where *n* is the number of cusps and *a* is the radius of the rolling circle.

Chapters 6 and 7 showed that the area of the Cardioid and Nephroid were $6\pi a^2$ and $12\pi a^2$, respectively, where *a* is the radius of the rolling circle. Table 8-2 tabulates the areas for the first six epicycloids (again, the calculations for the three-cusped through the six-cusped are left as an exercise for the reader).

Table 8-2: Epicycloid Areas

Number of Cusps	Name of the Curve	Area
1	Cardioid	$6\pi a^2$
2	Nephroid	$12\ \pi a^2$
3	3-Cusped Epicycloid	$20\ \pi a^2$
4	4-Cusped Epicycloid	$30\ \pi a^2$
5	Ranunculoid	$42\ \pi a^2$
6	6-Cusped Epicycloid	$56\ \pi a^2$

Again, one can quite readily intuit (from Table 8-2) that the area as a function of the number of cusps can be written as $A = \pi a^2 \cdot (n + 1)\,(n + 2)$.

If r denotes the distance from the origin to the curve, then

$$r = \sqrt{a^2 + 2ab + 2b^2 - 2b(a+b)\cos\tfrac{at}{b}}\,.$$

If p denotes the distance from the origin to the tangent line of the epicycloid, then

$$p = -(a+2b)\sin\tfrac{at}{2b}\,.$$

8.3.3 Curvature of the Epicycloid

If ρ represents the radius of curvature of the Epicycloid, then

$$\rho = \frac{4b(a+b)}{a+2b}\cdot\sin\frac{a}{2b}t.$$

If (α, β) represents the coordinates of the center of curvature of the general epicycloid, then,

$$\alpha = \frac{a}{a+2b}\left[(\mathrm{a}+\mathrm{b})\cos t + b\cos\tfrac{a+b}{b}t\right] \quad \text{and} \quad \beta = \frac{a}{a+2b}\left[(a+b)\sin t + b\sin\tfrac{a+b}{b}t\right]$$

8.3.4 Angles of the Epicycloid

If ψ is the angle between the tangent and the radius vector at the point of tangency to the general Epicycloid, then

$$\tan\psi = \frac{a+2b}{a}\cdot\frac{1-\cos\frac{a}{b}t}{\sin\frac{a}{b}t}.$$

If ϕ denotes the tangential angle to the general Epicycloid, then

$$\tan\phi = \frac{\cos t - \cos\left(\frac{a+b}{b}t\right)}{\sin\left(\frac{a+b}{b}t\right) - \sin t}.$$

If θ denotes the radial angle for the general Epicycloid, then

$$\tan\theta = \frac{(a+b)\sin t - b\sin\left(\frac{a+b}{b}t\right)}{(a+b)\cos t - b\cos\left(\frac{a+b}{b}t\right)}.$$

8.4 Geometric Properties of the Epicycloid

The general Epicycloid is always symmetric about the x-axis; however, it is also symmetric about the y-axis if the quantity $(a + b) / b$ is an odd integer. It is completely contained within a circle defined by $|r| \leq a + 2b$.

8.5 Dynamic Geometry of the Epicycloid

The next five subsections present some of the dynamic geometry constructions for the Epicycloid.

8.5.1 An Epicycloid Toy

Do you want to draw designs like those of Figure 8-3? If you do, carefully follow the steps of the construction delineated below in Table 8-3 and you will reproduce a marvelous mechanism that can provide many stimulating hours of enjoyment.

Table 8-3: An Epicycloid Toy

1. Draw horizontal line AB	11. Let G be a random point on the circumference of circle AE
2. Draw circle AB with center at A and passing through point B	12. Draw circle DF with center at D and passing through point F
3. Construct $P_1 \perp$ to line AB through point A	13. Let H be a random point on the circumference of circle DF
4. Construct $P_2 \perp$ to line AB through point B	14. Create circle C_1 by translating circle DF by vector D → G
5. Let C be a random point on the circumference of circle AB	15. Draw line segment DH
6. Let D be a random point on perpendicular P_2	16. Create S_1 by translating line segment DH by vector D → G
7. Let E be a random point on line AB	17. Let I be the intersection of elements C_1 and S_1
8. Construct $P_3 \perp$ to line AB through point E	18. Trace point I and change its color
9. Draw circle AE with center at A and passing through point E	19. Simultaneously animate point H on circle DF and Point G on circle AE
10. Let F be a random point on perpendicular P_3	

To see your tracings better, it is recommended that you hide the following construction elements after completing the construction: Points A, C, D, F, and H; all three perpendiculars, the line AB, and the line segment DH; and the two circles, AE and DF. Note that one can obtain different Epicycloids by dragging point E. Have fun!

8.5.2 An Epicycloid of Three Cusps

The simple, but spectacular, construction delineated below in Table 8-4 of a Three-Cusped Epicycloid contains a curve that we have not yet addressed. See if you can spot it.

Table 8-4: A Three-Cusped Epicycloid

1. Draw circle AB with center at A and passing through point B	8. Let C_2 be the image when C_1 is rotated about A' by ∠CAB
2. Let C be a random point on the circumference of circle AB	9. Let C_3 be the image when C_2 is rotated about A' by ∠CAB
3. Draw line segment AC	10. Draw circle CC_3 with center at C and passing through C_3
4. Construct $P_1 \perp$ to line segment AC through point C	11. Trace circle CC_3 and change its color
5. Dilate circle AB about point C by a factor of ⅓	12. Let C_4 be the image when point C_3 is reflected by mirror P_1
6. Let A' be the image as A is dilated about C by a factor of ⅓	13. Animate point C around circle AB
7. Let C_1 be the image when C is rotated about A' by ∠CAB	

Actually, this remarkable little construction incorporates two curves that we have not yet encountered. If you turn on the trace of point C_3, a curve called the Deltoid is drawn. If you trace point C_2, an ellipse is drawn. Both of these curves will be addressed in subsequent chapters. Of course, point C_4 traces a three-cusped Epicycloid, just as circle CC_3 generates an envelope that is a three-cusped Epicycloid. As-a-matter-of-fact, the outside envelope generated by circle CC_3 is the Epicycloid while the inside envelope is the Deltoid alluded to above. What a fascinating, elegant construction! See Figure 8-4 for a snapshot of this construction.

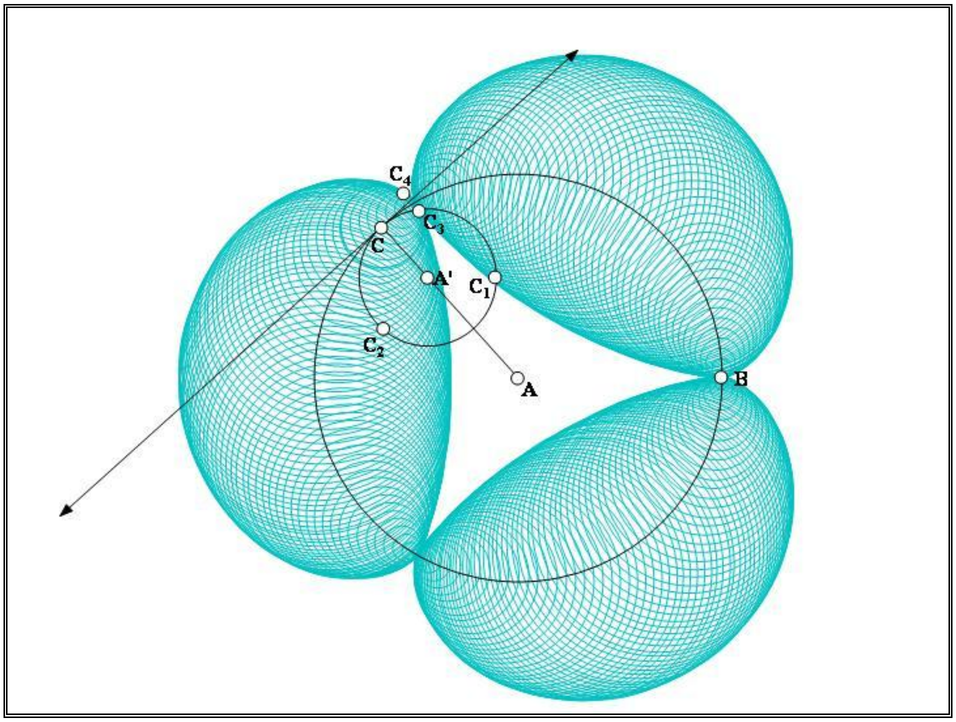

Figure 8-4: A Three-Cusped Epicycloid

8.5.3 A Compass-Only Three-Cusped Epicycloid

At the risk of being redundant, a construction for a three-cusped Epicycloid follows in Table 8-5. Yes, the same result was presented in the previous section. However, this time it is done as a GSP-version of a compass-only construction. Truly remarkable!

Table 8-5: A Three-Cusped Epicycloid by Compass Only

1. Draw circle AB with center at A and passing through point B	14. Draw circle FB' with center at F and passing through point B'
2. Let C be a random point on the circumference of circle AB	15. Let points G and H be the intersections of circles FB' and B'C
3. Draw circle BC with center at B and passing through point C	16. Draw circle GH with center at G and passing through point H
4. Draw line segment AB*	17. Let I be the unlabeled intersection of circle GH and circle FB'
5. Let C' be the image as C is reflected across line segment AB*	18. Draw circle IC with center at I and passing through point C
6. Draw circle C'B with center at C' and passing through point B	19. Let points J and K be the intersections of circles IC and CB'
7. Draw line segment AC'*	20. Draw circle JC with center at J and passing through point C
8. Let B' be the image as B is reflected across line segment AC'*	21. Draw circle KC with center at K and passing through point C
9. Draw circle B'C with center at B' and passing through point C	22. Let point L be the unlabeled intersection of circles KC and JC
10. Draw circle CB' with center at C and passing through point B'	23. Draw circle CL with center at C and passing through point L
11. Let points D and E be the intersections of circles B'C and CB'	24. Trace circle CL and change its color
12. Draw circle DE with center at D and passing through point E	25. Animate point C around circle AB
13. Let F be the unlabeled intersection of circles DE and B'C	

* Steps 4, 5, 7, and 8 are the only non-compass steps in the construction

8.5.4 An *n*-Cusped Epicycloid

The construction following in Table 8-6 is that of a five-cusped Epicycloid (e.g., a Ranunculoid). However, following the construction steps, it will be shown how to modify specific steps of the construction in order to change the construction into one for an Epicycloid of any number of cusps desired, i.e., *n* cusps.

Table 8-6: A Five-Cusped (or *n*-Cusped) Epicycloid

1. Create an *x-y* axis with origin at A and unit point B(1, 0)	9. Draw circle C'C with center at C' and passing through point C
2. Draw circle AB with center at A and passing through point B	10. Let C_1 be the image when C is rotated about C' by ∠BAC
3. Let C be a random point on the circumference of circle AB	11. Let C_2 be the image when C_1 is rotated about C' by ∠BAC
4. Measure ∠BAC	12. Let C_3 be the image when C_2 is rotated about C' by ∠BAC
5. Draw line segment AC	13. Let C_4 be the image when C_3 is rotated about C' by ∠BAC
6. Measure the length of line segment AC	14. Let C_5 be the image when C_4 is rotated about C' by ∠BAC
7. Calculate AC / 5	15. Trace point C_5 and change its color
8. Let C' be the image as C is translated by ∠BAC and AC / 5	16. Animate point C around circle AB

Make sure that the angle units in the GSP preference table are set for radian measure before executing the above animation. To alter the steps in the above table for the construction of an *n*-cusped epicycloid (where *n* is any integer), do the following: Change step 7 to read "Calculate AC / *n*." Change step 8 to read "Let C' be the image when C is translated by ∠BAC and AC / *n*." Replace steps 11-14 with the steps that result from executing the following pseudo-language loop

begin loop
for $i = 2$ to n
    $9 + i$. Let C_i be the image when point C_{i-1} is rotated about point C′ by ∠BAC.
end loop

Finally, change what is now step 15 (and will be step $9 + n$) to read "Trace point C_n and change its color."

8.5.5 Can We Build a Better Mousetrap?

Section 8.5.1 (An Epicycloid Toy) presented an "adjustable" Epicycloid construction; that is, a construction where a point can be dragged that changes the radius of the fixed circle, thereby changing the ratio of the radius of the fixed circle to that of the revolving circle and thence, as we have learned, changing the number of cusps in the traced Epicycloid. However, that construction suffers from the inability to adjust the ratio finely enough to result in a closed Epicycloid (i.e., one where the tracing point eventually returns to its starting point), in other words, making the ratio a rational number. The following construction (Table 8-7) is an *attempt* to design a better adjustable Epicycloid construction, one in which the ratio can be fine-tuned to result in a closed Epicycloid.

Table 8-7: A Better Mousetrap?

1. Draw circle AB with center at A and passing through point B	10. Let H be a random point on the circumference of circle GG'
2. At top of screen, draw line segment CD screen wide	11. Draw line segment GH
3. Let E be a random point on line segment CD	12. Construct the parallel to line segment GH through point F
4. Let F be a random point on the circumference of circle AB	13. Let point I be the intersection of the parallel and circle FF'
5. Let F' be the image when F is translated by vector C → E	14. Trace point I and change its color
6. Draw circle FF' with center at F and passing through point F'	15. Let m_1 be a measure of the radius of circle AB
7. Let G be a random point anywhere in the plane	16. Let m_2 be a measure of the radius of circle GG'
8. Let G' be the image when G is translated by vector C → E	17. Calculate m_2 / m_1
9. Draw circle GG' with center at G and passing through point G'	18. Animate points H and F on circles GG' and AB, respectively

The idea here is to construct an auxiliary line segment (i.e., line segment CD) upon which a variable point (point E) can be dragged from one end of the segment to the other. If the distance from one end of the segment to the variable point is then marked as a vector and used as the radius of the fixed circle, maybe a fine enough adjustment can be made to the ratio under consideration as the variable point is dragged, particularly if we calculate that ratio so that we can see at what location on the segment the ratio become a whole number. At least that's the concept!

Well, it was a good idea, anyway! It is very difficult to adjust the ratio to an integer by sliding point E along line segment CD. However, you can get very close. Figure 8-5 shows the ratio to be very close to 3 which, of course, represents a two-cusped Epicycloid (i.e., the Nephroid).

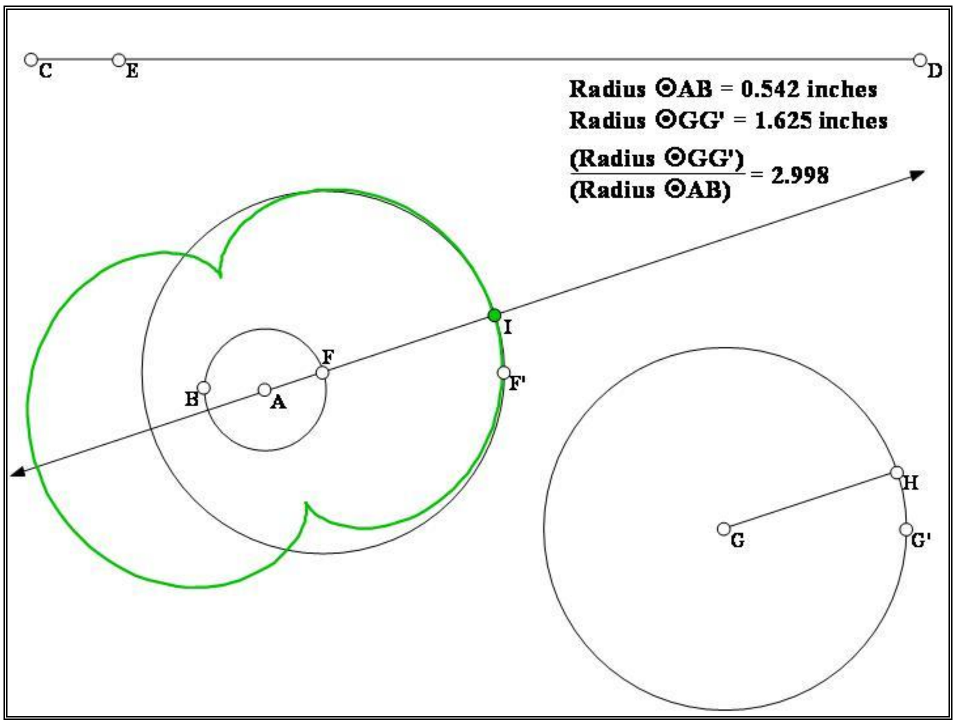

Figure 8-5: A Better Mousetrap?

8.5.6 Yes, We Can Build a Better Mousetrap

Table 8-8 presents an adjustable Epicycloid construction that can be adjusted finely enough to result in closed Epicycloids, but, interestingly enough, only when the ratio of $(a + b) / b$ is an integer. When the ratio is merely rational (but non-integer), GSP doesn't perform very well.

Readers should realize that the following construction (Table 8-8) is not a true geometric construction but utilizes the graphing capability of GSP to render its construction elements. However, here we will show how to use it to construct the center of curvature, the evolute, the osculating circle, as well as the tangent line—a simple but versatile use of a dynamic geometry application.

Table 8-8: Yes, a Better Mousetrap

1. Create an x-y axis with origin at A and unit point B(1, 0)	11. Calculate $y = (a + b) \cdot \sin t - b \cdot \sin [(a + b)\ t / b]$
2. Draw circle BC centered at B and passing through point C	12. Let G be the result of plotting the point (x, y)
3. Let D be a random point on the circumference of circle BC	13. Let $\alpha = [a / (a + 2b)] \cdot \{[(a + b) \cdot \cos t + b \cdot \cos [(a + b)\ t / b]\}$
4. Draw line segments BD and BC	14. Let $\beta = [a / (a + 2b)] \cdot \{[(a + b) \cdot \sin t + b \cdot \sin [(a + b)\ t / b]\}$
5. Let t be the measure of $\angle$CBD	15. Let point H be the result of plotting (alpha, beta)
6. Let E and F be two random points on the x-axis	16. Trace point H and change its color, say red
7. Let the x-coordinate of point E be a	17. Draw circle HG with center at H and passing through point G
8. Let the x-coordinate of point F be b	18. Trace circle HG and change its color, say green
9. Calculate $(a + b) / b$	19. Animate point D around circle BC
10. Calculate $x = (a + b) \cdot \cos t - b \cdot \cos [(a + b)\ t / b]$	

Point G (although we did not trace its path) will trace an Epicycloid. The number of cusps will depend on the value of $(a + b) / b$, which can be adjusted to an integer value by sliding point E and/or F along the x-axis. However, you must make sure that the value of $(a + b) / b$ is a positive integer. If it is negative, you will not obtain an Epicycloid but rather a curve called a Hypocycloid. Now, point H, which is the center of curvature for the curve generated by point G, will trace the evolute, which in the case of an Epicycloid is also an Epicycloid. Circle HG is the osculating circle for the Epicycloid, and by tracing it we obtain the Epicycloid as the envelope of those circles. We've got everything but the kitchen sink in this construction; we might as well add the kitchen sink—I mean the tangent line to the Epicycloid.

Table 8-8 (Continued): Yes, a Better Mousetrap

20. Calculate 0.000	22. Let point I be the result of plotting $(0, c)$
21. Calculate $c = (a + 2b)[\cos(at/b) - 1] / \{\sin[(a + b)t/b] - \sin t\}$	23. Draw line GI and change its color and make it thick

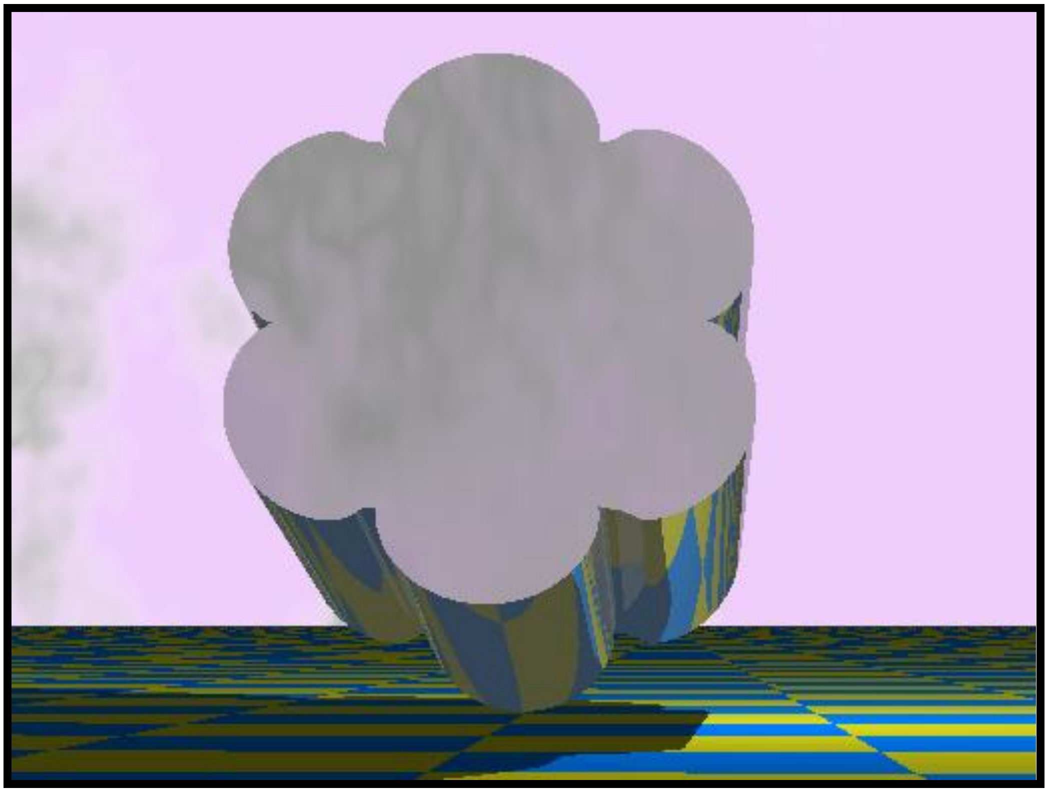

Figure 8-6: A Six-Cusped Epicycloid in Three Dimensions

The cross-section of the pseudo-cylinder above is a six-cusped Epicycloid. It was rendered by extruding the cross-section into the third dimension using a technique referred to as treating the object as a lathe object. The resulting object was then placed over the blue and yellow checkered plane which meets a pinkish-purple clouded sky at the horizon. The object's surface was given a pure reflective finish and one can see the plane reflected in its side and the sky and clouds reflected in its top. Light sources were then placed so as to cast the object's shadow onto the plane.

Chapter 9 – The Epitrochoid

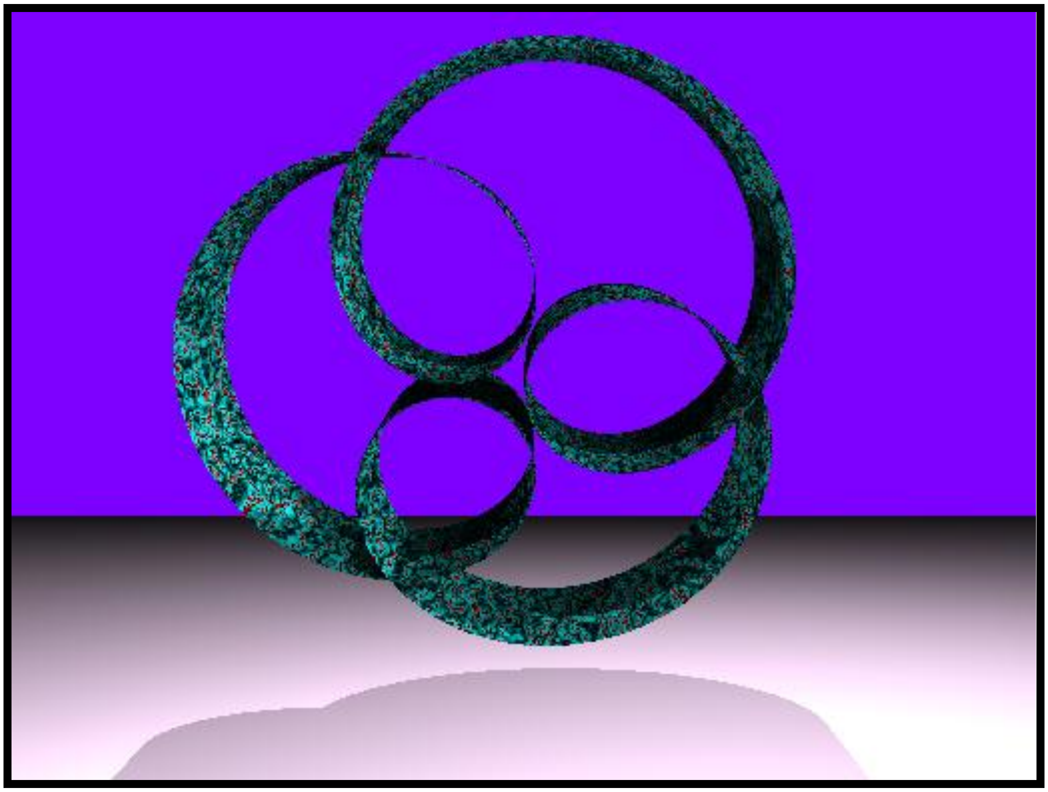

Figure 9-1: A Three-Dimensional Version of an Epitrochoid

The cross-section of the object in the above figure is that of an Epitrochoid with parameters (a, b, h) = (6, 2, 7). It was created by extruding a plane Epitrochoid with those parameters into the third dimension (i.e., the dimension normal to the plane of the page). It was then given a finish of green marble flecked with red and placed over an infinite gray plane which meets the slate blue sky at the horizon. Light sources were placed so as to illuminate the object and cast part of its shadow on the gray plane.

9.1 Introduction

The Epitrochoid was first described by Albrecht Dürer in 1525, who called the curve "spider lines" because he thought the curve bore resemblance to an arachnid. Olaf Roemer also studied Epitrochoids in 1674 in connection with his research concerning gear teeth. Through the ensuing centuries the curve was examined by a variety of mathematicians including Leibniz, Newton, L'Hopital, Desargues, and the Bernouillis. Today, Epitrochoids can be found in rotary combustion engines by observing the path that the rotor tip of the eccentric shaft traces out upon revolving.

The Epitrochoid is the locus of a point, P, that is rigidly attached to a small circle of radius b which rolls without slippage around the outside of a larger circle of radius a. Doesn't this sound just like the Epicycloids that were studied in the previous chapter? Indeed—however, for Epicycloids the point P was confined to the circumference of the rolling circle. Not so for the Epitrochoid. The point P may be internal to the rolling circle or may be external to it. (For the external case, consider it to be on an extension of the radius of the rolling circle.) Obviously, this makes Epicycloids merely a special case of the Epitrochoid (namely when P is on the circumference).

9.2 Equations and Graph of the Epitrochoid

If we let the distance from the center of the moving circle to the point P be h then the point P at t = 0 can be represented in coordinate form by its distance from the origin (see Figure 9-2). That is, $P = (a + b - h, 0)$. As the smaller circle revolves counterclockwise around the larger circle, point P moves to the location shown in the

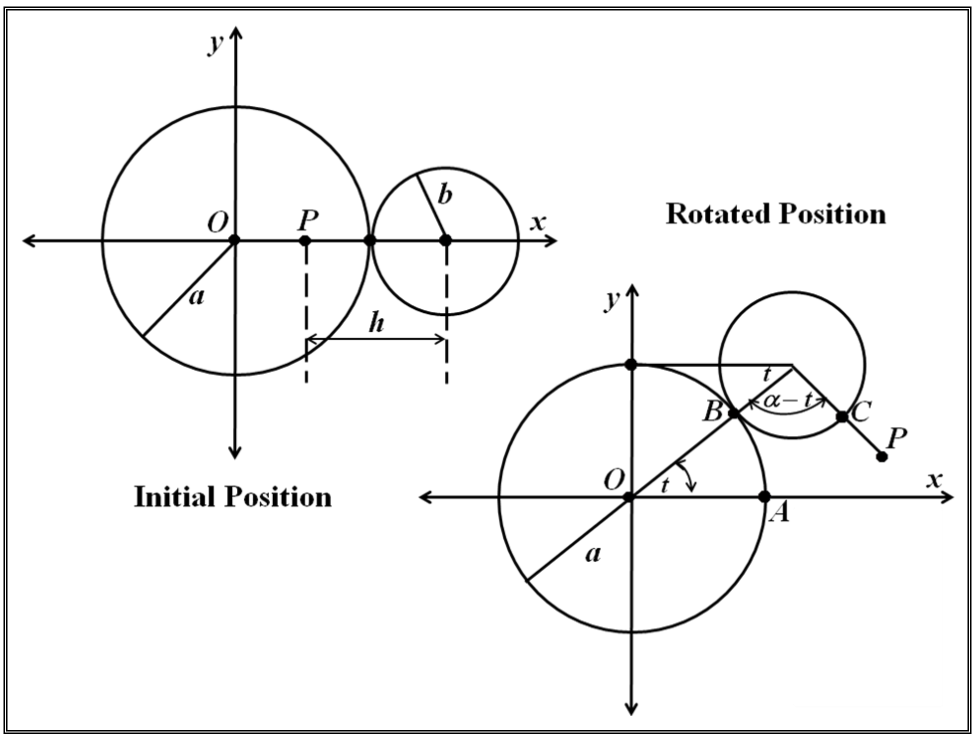

Figure 9-2: Epitrochoid Position of Point *P* at Time *t*

rotated position of Figure 9-2. At this time, the coordinates of point P can be described with the equation

$$P = [(a + b)\cos t - h\cos\alpha, (a + b)\sin t - h\sin\alpha].$$

However, as the small circle rolls around the larger one, it travels the same arc-length distance, that is, arc AB = arc BC, or $at = b(\alpha - t)$. Hence,

$$\alpha = \frac{a+b}{b}t.$$

We therefore have the parametric equations of the Epitrochoid, namely

$$\begin{aligned} x &= (a+b)\cos t - h\cos\tfrac{a+b}{b}t \\ y &= (a+b)\sin t - h\sin\tfrac{a+b}{b}t \end{aligned} \qquad -\pi < t < \pi \qquad \textbf{Equation 9-1}$$

Note how similar this equation is to that of the parametric equation for the Epicycloid (Equation 8-1). In fact, the only difference is the factor h in the second term of each component of Equation 9-1 versus b in Equation 8-1. And, of course, in the case of an Epicycloid, $h = b$, so this is consistent.

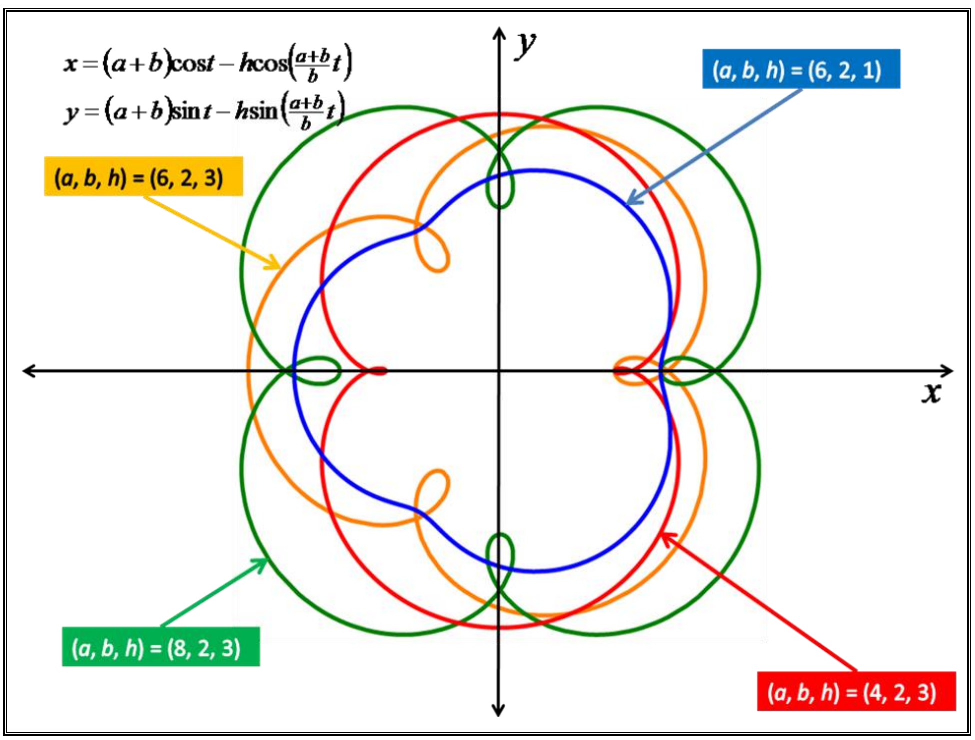

Figure 9-3: Graphs of a Variety of Epitrochoids

Figure 9-3 represents a graph of four different Epitrochoids. Note that the three parameters, (a, b, h) completely specify the curve. Further, if $b < h$, the curve will have

$\frac{a+b}{b}-1$ inner loops provided that $\frac{a+b}{b}$ is an integer. Of course, if $b = h$, as we have already learned, the curve degenerates to that of an Epicycloid. Finally, if $b > h$, the curve has no loops and takes a general form like that shown in the blue graph of Figure 9-3.

The equation of the tangent line to the Epitrochoid at the point $t = q$ is

$$y = \frac{\left(b\cos q - h\cos\frac{a+b}{b}q\right)\cdot x}{h\sin\frac{a+b}{b}q - b\sin q} + \frac{h(a+2b)\cos\frac{a}{b}q - \left(h^2 + ab + b^2\right)}{h\sin\frac{a+b}{b}q - b\sin q} \qquad \textbf{Equation 9-2}$$

9.3 Analytical and Physical Properties of the Epitrochoid

Using the parametric representation of the Epitrochoid given in Equation 9-1, the following subparagraphs delineate further properties of the Epitrochoid.

9.3.1 Derivatives of the Epitrochoid

- $\dot{x} = \dfrac{a+b}{b}\left[h\sin\left(\frac{a+b}{b}t\right) - b\sin t\right]$.

- $\ddot{x} = \dfrac{h(a+b)^2}{b^2}\cos\left(\frac{a+b}{b}t\right) - (a+b)\cos t$.

- $\dot{y} = \dfrac{a+b}{b}\left[b\cos t - h\cos\left(\frac{a+b}{b}t\right)\right]$.

- $\ddot{y} = \dfrac{h(a+b)^2}{b^2}\sin\left(\frac{a+b}{b}t\right) - (a+b)\sin t$.

- $y' = \dfrac{b\cos t - h\cos\left(\frac{a+b}{b}t\right)}{h\sin\left(\frac{a+b}{b}t\right) - b\sin t}$.

- $y'' = \dfrac{b^3 + h^2(a+b) - bh(a+2b)\cos\frac{a}{b}t}{(a+b)\left(h\sin\frac{a+b}{b}t - b\sin t\right)^3}$.

9.3.2 Metric Properties of the Epitrochoid

It makes no sense to talk about the area of an Epitrochoid if the curve is not closed or if the curve loops back upon itself. For closure, we have learned that $\frac{a+b}{b}$ must be an integer; let us call that integer n. To eliminate the loops, we have learned that h must be less than or equal to b (i.e., $h \le b$). Therefore, the parametric equation of the Epitrochoid, for these conditions becomes

$$x = bn\cos t - h\cos nt \quad \text{and} \quad y = bn\sin t - h\sin nt \qquad \textbf{Equation 9-3}$$

and we may calculate the area using the formula

$$A = \frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt\,.$$

Now, making the laborious calculation, we find that the value of the integrand is,

$$x\frac{dy}{dt} - y\frac{dx}{dt} = b^2n^2 + h^2n - bhn(n+1)\cos(n-1)t.$$

Therefore,

$$A = \frac{b^2n^2}{2}\int_0^{2\pi}dt + \frac{h^2n}{2}\int_0^{2\pi}dt - \frac{bhn(n+1)}{2}\int_0^{2\pi}\cos(n-1)t\,dt\,.$$

Integrating and evaluating the results, we find that

$$A = \pi b^2n^2 + \pi h^2n - \frac{bhn(n+1)}{2(n-1)}\sin 2\pi n$$

However, $\sin 2\pi n$ is always zero since n is an integer, hence we find that

$$A = \pi n\left(b^2n + h^2\right) = \pi(a+b)^2 + \frac{\pi h^2}{b}(a+b), \quad \text{for } 0 < h \le b.$$

If r denotes the distance from the origin to the curve, then

$$r = \sqrt{(a+b)^2 + h^2 - 2h(a+b)\cos\tfrac{a}{b}t}\,.$$

If p denote the distance from the origin to the tangent line, then

$$p = \frac{h(a+2b)\cos\frac{a}{b}t - h^2 - b(a+b)}{\sqrt{h^2 + b^2 - 2bh\cos\frac{a}{b}t}}\,.$$

9.3.3 Curvature of the Epitrochoid

If ρ represents the radius of curvature of the Epitrochoid, then

$$\rho = \frac{(a+b)\left(h^2 + b^2 - 2bh\cos\frac{a}{b}t\right)^{3/2}}{h^2(a+b) + b^3 - bh(a+2b)\cos\frac{a}{b}t}\,.$$

If (α, β) denotes the coordinates of the center of curvature for the Epitrochoid, then

$$\alpha = \frac{ah^2(a+b)\cos t - abh(a+b)\cos t\cos\frac{a}{b}t + ab^2h\cos\frac{a+b}{b}t - abh^2\cos\frac{a}{b}t\cos\frac{a+b}{b}t}{h^2(a+b) + b^3 - bh(a+2b)\cos\frac{a}{b}t} \quad \text{and}$$

$$\beta = \frac{ah^2(a+b)\sin t - abh(a+b)\sin t\cos\frac{a}{b}t + ab^2 h\sin\frac{a+b}{b}t - abh^2\sin\frac{a+b}{b}t\cos\frac{a}{b}t}{h^2(a+b)+b^3-bh(a+2b)\cos\frac{a}{b}t}.$$

9.3.4 Angles for the Epitrochoid

If ψ is the angle between the tangent and the radius vector at the point of tangency to the general Epitrochoid, then

$$\tan\psi = \frac{(a+b)b \;\; + h^2 - h(a+2b)\cos\frac{a}{b}t}{ah\sin\frac{a}{b}t}.$$

If ϕ denotes the tangential angle, then

$$\tan\phi = \frac{b\cos t - h\cos\left(\frac{a+b}{b}t\right)}{h\sin\left(\frac{a+b}{b}t\right) - b\sin t}.$$

If θ denotes the radial angle, then

$$\tan\theta = \frac{(a+b)\sin t - h\sin\frac{a+b}{b}t}{(a+b)\cos t - h\cos\frac{a+b}{b}t}.$$

9.4 Geometric Properties of the Epitrochoid

As already alluded to, for $\frac{a+b}{b} = n$ an integer, the Epitrochoid is closed and there are $n - 1$ inner loops whenever $h > b$. When $h < b$, the $n - 1$ loops become $n - 1$ "indentations" in the overall line of the curve but the curve does not cross itself as it must to form a loop. Further, these "indentations" are not cusps, as the derivative of the curve exists in every neighborhood of the "indentation." The curve is always symmetric about the *x*-axis, and if *n* is an odd integer, it is also symmetric about the *y*-axis. The curve is completely contained within a circle defined by $|r| \le a+b+h$.

9.5 Dynamic Geometry of the Epitrochoid

The next two subsections present two dynamic geometry constructions for the Epitrochoid.

9.5.1 The Geometry of the Epitrochoid Illuminated

If you think the explanation involving "indentations" in section 9.4 leaves something to be desired, perform the following simple GSP construction and the idea of the "indentation" should become clear; Table 9-1 contains this construction.

Table 9-1: The Geometry of the Epitrochoid Illuminated

1. Draw circle AB with center at A and passing through point B	6. Let C_2 be the image of C_1 rotated about point C' by ∠BAC
2. Let C be a random point on the circumference of circle AB	7. Draw line C'C_2
3. Let C' be the image of C dilated about A by a factor of 5.0	8. Let D be a random point on line C'C_2
4. Draw circle C'C with center at C' and passing through point C	9. Construct the locus of D while point C traverses circle AB
5. Let C_1 be the image of C rotated about point C' by ∠BAC	10. Change the color and line thickness of the locus

Now, drag point D along line C'C_2 and note how the locus of point D changes. When point D and point C' coincide, the locus is a circle. As point D moves away from point C', the locus becomes slightly elongated and flattened on two opposing sides. As point D continues to move away from point C', the flattened portions becomes two "indentations" which, as point D continues moving, becomes two cusps (forming a Nephroid) and then eventually two loops. Of course, the flattened and/or "indentation" configuration corresponds to an Epitrochoid with parameter $h < b$. The cusp configuration corresponds to an Epitrochoid with parameter $h = b$, or as we have learned, an Epicycloid. Finally, the loop configuration corresponds to an Epitrochoid with parameter $h > b$.

9.5.2 An Epitrochoid to Play With

The construction found in Table 9-2 is similar to the construction in the "Can We Build a Better Mousetrap?" section of Chapter 8, except that the tracing point can be dragged to simulate different values of the parameter h.

Table 9-2: An Epitrochoid to Play With

1. Draw circle AB with center at A and passing through point B	10. Let H be a random point on the circumference of circle GG'
2. Draw segment CD across the entire width of the screen top	11. Draw line segment GH
3. Let E be a random point on line segment CD	12. Construct the parallel to line segment GH through point F
4. Let F be a random point on the circumference of circle AB	13. Let I be a random point on the parallel line
5. Let F' be the image when F is translated by vector C → E	14. Trace point I and change its color
6. Draw circle FF' with center at F and passing through point F'	15. Let m_1 be a measure of the radius of circle AB
7. Let G be a random point anywhere in the plane	16. Let m_2 be a measure of the radius of circle GG'
8. Let G' be the image when G is translated by vector C → E	17. Calculate m_2 / m_1
9. Draw circle GG' with center at G and passing through point G'	18. Animate points H and F on circles GG' and AB, respectively

By dragging point E along line segment CD so that the ratio of m_2 to m_1 is an integer (or as close to an integer as one can obtain), one can cause point I (the tracing point) to trace a closed curve (or very close to it). By dragging point I along the parallel so that it is internal to circle FF' and then adjusting the ratio to be less than one (< 1), we can obtain Epitrochoids with the so called "indentations." As a matter of fact, adjusting the ratio as close to 0.25 while keeping point I internal to circle FF', we can trace the Epitrochoid shape found in the rotary combustion engine (i.e., the path that the rotor tip of the eccentric follows when revolving).

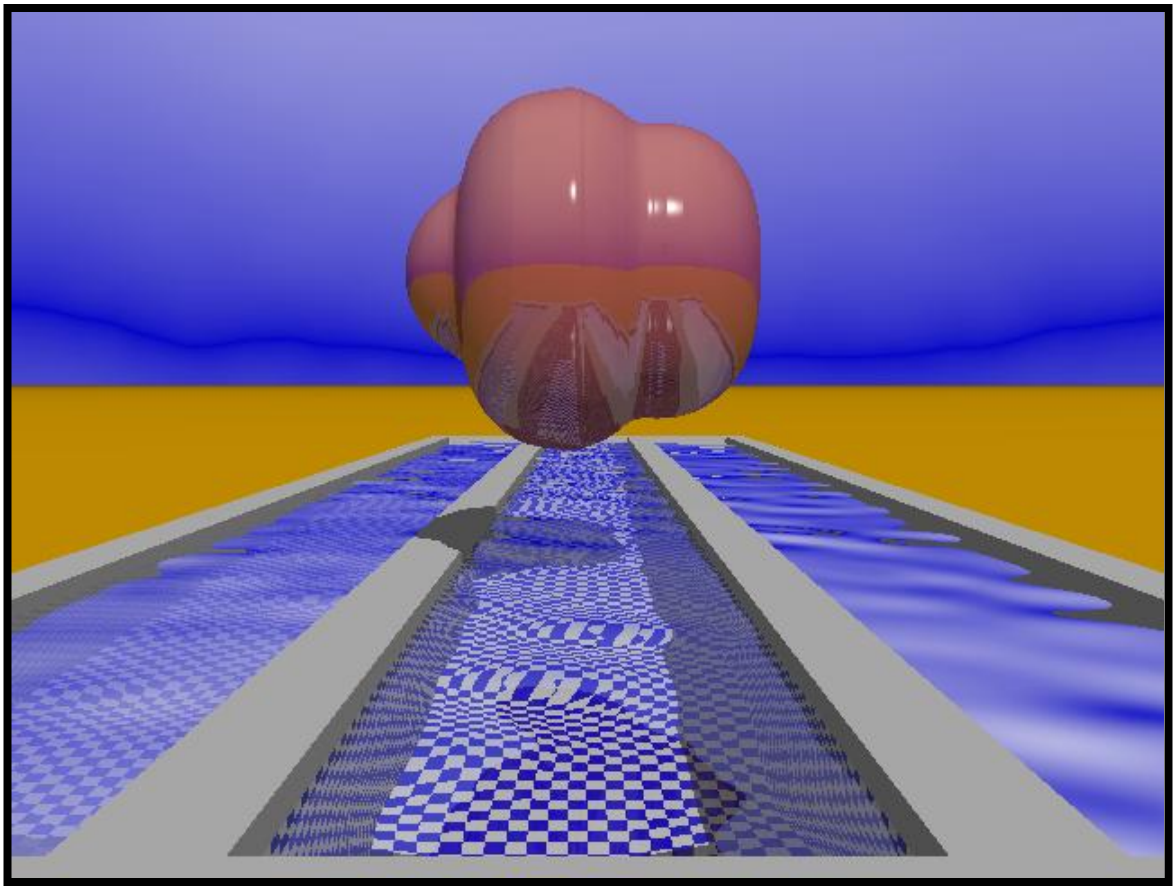

Figure 9-4: An Epitrochoid Solid of Revolution

In order to obtain the object above, an Epitrochoid with parameters (a, b, h) = (20, 4, 2) was revolved about the x-axis. It was then placed above the reflecting pool and a light source was located so as to cast the object's shadow onto the pool. Note how the object's shiny surface also reflects the pool itself as well as the horizon line, sky, and ground.

Chapter 10 – The Deltoid

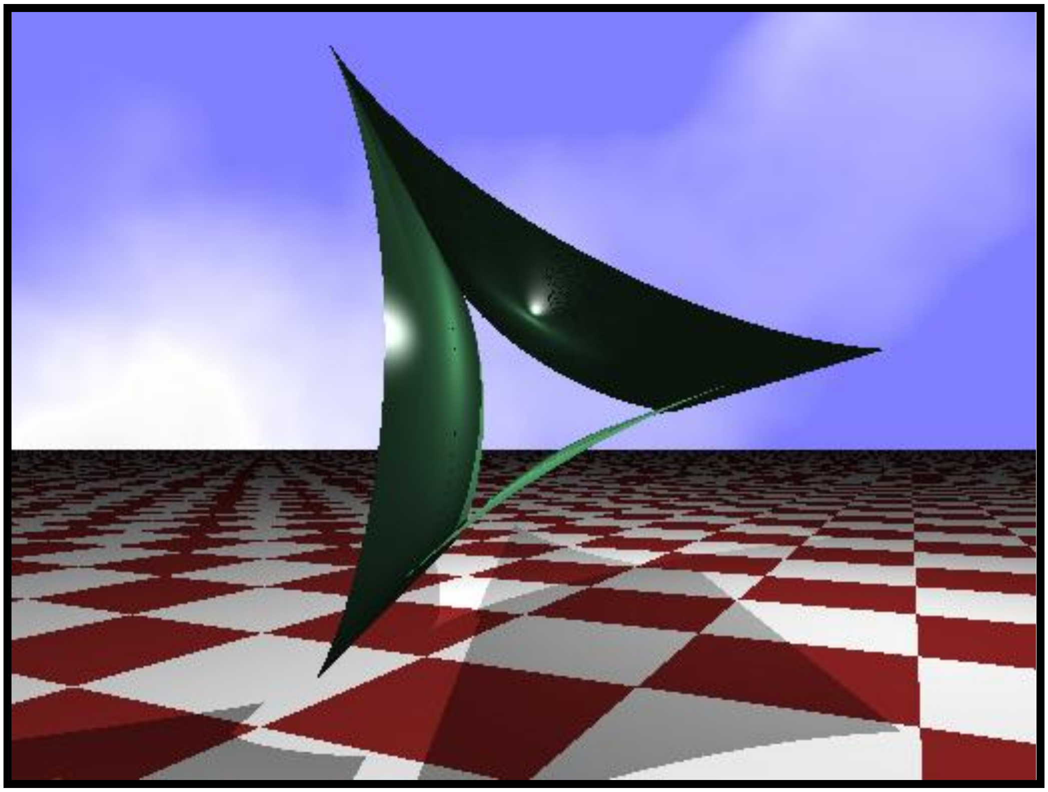

Figure 10-1: The Deltoid in Three Dimensions

This three-dimensional version of the Deltoid was rendered by extruding into the third dimension using its parametric representation. The resulting object was then given a shiny, aquamarine finish and situated over the red and white checkered plane. The plane meets a bright-blue sky at the horizon. Light sources were placed so as to cast shadows of the object on the plane in various locations. Also note the light sources reflecting off the object itself—one on the outer surface and one on the inner surface.

10.1 Introduction to the Deltoid

Conceived by Leonhard Euler in 1745, the Deltoid (sometimes called the Tricuspoid) was studied in connection with caustic curves. It was also investigated by Steiner in 1856 and is sometimes called Steiner's Hypocycloid. In point of fact, the Deltoid is a member of a family of curves called Hypocycloids. This variety of cycloid is obtained as the locus of a point attached to the circumference of one circle rolling along the circumference of another circle, but rolling interior to it. In other words, Hypocycloids are very much like the Epicycloids that we studied Chapter 8, but are produced by a rotating circle interior to the fixed circle instead of exterior. The Deltoid is the specific Hypocycloid where the radius of the fixed circle is three times as large as the radius of the rolling circle.

10.2 Equations and Graph of the Deltoid

To find the parametric equations for the Deltoid, let a be the radius of the rolling circle and $3a$ that of the fixed circle, as shown in Figure 10-2. The fixed circle has equation $x^2 + y^2 = 9a^2$, and for $t = 0$, the point P has coordinates $P = (3a, 0)$. Then, after the moving circle has rolled through an angle t (that is, the line O_1O_2 connecting the centers of the two circles makes an angle t with the x-axis), the point P has rolled around O_2 through an angle $3t - t = 2t$. If the coordinates of point O_2 are $O_2 = (O_x, O_y)$, then $P = (O_x, O_y) + (a\cos 2t, -a\sin 2t)$. But, $(O_x, O_y) = (2a\cos t, 2a\sin t)$. Therefore, the parametric representation for the Deltoid is

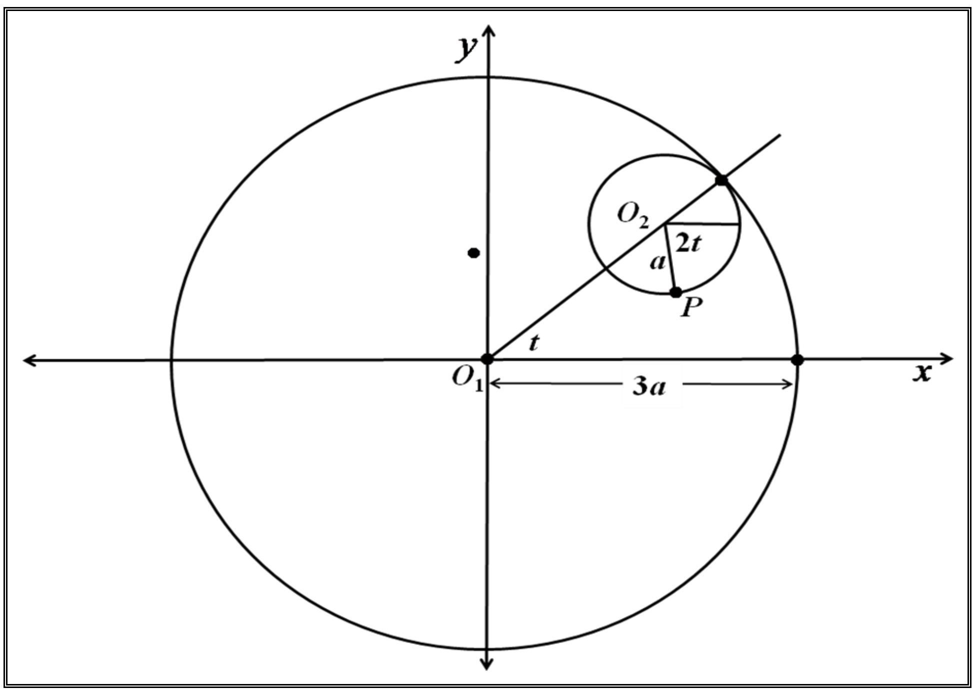

Figure 10-2: Derivation of the Deltoid Equations

$$(x, y) = a(2\cos t + \cos 2t, 2\sin t - \sin 2t), \quad -\pi < t < \pi \qquad \textbf{Equation 10-1}$$

Eliminating the parameter t from these two equations yields the Cartesian form of the Deltoid as

$$\left(x^2 + y^2 + 12ax + 9a^2\right)^2 = 4a\left(2x + 3a\right)^3 \quad \textbf{Equation 10-2}$$

If the pedal point is taken as the center of the fixed circle, then the pedal equation is

$$r^2 + 8p^2 = 9a^2 \quad \textbf{Equation 10-3}$$

Similarly, the Whewell equation is

$$3s = 8a\cos 3\varphi \quad \textbf{Equation 10-4}$$

The Cesáro equation is

$$9s^2 + \rho^2 = 64a^2 \quad \textbf{Equation 10-5}$$

Finally, the equation of the tangent line to the Deltoid at the point $t = q$ is

$$\sin q \cdot y = \left(\cos q - 1\right) \cdot x + a\left(1 - \cos q\right)\left(1 + 2\cos q\right). \quad \textbf{Equation 10-6}$$

Figure 10-3 depicts the graph of the Deltoid.

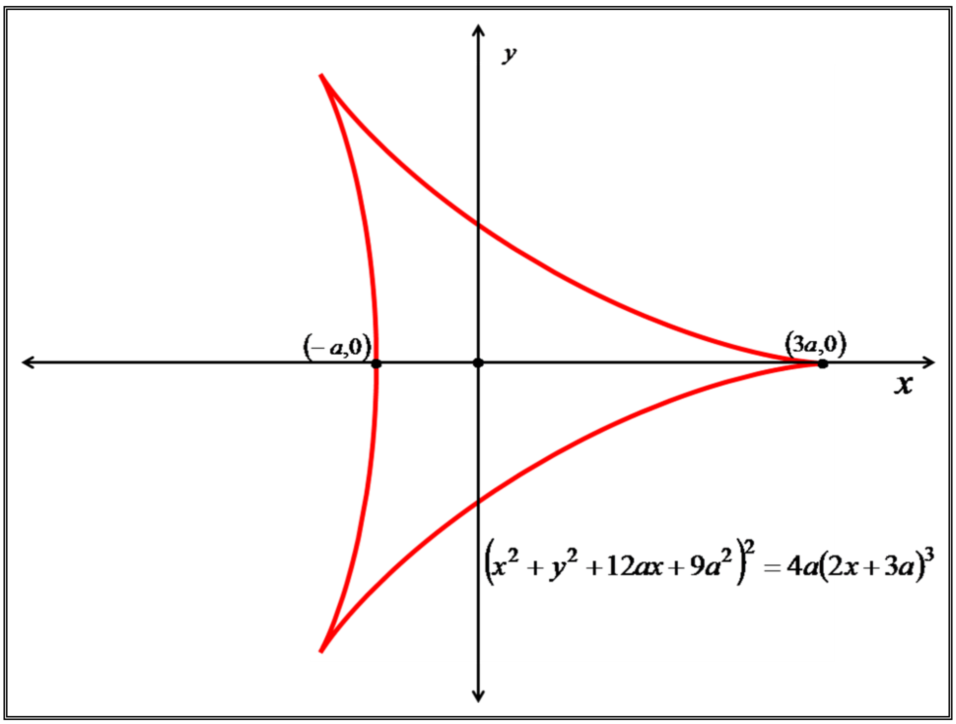

Figure 10-3: The Graph of the Deltoid

10.3 Analytical and Physical Properties of the Deltoid

Using the parametric representation of the Deltoid given in Equation 10-1, i.e., $x = 2a\cos t + a\cos 2t$ and $y = 2a\sin t - a\sin 2t$, the following subsections present the Deltoid's derivatives, metric properties, curvature, and angles of interest.

10.3.1 Derivatives of the Deltoid

- $\dot{x} = -2a\sin t(1 + 2\cos t)$.
- $\ddot{x} = 4a - 2a\cos t(1 + 4\cos t)$.
- $\dot{y} = 2a + 2a\cos t(1 - 2\cos t) = 2a(1 + 2\cos t)(1 - \cos t)$.
- $\ddot{y} = 2a\sin t(4\cos t - 1)$.
- $y' = \dfrac{\cos t - 1}{\sin t} = \tan\left(\pi - \dfrac{t}{2}\right)$.
- $y'' = \dfrac{1 - \cos t}{2a\sin^3 t(1 + 2\cos t)}$.

10.3.2 Metric Properties of the Deltoid

The Deltoid's length can be calculated by considering one branch of the curve, calculating its length, and then multiplying that result by three since each of the three branches are of equal length. Therefore, the length of the branch from $t = 0$ to $t = 2\pi/3$ is

$$s = \int_0^{2\pi/3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt .$$

$$\frac{dx}{dt} = -2a(\sin t + \sin 2t) \quad \text{and} \quad \frac{dy}{dt} = 2a(\cos t - \cos 2t)$$

$$\left(\frac{dx}{dt}\right)^2 = 4a^2\sin^2 t + 8a^2\sin t\sin 2t + 4a^2\sin^2 2t$$

$$\left(\frac{dy}{dt}\right)^2 = 4a^2\cos^2 t - 8a^2\cos t\cos 2t + 4a^2\cos^2 t .$$

Adding these last two expressions together yields

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 8a^2(1 - \cos 3t).$$

However, $1 - \cos 3t = 2\sin^2(3t/2)$. We therefore have

$$s = 4a\int_0^{2\pi/3} \sin\frac{3t}{2}\, dt = -\frac{8a}{3}\int_0^{2\pi/3} d\left(\cos\tfrac{3t}{2}\right) = -\frac{8a}{3}\left[\cos\tfrac{3t}{2}\right]_0^{2\pi/3} = \frac{16a}{3}$$

We may therefore conclude that the total length of the Deltoid is $16a$, where a is the radius of the rolling circle.

The area of the Deltoid may be computed from the formula

$$A = \frac{1}{2}\int_{\alpha}^{\beta}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt,$$

where, in this case, $\alpha = 0$ and $\beta = 2\pi$. From the previous calculation of the Deltoid's length, we have expressions for dy/dt and dx/dt. Therefore,

$$x\frac{dy}{dt} = 4a^2\cos^2 t - 2a^2\cos t\cos 2t - 2a^2\cos^2 2t, \quad \text{and}$$

$$y\frac{dx}{dt} = -4a^2\sin^2 t - 2a^2\sin t\sin 2t + 2a^2\sin^2 2t.$$

Forming the difference from these last two expressions yields

$$x\frac{dy}{dt} - y\frac{dx}{dt} = 2a^2 - 2a^2(\cos t\cos 2t - \sin t\sin 2t) = 2a^2(1 - \cos 3t).$$

Therefore, the area is

$$A = \frac{1}{2}\int_0^{2\pi} 2a^2(1-\cos 3t)dt = a^2\int_0^{2\pi} dt - \frac{a^2}{3}\int_0^{2\pi} d(\sin 3t) = 2\pi a^2.$$

If p is the distance from the origin to the Deltoid's tangent, then

$$p = \frac{1}{2}a(1 + 2\cos t)\sqrt{2(1-\cos t)}.$$

If r is the distance from the origin to the curve, then

$$r = a\sqrt{5 + 4\cos 3t}.$$

10.3.3 Curvature of the Deltoid

If ρ is the radius of curvature of the Deltoid, then

$$\rho = -8a\sin\frac{3t}{2}.$$

If (α, β) are the coordinates of the center of curvature of the Deltoid, then

$$\alpha = 3a\left(1 + 2\cos t - 2\cos^2 t\right) \quad \text{and} \quad \beta = 6a\sin t(1 + \cos t).$$

10.3.4 Angles for the Deltoid

If θ is the radial angle of the Deltoid, then

$$\tan\theta = \frac{2\sin t(1-\cos t)}{2\cos t(1+\cos t)-1}.$$

If ψ is the tangential-radial angle of the Deltoid, then

$$\tan\psi = \frac{(1+2\cos t)(1-\cos t)}{3\sin t(1-2\cos t)}.$$

If ϕ is the tangential angle of the Deltoid, then

$$\phi = \pi - t/2.$$

10.4 Geometric Properties of the Deltoid

- Intercepts: (3*a*, 0); (-*a*, 0); $\left(0, \pm a\sqrt{6\sqrt{3}-9}\right)$.
- Extent: $-\frac{3a}{2} \le x \le 3a;\quad -\frac{3\sqrt{3}}{2}a \le y \le \frac{3\sqrt{3}}{2}a.$
- Symmetry: The Deltoid is symmetric about the *x*-axis and the lines $y = \pm\sqrt{3}x$.
- Cusps: (3*a*, 0); $\left(-\frac{3}{2}a,\ \pm\frac{3\sqrt{3}}{2}a\right)$.

10.5 Dynamic Geometry of the Deltoid

The following subsections provide a variety of different and interesting constructions of the Deltoid.

10.5.1 The Deltoid as a Hypocycloid

As briefly addressed in Section 10.1, the Deltoid is defined as a specific Hypocycloid, one where the radii of the fixed circle and rotating circle are in the ratio of 3 to 1. Our first dynamic geometry construction, found in Table 10-1, is based on this very definition.

Table 10-1: The Deltoid as a Hypocycloid

1. Draw circle AB with center at A and passing through point B	6. Let C_2 be the image when C_1 is rotated about A' by ∠CAB
2. Let C be a random point on the circumference of circle AB	7. Let C_3 be the image when C_2 is rotated about A' by ∠CAB
3. Let A' be the image when A is dilated about C by ⅓	8. Trace point C_3 and change its color
4. Draw circle A'C with center at A' and passing through point C	9. Animate point C around circle AB
5. Let C_1 be the image of C rotated about point A' by ∠CAB	

Step 1, of course, is to obtain the fixed circle while steps 2 through 4 construct the moving circle whose radius is one-third that of the fixed circle. Steps 5 through 7 then perform the necessary functions in order to simulate a point on the circumference of the moving circle that rolls without slipping around the interior of the fixed circle. Note that point C_2 will trace an ellipse.

10.5.2 The Pedal Curve of the Deltoid

Remember, the pedal curve of a given curve C is merely the locus of the intersection point created by dropping a perpendicular from the pedal point to a tangent of C. Therefore, if we can construct a tangent to the Deltoid, it will be "duck soup" to construct the Deltoid's pedal curve. Such a construction is found below in Table 10-2.

Table 10-2: The Pedal Curve of the Deltoid

1. Draw circle AB with center at A and passing through point B	9. Construct the locus of C_3 while point C traverses circle AB
2. Let C be a random point on the circumference of circle AB	10. Draw line segment CC_3
3. Dilate circle AB about point C by a factor of ⅓	11. Construct $P_1 \perp$ to line segment CC_3 through point C_3
4. Let A_1 be the image when A is dilated about C by ⅓	12. Let D be a random point in the plane
5. Draw line segment AC	13. Construct $P_2 \perp$ to P_1 through point D
6. Let C_1 be the image when C is rotated about A_1 by $\angle CAB$	14. Let point E be the intersection of perpendiculars P_1 and P_2
7. Let C_2 be the image when C_1 is rotated about A_1 by $\angle CAB$	15. Trace point E and change its color
8. Let C_3 be the image when C_2 is rotated about A_1 by $\angle CAB$	16. Animate point C around circle AB

In the above construction, point D acts as the pedal point. You can drag it anywhere in the plane so as to change the pedal curve. It is particularly instructive to drag it to where it is internal to the Deltoid and specifically when it is coincident with point A, the center of circle AB. If instead of tracing point E, you construct the locus of point E as point C traverses circle AB (i.e., the locus of the pedal curve), then when you drag point D, you can see how the pedal curve changes.

10.5.3 The Deltoid as an Envelope of Simson Lines

First of all, what is a Simson line? Let there be any triangle inscribed in a circle and let the point P be any random point on the circumference of the circle. On one of the sides of the triangle mark a point Q_1, such that the line PQ_1 is perpendicular to the side so chosen (one may have to extend the side of the triangle in order to find the point Q_1). Now do the same thing for the other two sides of the triangle with points Q_2 and Q_3, respectively. It turns out that the points Q_1, Q_2, and Q_3 are collinear and the line passing through them is called a Simson line. The envelope of the set of all Simson lines (the set composed from all points, P) form a Deltoid, as can be seen from the construction that follows in Table 10-3.

Table 10-3: The Deltoid as an Envelope of Simson Lines

1. Draw circle AB with center at A and passing through point B	8. Let point H be the intersection of line EF and perpendicular P_2
2. Let C, D, E, and F be four random points on circle AB	9. Draw line DF
3. Draw line DE	10. Construct $P_3 \perp$ to line DF through point C
4. Construct $P_1 \perp$to line DE through point C	11. Let point I be the intersection of line DF and perpendicular P_3
5. Let G be the intersection of line DE and perpendicular P_1	12. Draw line GH
6. Draw line EF	13. Trace line GH and change its color
7. Construct $P_2 \perp$ to line EF through point C	14. Animate point C around circle AB

Note that in this construction, we never really drew the triangle, however, it is implied (it is ΔDEF). Note also, that when we draw line GH (step 12), that line also passes through point I, i.e., G, H, and I are all collinear. See Figure 10-4 for a snapshot of this construction.

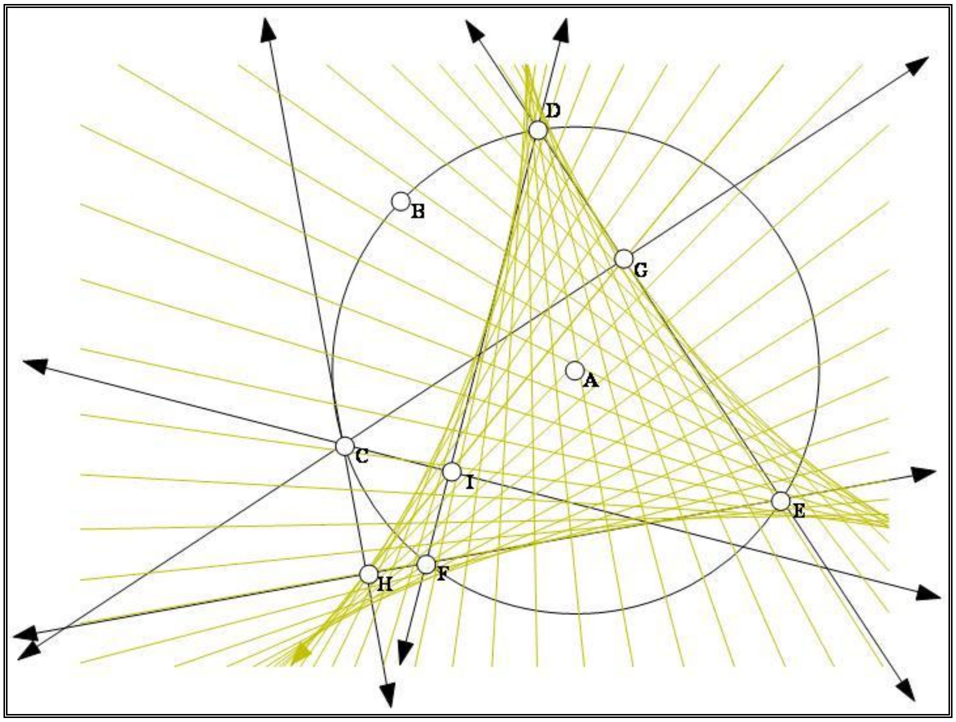

Figure 10-4: The Deltoid as an Envelope of Simson Lines

10.5.4 The Evolute of the Deltoid

As addressed earlier (Chapter 1), the evolute of a curve is the locus of its center of curvature. The construction delineated below in Table 10-4 is the evolute of the Deltoid, which, it turns out, is another Deltoid. Obviously, to use this construction as a means of generating a Deltoid requires one to possess another construction of the Deltoid in order to begin constructing its evolute. Bottom line—obviously this is not a very good "generation from scratch" technique!

Table 10-4: The Evolute of the Deltoid

1. Draw circle AB with center at A and passing through point B	8. Draw line CC_2
2. Let C be a random point on the circumference of circle AB	9. Let C_3 be the image when C_2 is rotated about point C_1 by 180º
3. Let m_1 be the measure of angle ∠BAC	10. Draw line AC_3
4. Calculate $m_2 = -3m_1$	11. Let point D be the intersection of lines AC_3 and CC_2
5. Let C_1 be the image when C is dilated about A by ⅔	12. Trace point D and change its color
6. Draw circle C_1C with center at C_1 and passing through point C	13. Animate point C around circle AB
7. Let C_2 be the image when point C is rotated about C_1 by $\angle m_2$	

When executing this construction, make sure that the unit for angle measurement in the preferences window under the display menu of GSP is set for either radian measure or directed degree measure. (Of course, if it is set for radian measure, then step 9 should really read "Let C_3 be the image when C_2 is rotated about C_1 by 3.14159 … radians".) Note that point D, the intersection point of lines AC_3 and CC_2 (step 11), is the center of curvature for the Deltoid traced by point C_3. Also note that tracing point C_2 produces the Deltoid for which the trace of point D is the evolute. And note how the evolute is circumscribed about the fixed circle (circle AB). Very neat!

10.5.5 The Deltoid as an Envelope of Osculating Circles

The previous construction, the evolute of the Deltoid, shows us how to locate the center of curvature. From that, it is a simple matter to construct the Deltoid's osculating circle. For this construction, execute steps 1 – 11 of section 10.5.4 and then perform the following three steps:

12. Draw circle DC_2 with center at point D and passing through point C_2.
13. Trace circle DC_2 and change its color.
14. Animate point C around circle AB.

Figure 10-5 illustrates this construction.

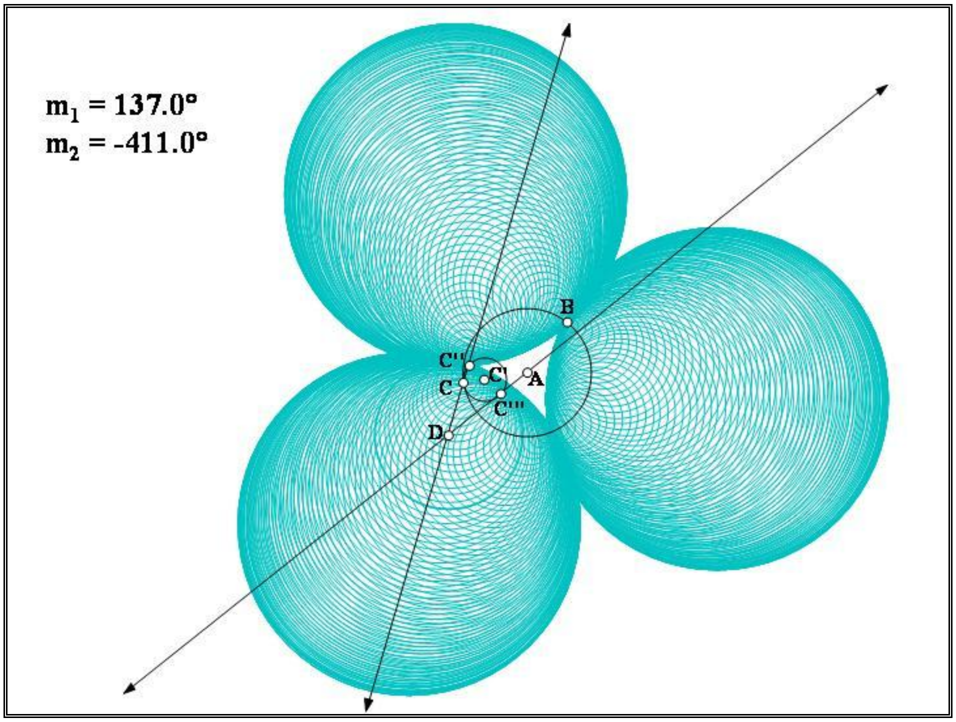

Figure 10-5: The Deltoid as an Envelope of Osculating Circles

The reader might ponder why this construction is even included in the text, given that it is so similar to the previous construction. After all, they are basically the same construction. If one is at all serious about learning about these curves, the previous construction is instructive because it shows that the evolute of the Deltoid is another Deltoid and, lo and behold, this is true for all Hypocycloids; that is, their evolute is another version of the same curve. However, this construction is included because it creates such a beautiful picture when the animation is executed – your author couldn't bear to leave it out!

10.5.6 A Rotating Deltoid

The construction delineated in Table 10-6 is quite elaborate and beautiful. It is worth reproducing.

Table 10-5: A Rotating Deltoid

1. Draw horizontal line AB	16. Draw circle GF' with center at G and passing through point F'
2. Let C be a random point on line AB	17. Draw circle GF" with center at G and passing through F"
3. Construct $P_1 \perp$ to line AB through point A	18. Let G' be the image when G is dilated about F" by ⅓
4. Draw circle AC with center at A and passing through point C	19. Draw circle G'F" with center at G' and passing through F"
5. Let D be a random point on the circumference of circle AC	20. Let H be a random point on circle GF"
6. Construct $P_2 \perp$ to P_1 through point D	21. Rotate circle G'F" about point G by ∠F"GH
7. Construct $P_3 \perp$ to line AB through point D	22. Let E_1 be the image when E is rotated about G by ∠F"GH
8. Let E be the intersection of line AB and perpendicular P_3	23. Let G" be the image when G' is rotated about G by ∠F"GH
9. Let point F be the intersection of perpendiculars P_1 and P_2	24. Let E_2 be the image when E_1 is rotated about G" by ∠HGF"
10. Draw line segment EF	25. Let E_3 be the image when E_2 is rotated about G" by ∠HGF"
11. Let F' be the image when F is dilated about A by ½	26. Let E_4 be the image when E_3 is rotated about G" by ∠HGF"
12. Let E' be the image when E is dilated about A by ½	27. Construct the locus of point E_4 as H traverses circle GF"
13. Draw line segment E'F'	28. Let points I and J be the intersections of circle AC and P_1
14. Let F" be the image when F' is rotated about point E' by 180º	29. Animate point D around circle AC
15. Let G be the midpoint of line segment E'F'	

Note that as the Deltoid rotates it remains tangent to line AB and perpendicular P_1. Further, point G' describes an ellipse, and line segment EF and line segment E'F' both describe a curve we have not yet examined called an Astroid (Chapter 11). Changing either one or both of these line segments to dashed segments, changing their color, and tracing them as the animation runs, paints quite an interesting picture.

10.5.7 The Deltoid as an Envelope of Straight Lines

In section 10.5.3 we showed how the Deltoid can be generated as an envelope of Simson lines. It's not only Simson lines that do the job, as can be seen from the construction of Table 10-6.

Table 10-6: The Deltoid as an Envelope of Straight Lines

1. Draw horizontal line AB	6. Let C' be the image when point C is reflected across line AB
2. Draw circle AB with center at A and passing through point B	7. Construct $P_2 \perp$ to line BC through point C'
3. Let C be a random point on the circumference of circle AB	8. Trace perpendicular P_2 and change its color
4. Construct $P_1 \perp$ to line AB through point C	9. Animate point C around circle AB
5. Draw line BC	

Alternately, here is a different construction for the Deltoid as an envelope of straight lines (see Table 10-7).

Table 10-7: An Alternate Straight Line Envelope Construction of the Deltoid

1. Draw horizontal line AB	6. Let B' be the image when B is reflected across P_1
2. Draw circle AB with center at A and passing through point B	7. Draw line B'C
3. Let C be a random point on the circumference of circle AB	8. Trace line B'C and change its color
4. Draw line BC	9. Animate point C around circle AB
5. Construct $P_1 \perp$ to line AB through point C	

Combining these two "straight-line constructions" in one sketch results in one Deltoid being superimposed upon the other Deltoid, but separated by 60°, thereby making quite an interesting graphic. (These two constructions are really not different from one another in terms of the geometry involved but are rather slight variations of one another, i.e., in terms of the line used to trace the Deltoid.)

10.5.8 A Deltoid-Nephroid Gear

Table 10-8 below presents a rotating Deltoid coupled, in this case, to a Nephroid. Note the similarities and differences between this construction and the rotating Deltoid of section 10.5.6.

Table 10-8: A Deltoid-Nephroid Gear

1. Draw circle AB with center at A and passing through point B	12. Let C_1 be the image when C is rotated about A_1 by $\angle BA_1D$
2. Let A_1 be the image when A is dilated about B by a factor of 3	13. Let C_2 be the image when C is rotated about A_2 by $\angle BA_2E$
3. Let A_2 be the image when A is dilated about B by –2	14. Let A_4 be the image when A is rotated about A_2 by $\angle BA_2E$
4. Create circle O_1 by dilating circle AB about B by a factor of 3	15. Let C_3 be the image when C_1 is rotated about A_3 by $\angle DA_1B$
5. Create circle O_2 by dilating circle AB about B by –2	16. Let C_4 be the image when C_2 is rotated about A_4 by $\angle BA_2E$
6. Let C be a random point on the circumference of circle AB	17. Let C_5 be the image when C_3 is rotated about A_3 by $\angle DA_1B$
7. Draw line segment BC	18. Let C_6 be the image when C_4 is rotated about A_4 by $\angle BA_2E$
8. Construct $P_1 \perp$ to segment BC through point C	19. Let C_7 be the image when C_5 is rotated about A_3 by $\angle DA_1B$
9. Let D be a random point on the circumference of circle O_1	20. Construct the locus of C_6 while point E traverses circle O_2
10. Let E be a random point on the circumference of circle O_2	21. Construct the locus of C_7 while point D traverses circle O_1
11. Let A_3 be the image when A is rotated about A_1 by $\angle BA_1D$	22. Animate point C around circle AB

As a suggestion, simultaneously animate point C on circle AB, point D on circle O_1, and point E on circle O_2. Then trace points C_1, C_2, C_3, C_4, and C_5. Point C_2 traces another Nephroid, point C_4 traces a 3-looped Epitrochoid, and point C_5 traces a Cardioid, while points C_1 and C_3 trace curves called Hypotrochoids (see Chapter 13).

10.5.9 The Deltoid from a Compass-Only Construction

This construction is essentially the same construction that was presented in section 8.5.3. However, instead of creating a three-cusped Epicycloid with the traced element, we now trace an element that yields the Deltoid directly (see Table 10-9).

Table 10-9: The Deltoid from a Compass-Only Construction

1. Draw circle AB with center at A and passing through point B	13. Let F be the unlabeled intersection of circles DE and B'C
2. Let C be a random point on the circumference of circle AB	14. Draw circle FB' with center at F and passing through point B'
3. Draw circle BC with center at B and passing through point C	15. Let points G and H be the intersections of circles FB' and B'C
4. Draw line segment AB	16. Draw circle GH with center at G and passing through point H
5. Let C' be the image of C reflected across line segment AB	17. Let I be the unlabeled intersection of circle GH and circle FB'
6. Draw circle C'B with center at C' and passing through point B	18. Draw circle IC with center at I and passing through point C
7. Draw line segment AC'	19. Let points J and K be the intersections of circles IC and CB'
8. Let B' be the image of B reflected across line segment AC'	20. Draw circle JC with center at J and passing through point C
9. Draw circle B'C with center at B' and passing through point C	21. Draw circle KC with center at K and passing through point C
10. Draw circle CB' with center at C and passing through point B'	22. Let L be the unlabeled intersection of circle KC and circle JC
11. Let points D and E be the intersections of circles B'C and CB'	23. Trace point L and change its color
12. Draw circle DE with center at D and passing through point E	24. Animate point C around circle AB

Instead of tracing point L, construct the locus of point L as point C traverses circle AB and then add the following steps to the construction.

25. Let point M be the unlabeled intersection of circle DE and circle CB'.
26. Draw circle ML.

Now rerun the animation and you will see that circle ML is the osculating circle to the Deltoid, constructed with a compass-only methodology. Spectacular!

10.5.10 Orthogonal Tangents to the Deltoid

Given any tangent to the Deltoid, one can always draw a second tangent that is perpendicular to the given tangent. See Table 10-10.

Table 10-10: Orthogonal Tangents to the Deltoid

1. Draw circle AB with center at A and passing through point B	14. Trace line segment CC_4 and change its color
2. Let C be a random point on the circumference of circle AB	15. Let C_5 be the image when C_4 is rotated about point A by 180º
3. Dilate circle AB about point A by a factor of ⅓	16. Construct $P_1 \perp$ to segment CC_4 through point C_4
4. Dilate circle AB about point C by a factor of ⅓	17. Change the color of Perpendicular P_1
5. Let A' be the image when A is dilated about C by ⅓	18. Let C_6 be the image when C_5 is rotated about A'' by 180º
6. Draw line segment AC	19. Let D be the intersection of line segment AC and P_1
7. Let C_1 be the image when C is rotated about point A by 180º	20. Draw line segment C_1C_6
8. Let A'' be the image when A' is rotated about point A by 180º	21. Trace line segment C_1C_6 and change its color
9. Let C_2 be the image when C is rotated about A' by ∠CAB	22. Construct $P_2 \perp$ to line segment C_1C_6 through point C_6
10. Let C_3 be the image when C_2 is rotated about A' by ∠CAB	23. Change the color of perpendicular P_2
11. Let C_4 be the image when C_3 is rotated about A' by ∠CAB	24. Let point E be the intersection of perpendiculars P_1 and P_2
12. Construct the locus of C_4 while point C traverses circle AB	25. Animate point C around circle AB
13. Draw line segment CC_4	

Figure 10-6 portrays a snapshot of this construction. In Figure 10-6, the two cyan colored lines are the two orthogonal tangents. They are tangent to the dark blue Deltoid at points C_4 and C_6, respectively. They intersect at point E and when the animation is executed point E revolves around the inner circle that was constructed in step 3. The locus of the intersection point of tangents to a curve meeting at a constant angle is an isoptic of the curve. In this case, the tangents meet at a constant 90° and trace a circle. Therefore, the circle is a 90°-isoptic (sometimes called an orthoptic) of the Deltoid. Line segments C_1C_6 and CC_4 are traced simply to fill in the space between the Deltoid and the outer circle; it makes for a more spectacular picture. Note that the Deltoid will also have two orthogonal normals. This same construction can be used to create the normals. Simply construct perpendiculars to the two tangent lines through the points of tangency. They are not shown in Figure 10-6.

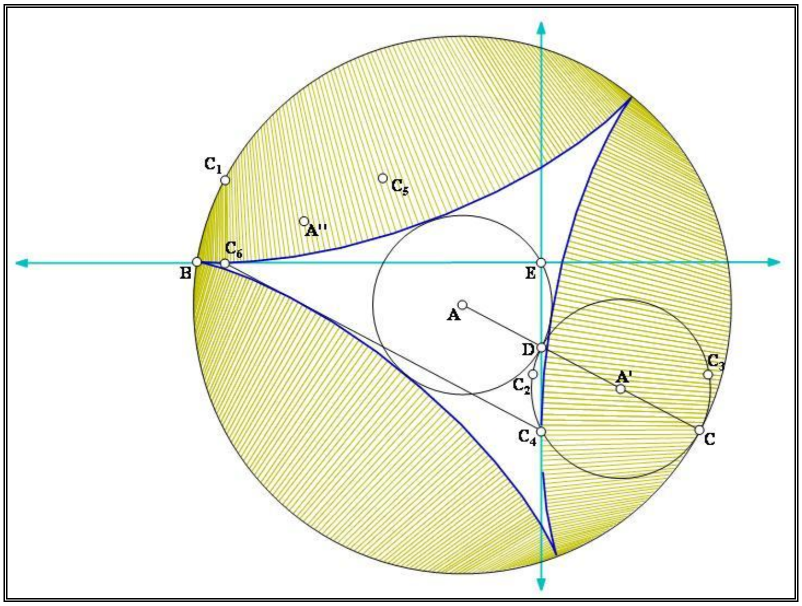

Figure 10-6: Orthogonal Tangents to the Deltoid

10.5.11 Two Deltoids for the Price of One

In sections 10.5.4 and 10.5.5 we presented the Evolute of the Deltoid and the Deltoid as an envelope of its osculating circle. As alluded to earlier, these two previous constructions are really the same; that is, the evolute is the locus of the centers of curvature and that locus is the same as the locus of the centers of the osculating circles. There is really nothing more to be learned by combining them into one construction; however, the resulting picture and dynamic geometry animation are truly beautiful. As a result, this combined construction is presented in Table 10-11 and shown in Figure 10-7. Note, however, that we do not actually construct the osculating circle, only its radius.

Table 10-11: Two Deltoids for the Price of One

1. Draw circle AB with center at A and passing through point B	9. Draw line A'C_3
2. Let C be a random point on the circumference of circle AB	10. Construct $P_1 \perp$ to line CC_3 through point C
3. Let A' be the image when A is dilated about C by ⅓	11. Let point D be the intersection of P_1 and line A'C_3
4. Let C_1 be the image when C is rotated about A' by ∠CAB	12. Draw line AD
5. Let C_2 be the image when C_1 is rotated about A' by ∠CAB	13. Let point E be the intersection of lines AD and CC_3
6. Let C_3 be the image when C_2 is rotated about A' by ∠CAB	14. Draw line segment EC_3
7. Trace point C_3 and change its color	15. Trace line segment EC_3 and change its color
8. Draw line CC_3	16. Animate point C around circle AB

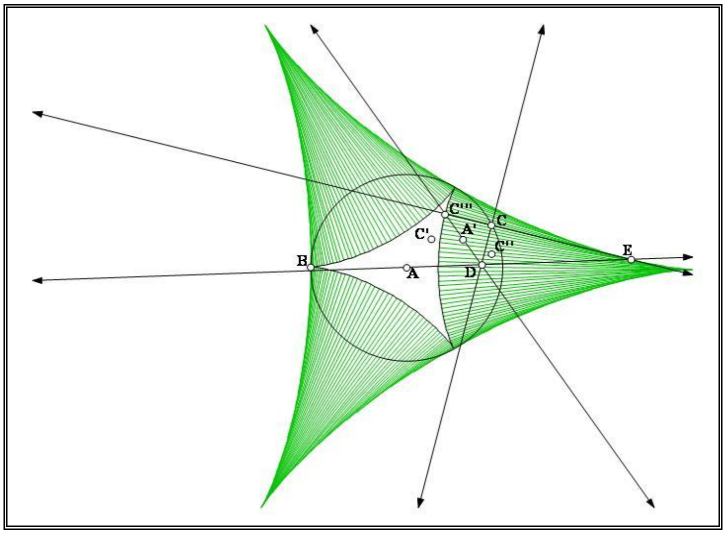

Figure 10-7: Two Deltoids for the Price of One

Note how the larger Deltoid, e.g., the one formed by the trace of line segment EC_3, is offset from the smaller Deltoid by 90º. Also note that the cusps of the smaller Deltoid bisect the respective sides of the larger Deltoid. Finally, note how the stationary circle is inscribed in the larger Deltoid and circumscribes the smaller Deltoid. This particular construction forms a graphic that is so symmetric, esthetically pleasing, and interesting to look at, it's very surprising that it has not been chosen by some large corporation as their company logo.

10.5.12 The Deltoid and a Three-Cusped Epicycloid as Gears

The construction below in Table 10-12 is that of both a Deltoid and a three-cusped Epicycloid interacting as though they were mechanical gears in an elaborate Rube Goldberg machine of some kind.

Table 10-12: The Deltoid-Epicycloid as Gears

1. Draw circle AB with center at A and passing through point B	13. Let C_2 be the image when C_1 is rotated about A_3 by $\angle DA_1B$
2. Let A_1 be the image when A is dilated about B by a factor of 3	14. Let C_3 be the image when C is rotated about A_2 by $\angle BA_2E$
3. Let C be a random point on the circumference of circle AB	15. Let A_4 be the image when A is rotated about A_2 by $\angle BA_2E$
4. Dilate circle AB about B by a factor of 3 to create circle O_1	16. Let C_4 be the image when C_2 is rotated about A_3 by $\angle DA_1B$
5. Let A_2 be the image when A_1 is rotated about point B by 180º	17. Let C_5 be the image when C_3 is rotated about A_4 by $\angle BA_2E$
6. Let D be a random point on the circumference of circle O_1	18. Let C_6 be the image when C_4 is rotated about A_3 by $\angle DA_1B$
7. Rotate circle O_1 about point B by 180º to create circle O_2	19. Let C_7 be the image when C_5 is rotated about A_4 by $\angle BA_2E$
8. Draw line segment BC	20. Let C_8 be the image when C_7 is rotated about A_4 by $\angle BA_2E$
9. Let C_1 be the image when C is rotated about A_1 by $\angle BA_1D$	21. Construct the locus of C_6 while point D traverses circle O_1
10. Let A_3 be the image when A is rotated about A_1 by $\angle BA_1D$	22. Construct the locus of C_8 while point E traverses circle O_2
11. Let E be a random point on circle O_2	23. Animate point C around circle AB
12. Construct $P_1 \perp$ to line segment BC through point C	

The perpendicular to segment BC is tangent to both the Deltoid and the Epicycloid. The Deltoid rotates about its center, point A_1, and the Epicycloid rotates about its center, point A_2. The cusps of the Deltoid and Epicycloid coincide as this rotation takes place. It's as if they were gears, albeit strange looking gears, in some large, intricate machine. The remarkable thing about this is that if one actually manufactured gears with the cross-section of these two curves, they would, indeed, operate correctly. It is also instructive to simultaneously animate point C around circle AB, point D around circle O_1, and point E around circle O_2, while tracing points C_1, C_2, C_3, C_4, C_5, and C_7. You'll find that point C_3 traces a three-cusped Epicycloid, point C_4 traces a Cardioid, point C_5 traces a four-looped Epitrochoid, and point C_7 traces a five-looped Epitrochoid. Points C1 and C2 trace curves called Hypotrochoids (see Chapter 13).

It's not a coincidence that Epicycloids and Hypocycloids appear to satisfy requirements for gears. All Cycloidal curves were first conceived by Roemer (circa 1674) while studying the best form for gear teeth. However, enough of these "gear" constructions; let's look at a couple of more pure Deltoid constructions!

10.5.13 Steiner's Deltoid

Here is another Deltoid generation construction where the Deltoid is formed from an envelope of straight lines. Through each point P on the circumcircle of ΔABC, construct a line parallel to the line obtained by reflecting line AP in the bisector of ∠BAC. (You get the same direction if line BP is reflected in the bisector of ∠ABC, and likewise for line CP reflected in the bisector of ∠ACB.) If you do this, you will find that the envelope of all such lines forms a Deltoid, as can be seen from the construction found in Table 10-13. This construction is sometimes called Steiner's Deltoid. Jakob Steiner (1796 – 1863) was a Swiss mathematician who extensively investigated, among many other things, the Deltoid curve.

Table 10-13: Steiner's Deltoid

1. Draw circle AB centered at A and passing through point B	7. Draw line CD
2. Let C, D, E, and F be random points on circle AB	8. Let line L_1 be the reflection of line CD across the bisector
3. Draw line segment DE	9. Construct line L_2 parallel to L_1 through point C
4. Draw line segment EF	10. Trace line L_2 and change its color
5. Draw line segment DF	11. Animate point C around circle AB
6. Construct the bisector of ∠EDF	

10.5.14 The Deltoid as a Hypocycloid – Again

Well, we started out this dynamic geometry section with a Deltoid construction based on the definition of a Hypocycloid, and we will finish this section with the same concept. As addressed in section 10.5.1, when the radius of the moving circle is one-third the radius of the fixed circle, a point on the circumference of the moving circle traces the Deltoid. Well, the Double Generation theorem of Daniel Bernoulli tells us that a Deltoid is also traced if the radius of the moving circle is two-thirds that of the fixed circle. Details of a construction based on this fact follow in Table 10-14.

Table 10-14: The Deltoid as a Hypocycloid – Again

1. Draw circle AB with center at A and passing through point B	6. Let C_3 be the image when C_2 is rotated about A' by ∠CAB
2. Let C be a random point on the circumference of circle AB	7. Let A" be the image when A is translated by vector A' → C_3
3. Let A' be the image when A is dilated about C by ⅓	8. Draw circle A"C_3 with center at A" and passing through C_3
4. Let C_1 be the image when C is rotated about A' by ∠CAB	9. Trace point C_3 and change its color
5. Let C_2 be the image when C_1 is rotated about A' by ∠CAB	10. Animate point C around circle AB

Note that this is really the same as the construction of section 10.5.1 except the smaller circle (the one with a radius of one-third that of the fixed circle) is not drawn and a two-thirds radius circle is drawn through the same tracing point (point C_3) as before. The center of that larger circle is located by translating point A in step 7. But, nevertheless, when the animation is run, one can see that a point on a two-thirds radius rolling circle does indeed trace the Deltoid.

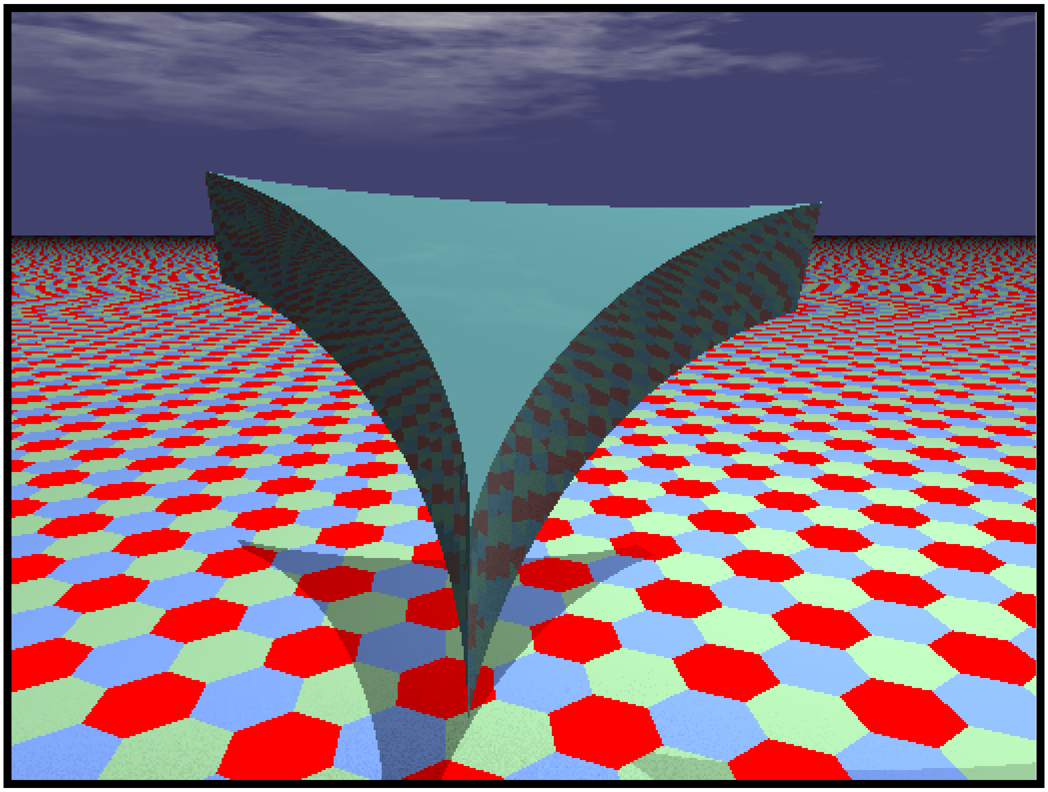

Figure 10-8: A Three-Dimensional Version of the Deltoid

Here, the Deltoid is rendered as a solid three-dimensional object using a technique called the "prism" methodology. It has been given a glossy, cadet-blue finish that reflects the infinite, hexagonally checkered plane on the two lateral sides visible in the picture. A light source is placed so as to cast the Deltoid's shadow below and onto the plane.

Chapter 11 – The Astroid

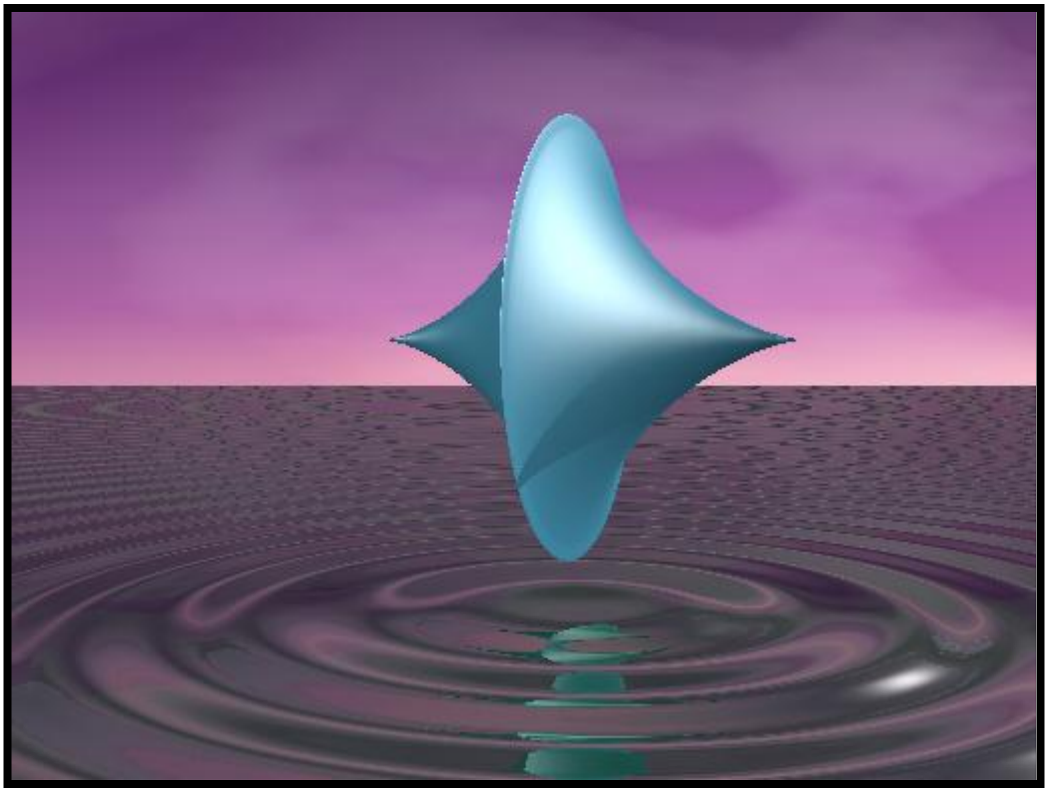

Figure 11-1: The Solid of Revolution Generated from the Astroid

The Astroid has been rotated about the x-axis to obtain the solid of revolution seen above. The resulting object has been given a light metallic-blue finish and placed over an infinite plain rendered to simulate water. The water surface has been rippled so that it appears as though the object has just emerged from the depths. The sky meeting the water at the horizon has been given a stormy-purplish color which is reflected in the water, giving it a purple color. A light source has been placed in the scene in order to partially reflect the object in the water and can be seen glaring off the water in the lower right.

11.1 Introduction to the Astroid

Chapter 10 introduced the concept of a Hypocycloid as the trace of a fixed point on the circumference of a circle rolling around the inside of the circumference of a second, stationary circle. It further stated that when the radius of the rolling circle was ⅓ of the radius of the stationary circle, the curve traced by the fixed point is called a Deltoid. It turns out that if the radius of the rolling circle is ¼ the radius of the stationary circle, the curve so traced is called an Astroid. Astroid (also sometimes referred to as the tetracuspid) of course means star-shaped.

11.2 Equations and Graph of the Astroid

To find the parametric equations for the Astroid, let $a/4$ be the radius of the rolling circle and a that of the fixed circle. Now, using an analogous argument to that in Chapter 10, section 10.2, where we derived the parametric equations for the Deltoid, we find that the parametric equations for the Astroid are:

$$(x, y) = \tfrac{a}{4}(3\cos t + \cos 3t, 3\sin t - \sin 3t), \quad -\pi < t < \pi \qquad \textbf{Equation 11-1}$$

However, a more compact form for the parametric representation can be obtained by writing these equations as

$$\begin{aligned} x &= \tfrac{3a}{4}\cos t + \tfrac{a}{4}\cos(2t + t) \\ y &= \tfrac{3a}{4}\sin t - \tfrac{a}{4}\sin(2t + t) \end{aligned}$$

and then expanding the cos (2t +t) and sin (2t + t) to obtain,

$$\begin{aligned} x &= \tfrac{3a}{4}\cos t + \tfrac{a}{4}(\cos 2t \cos t - \sin 2t \sin t) \\ y &= \tfrac{3a}{4}\sin t - \tfrac{a}{4}(\sin 2t \cos t + \sin t \cos 2t) \end{aligned}$$

Of course, these last expressions can be written as

$$\begin{aligned} x &= \tfrac{3a}{4}\cos t + \tfrac{a}{4}\cos t(\cos^2 t - \sin^2 t) - \tfrac{a}{4}\sin t(2\sin t \cos t) \\ y &= \tfrac{3a}{4}\sin t - \tfrac{a}{4}\cos t(2\sin t \cos t) - \tfrac{a}{4}\sin t(\cos^2 t - \sin^2 t) \end{aligned}$$

By multiplying out and collecting like terms, we finally arrive at

$$\begin{aligned} x &= a\cos^3 t \\ y &= a\sin^3 t \end{aligned} \qquad \textbf{Equation 11-2}$$

Equation 11-2 is a form from which it is quite easy to derive the Cartesian equation of the Astroid, that is,

$$x^{2/3} + y^{2/3} = a^{2/3} \qquad \textbf{Equation 11-3}$$

The pedal, Whewell, and Cesáro equations for the Astroid are, respectively

$$r^2 = a^2 - 3p^2 \qquad \textbf{Equation 11-4}$$

$$s = a\cos 2\varphi \qquad \textbf{Equation 11-5}$$

and

$$\rho^2 + 4s = 4a^2 \qquad \textbf{Equation 11-6}$$

Finally, the equation of the Astroid's tangent at the point $t = q$ is

$$y + \tan q \cdot x = a\sin q \qquad \textbf{Equation 11-7}$$

The graph of the Astroid is seen in Figure 11-2.

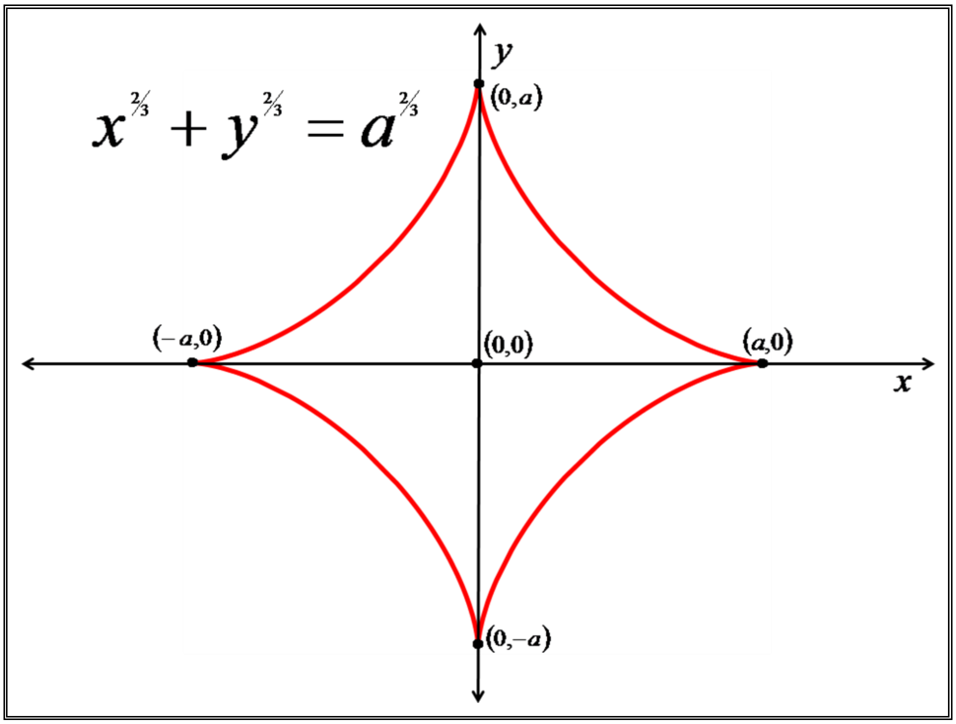

Figure 11-2: Graph of the Astroid

11.3 Analytical and Physical Properties of the Astroid

Based on the Astroid's parametric representation found in Equation 11-2, that is, $x = a\cos^3 t$ and $y = a\sin^3 t$, the following subsections contain an analysis of the Astroid.

11.3.1 Derivatives of the Astroid

- $\dot{x} = -3a\cos^2 t \cdot \sin t$
- $\ddot{x} = -3a\cos t\left(1 - 3\sin^2 t\right)$
- $\dot{y} = 3a\sin^2 t \cdot \cos t$
- $\ddot{y} = 3a\sin t\left(3\cos^2 t - 1\right)$

- $y' = -\tan t$

- $y'' = \dfrac{\sec^4 t}{3a \sin t}$

11.3.2 Metric Properties of the Astroid

The Astroid's length may be calculated using the parametric representation and the formula

$$ds = \sqrt{(dx)^2 + (dy)^2}\ .$$

Hence, $dx = -3a\cos^2 t \cdot \sin t \cdot dt$ and $dy = 3a\sin^2 t \cdot \cos t \cdot dt$. Squaring these two expressions and adding gives us

$$(dx)^2 + (dy)^2 = 9a^2 \sin^2 t \cos^2 t \cdot dt\,.$$

And, obviously,

$$ds = 3a \sin t \cos t \cdot dt\ .$$

Therefore, the total length of the Asteroid is

$$s = 4\int_0^{\pi/2} 3a \sin t \cos t dt = 12a \int_0^{\pi/2} \sin t \cdot d(\sin t) = \frac{12a}{2} \sin^2 t \Big|_0^{\pi/2} = 6a.$$

The area of the Astroid can be calculated by considering a small incremental rectangle, whose area is $dA = y \cdot dx$. Since, from the Cartesian equation, $y = (a^{2/3} - x^{2/3})^{3/2}$, integrating between $x = 0$ and $x = a$, gives us the area of the Astroid in the first quadrant. By symmetry, we can then conclude that the total area of the Astroid is

$$A = 4\int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2} dx\,.$$

This integral can be evaluated by making the substitution $x = a\sin^3\theta$. Under this substitution, the integral for the area becomes

$$A = 12a^2 \int_0^{\pi/2} \sin^2\theta \cdot \cos^4\theta \cdot d\theta = 12a^2 \int_0^{\pi/2} \cos^4\theta \cdot d\theta - 12a^2 \int_0^{\pi/2} \cos^6\theta \cdot d\theta\,.$$

The first integral has a value of $3\pi/16$ and the second integral has a value of $5\pi/32$. Therefore, the total value of the area enclosed by the Astroid is

$$A = 12a^2\left(\frac{3\pi}{16} - \frac{5\pi}{32}\right) = \frac{3\pi a^2}{8}\,.$$

The volume of the solid of revolution that is formed when the Astroid is rotated about the x-axis can be calculated by considering the incremental volume of a circular disk. The volume of this disk is simply its area times its thickness or $dV = \pi y^2 dx$. Of course, y is the ordinate of the curve and because of symmetry considerations, we have as the total volume

$$V = 2\pi \int_0^a \left(a^{2/3} - x^{2/3}\right)^3 dx .$$

This integral can be evaluated by making the substitution $x = a\sin^3\theta$. Under this substitution, the integral is transformed to

$$V = 6\pi a^3 \int_0^{\pi/2} \cos^7\theta \sin^2\theta \cdot d\theta = 6\pi a^3 \left[\int_0^{\pi/2} \cos^7\theta \cdot d\theta - \int_0^{\pi/2} \cos^9\theta \cdot d\theta \right].$$

By writing the argument of the first integral as $(1 - \sin^2\theta)^3 \cdot \cos\theta \cdot d\theta$ and then expanding the cubed expression, one can obtain a series of integrals that are all powers of the sin function multiplied by $\cos\theta \cdot d\theta$. These easily integrate and one obtains as a final value of this first integral 16/35. By similar reasoning and manipulation, one obtains as a final value of the second integral 128/315. Hence, the total volume under consideration is

$$V = 6\pi a^3 \left[\frac{16}{35} - \frac{128}{315} \right] = \frac{32\pi \cdot a^3}{105} .$$

The surface area of the solid of revolution that is formed when the Astroid is rotated about the x-axis can easily be calculated using the formula

$$S = 2\pi \int_a^b y \cdot ds ,$$

where y is the ordinate of the curve and s is the arc length over the portion of the desired surface area. In section 11.3.2, we have already shown that the incremental arc length for the Astroid is $3a\sin t \cdot \cos t \cdot dt$. Due to symmetry considerations, we can integrate between 0 and $\pi/2$ and simply multiply the result by 2. Hence, the total surface area of the Astroid is

$$S = 4\pi \int_0^{\pi/2} a\sin^3 t \cdot 3a\sin t \cos t dt = 12\pi a^2 \int_0^{\pi/2} \sin^4 t \cos t dt = \frac{12\pi a^2}{5}$$

If p represents the distance from the origin to the tangent of the Astroid, then

$$p = -a\sin t \cos t .$$

If r denotes the distance from the origin to the Astroid, then

$$r = \sqrt{\sin^4 t - \sin^2 t \cos^2 t + \cos^4 t}\,.$$

11.3.3 Curvature of the Astroid

If ρ is the radius of curvature of the Asteroid, then

$$\rho = 3a \sin t \cos t.$$

If (α, β) are the coordinates of the center of curvature of the Astroid, then

$$\alpha = \frac{3a}{2}\cos t - \frac{a}{2}\cos 3t \quad \text{and} \quad \beta = \frac{3a}{2}\sin t + \frac{a}{2}\sin 3t\,.$$

11.3.4 Angles for the Astroid

If θ is the radial angle, then

$$\tan\theta = \tan^3 t\,.$$

If ψ is the tangential-radial angle, then

$$\tan\psi = \frac{-\sin 2t}{2\cos 2t}\,.$$

If ϕ is the tangential angle, then

$$\tan\phi = -\tan t$$

11.4 Geometric Properties of the Astroid

- Intercepts: $(a, 0)$; $(-a, 0)$; $(0, a)$; $(0, -a)$.
- Extrema: Same as intercepts.
- Extent: Same as intercepts.
- Symmetries: The Astroid is symmetric about the x-axis, the y-axis, and the origin.
- Cusps: Same as intercepts.

11.5 Dynamic Geometry of the Astroid

The following subsections contain constructions that can all be used to generate and demonstrate interesting properties of the Astroid.

11.5.1 The Astroid as a Hypocycloid

In section 11.1, we introduced the Astroid as a Hypocycloid in which the radius of the moving circle is one-fourth the radius of the stationary circle. Table 11-1 delineates a construction based on this concept.

Table 11-1: The Astroid as a Hypocycloid

1. Draw circle AB with center at A and passing through point B	6. Let B_2 be the image when B_1 is rotated about A' by $\angle CAB$
2. Let C be a random point on the circumference of circle AB	7. Let B_3 be the image when B_2 is rotated about A' by $\angle CAB$
3. Dilate circle AB about point C by a factor of ¼	8. Let B_4 be the image when B_3 is rotated about A' by $\angle CAB$
4. Let A' be the image when A is dilated about C by ¼	9. Trace point B_4 and change its color
5. Let B_1 be the image when B is dilated about C by ¼	10. Animate point C around circle AB

For fun, draw line segment AB_4 as a dashed line segment, trace it, and color it. When the animation is run, the trace of this line segment will fill in the area enclosed by the Astroid in an interesting pattern.

11.5.2 Concurrent Tangents of the Astroid

Given a tangent to the Astroid, it is always possible to find two additional tangents that are concurrent with each other and with the given tangent. The construction of Table 11-2 illustrates this property.

Table 11-2: Concurrent Tangents to the Astroid

1. Draw circle AB with center at A and passing through point B	13. Let C_6 be the image when C_5 is rotated about A_2 by $\angle CAB$
2. Let A_1 be the image when A is dilated about B by ½	14. Let C_7 be the image when C_6 is rotated about point A by 120º
3. Draw circle AA_1 with center at A and passing through A_1	15. Let C_8 be the image when C_7 is rotated about A_3 by –120º
4. Let C be a random point on the circumference of circle AB	16. Let C_9 be the image when C_8 is rotated about point A by 120º
5. Let A_2 be the image when A is dilated about C by ¼	17. Let C_{10} be the image when C_9 is rotated about A_4 by –120º
6. Let A_3 be the image when A_2 is rotated about point A by 120º	18. Draw line segments CC_6, C_1C_8, and C_3C_{10}
7. Let A_4 be the image when A_3 is rotated about point A by 120º	19. Construct $P_1 \perp$ to line segment CC_6 through point C_6
8. Let C_1 be the image when C is rotated about point A by 120º	20. Construct $P_2 \perp$ to line segment C_1C_8 through point C_8
9. Let C_2 be the image when C is rotated about A_2 by $\angle CAB$	21. Construct $P_3 \perp$ to line segment C_3C_{10} through point C_{10}
10. Let C_3 be the image when C_1 is rotated about point A by 120º	22. Construct the locus of C_6 while point C traverses circle AB
11. Let C_4 be the image when C_2 is rotated about A_2 by $\angle CAB$	23. Animate point C around circle AB
12. Let C_5 be the image when C_4 is rotated about A_2 by $\angle CAB$	

Note that circle AA_1 has nothing whatsoever to do with the actual construction. However, observe how the point of concurrency (the common intersection point of the tangents) is confined to the circumference of this circle. Three other circles can be drawn which also have nothing to do with the actual construction, but make the final construction look a little more like a finished product. They are circles A_3C_1, A_2C, and A_4C_3. Note how these circles remain tangent to concentric circles AB and AA_1. Finally, if one creates a random point on line segment CC_6 (or C_3C_{10} or C_1C_8) and traces this random point, it will trace a four-sided figure that approaches an Astroid as the point is made to move closer and closer to the point C_6 (or C_{10} or C_8) and, of course, degenerates into a circle as the point is made to move closer to the other end of the line segment. We will learn, a couple of chapters from now, that this four-sided figure is a Hypotrochoid.

11.5.3 The Astroid as an Envelope of Line Segments

Here is a very beautiful and interesting construction for the Astroid (see Table 11-3). Note that in this construction, circle AB, the animation circle, becomes inscribed in the Astroid generated by the traced line segment, whereas in the Hypocycloid (section 11.5.1) construction, the animation circle circumscribes the Astroid.

Table 11-3: The Astroid as an Envelope of Line Segments

1. Draw circle AB with center at A and passing through point B	7. Let D be the midpoint of line segment CB_3
2. Let C be a random point on the circumference of circle AB	8. Let C' be the image when C is translated by vector D → C
3. Let B_1 be the image when B is rotated about A by ∠CAB	9. Draw line segment B_3C'
4. Let B_2 be the image when B_1 is rotated about A by ∠CAB	10. Trace line segment B_3C' and change its color
5. Let B_3 be the image when B_2 is rotated about A by ∠CAB	11. Animate point C around circle AB
6. Draw line segment CB_3	

Trace point D for an interesting curve not yet encountered. Figure 11-3 displays a snapshot of this construction.

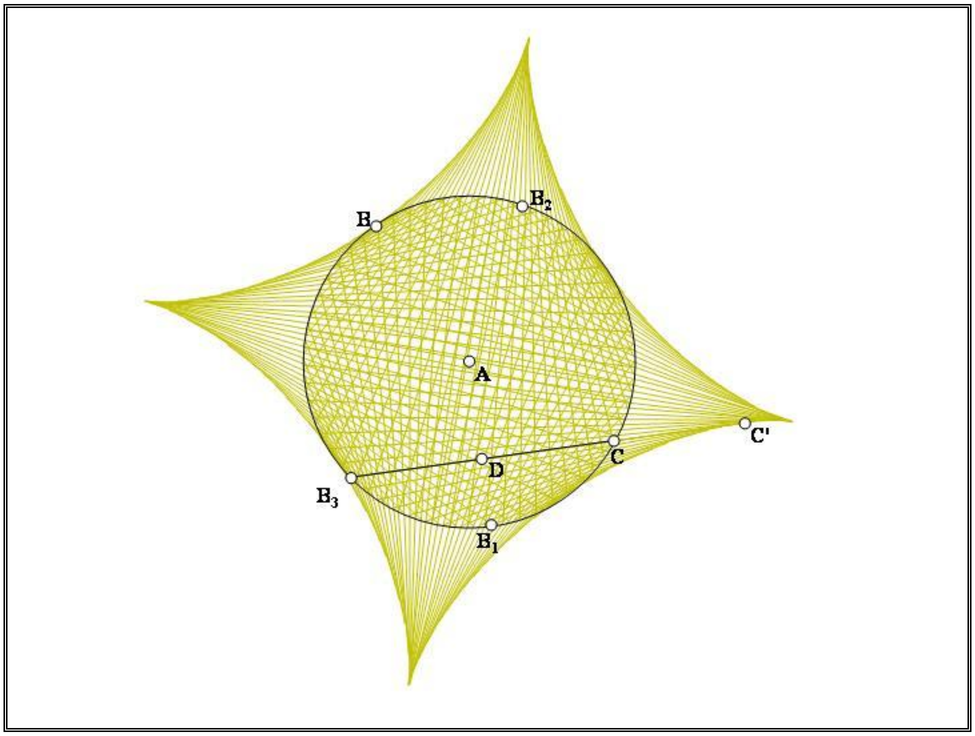

Figure 11-3: The Astroid as an Envelope of Line Segments

11.5.4 A Revolving Astroid and an Equilateral Triangle

Table 11-4: A Revolving Astroid and Equilateral Triangle

1. Draw horizontal line AB	17. Let G_1 be the image when G is dilated about point D by ⅓.
2. Let C and D be two random points on line AB	18. Let G_2 be the image when G_1 is dilated about point F_1 by ¼
3. Draw circle DC with center at D and passing through point C	19. Draw circle G_1G with center at G_1 and passing through G
4. Let E be a random point on the circumference of circle DC	20. Draw line segments D'H and HF
5. Rotate line AB about point D by –120º to obtain line L_1	21. Construct the interior of polygon D'HF and change its color
6. Rotate line AB about point D by +120º to obtain line L_2	22. Let I be a random point on the circumference of circle G_1G
7. Construct the parallel to line AB through point E	23. Draw circle G_2F_1 with center at G_2 and passing through F_1
8. Let point F be the intersection of the parallel line and line L_2	24. Let G_3 be the image when G_2 is rotated about G_1 by $\angle F_1G_1I$
9. Construct $P_1 \perp$ to line L_2 through point F	25. Draw circle G_3I with center at G_3 and passing through point I
10. Let D' be the image when D is translated by vector E → F	26. Let F_2 be the image when F is rotated about G_1 by $\angle F_1G_1I$
11. Construct $P_2 \perp$ to line AB through point D'	27. Let F_3 be the image when F_2 is rotated about G_3 by $\angle IG_1F_1$
12. Draw line segment FD'	28. Let F_4 be the image when F_3 is rotated about G_3 by $\angle IG_1F_1$
13. Let point G be the intersection of perpendiculars P_1 and P_2	29. Let F_5 be the image when F_4 is rotated about G_3 by $\angle IG_1F_1$
14. Construct $P_3 \perp$ to line L_1 through point G	30. Let F_6 be the image when F_5 is rotated about G_3 by $\angle IG_1F_1$
15. Let point H be the intersection of line L_1 and perpendicular P_3	31. Construct the locus of F_6 while point I traverses circle G_1G
16. Let F_1 be the image when F is dilated about G by 4/3	32. Animate point E around circle DC

Table 11-4 presents an interesting construction of an Astroid that appears to revolve about its center along with an equilateral triangle whose vertices are confined to the circumference of the Astroid. Although the Astroid appears to be rotating, note that point H performs simple harmonic motion along the first rotated line (L_1), point F executes simple harmonic motion along the second rotated line (L_2), and point D' does the same thing along line AB.

11.5.5 The Trammel of Archimedes

The Astroid has the following interesting property: If one defines the axes of an Astroid to be two mutually perpendicular lines that pass through the cusps of the Astroid, then the length of any tangent cut by these axes is constant. Because this length is constant, no matter which tangent is selected (for a given Astroid), one can construct a mechanical device made up of a fixed bar with ends sliding on two perpendicular tracks. The envelope of the bar will then generate the Astroid. Such a device is called the Trammel of Archimedes. A GSP construction for this device is shown in Table 11-5.

Table 11-5: The Trammel of Archimedes

1. Draw horizontal line AB	7. Let point D be the intersection of perpendiculars P_1 and P_3
2. Draw circle AB with center at A and passing through point B	8. Let E be the intersection of line AB and perpendicular P_2
3. Construct $P_1 \perp$ to line AB through point A	9. Draw line segment DE
4. Let C be a random point on the circumference of circle AB	10. Trace line segment DE and change its color
5. Construct $P_2 \perp$ to line AB through point C	11. Animate point C around circle AB
6. Construct $P_3 \perp$ to P_1 through point C	

Of course, line segment DE is the moving bar whose end points travel on the mutually perpendicular tracks, AB and DA. An interesting, alternate construction for the Trammel of Archimedes is also included here as Table 11-6.

Table 11-6: An Alternate Trammel of Archimedes Construction

1. Draw horizontal line segment AB	8. Let F and G be the intersections of circle DC and P_1
2. Let C be any random point not on line AB	9. Draw circle EE' with center at E and passing through point E'
3. Let D be the midpoint of line segment AB	10. Let H and I be the two intersections of circle EE' and P_1
4. Let E be a random point on line segment AB	11. Draw the two line segments EH and EI
5. Draw circle DC with center at D and passing through point C	12. Trace line segments EH and EI and change their color
6. Construct $P_1 \perp$ to line segment AB through point D	13. Animate point E along line segment AB
7. Let E' be the image when E is translated by vector D → B	

11.5.6 Three Deltoids Inside an Astroid

Here is an interesting configuration of the Deltoid and the Astroid.

Table 11-7: Three Deltoids and an Astroid

1. Draw horizontal line AB	15. Construct the locus of H while point D traverses circle AC
2. Let C be a random point on line AB	16. Let I be a random point on the circumference of circle GF
3. Construct $P_1 \perp$ to line AB through point A	17. Construct $P_5 \perp$ to line segment EF through point I
4. Draw circle AC with center at A and passing through point C	18. Let I' be the image when I is rotated about point G by 120°
5. Let D be a random point on the circumference of circle AC	19. Let point J be the intersection of line segment EF and P_5
6. Construct $P_2 \perp$ to line AB through point D	20. Construct $P_6 \perp$ to line segment EF through point I'
7. Construct $P_3 \perp$ to P_1 through point D	21. Let I'' be the image when I' is rotated about point G by 120°
8. Let E be the intersection of line AB and perpendicular P_2	22. Construct the locus of J while point D traverses circle AC
9. Let point F be the intersection of perpendiculars P_1 and P_3	23. Let point K be the intersection of line segment EF and P_6
10. Draw line segment EF	24. Construct $P_7 \perp$ to line segment EF through point I''
11. Construct $P_4 \perp$ to line segment EF through point D	25. Construct the locus of K while point D traverses circle AC
12. Let G be the midpoint of line segment EF	26. Let point L be the intersection of line segment EF and P_7
13. Let point H be the intersection of line segment EF and P_4	27. Construct the locus of L while point D traverses circle AC
14. Draw circle GF with center at G and passing through point F	28. Animate point I around circle GF

11.5.7 Two Astroids for the Price of One

The evolute of all Epicycloids and Hypocycloids is another curve of the same type. In other words, the evolute of the Astroid is another Astroid. Therefore, as we have learned previously, the trace of the center of curvature of an Astroid should be its evolute, another Astroid. Similarly, the evolute can be drawn as the envelope of the Astroid's normals. These two ideas are incorporated into the construction that is listed below in Table 11-8 and portrayed in Figure 11-4.

Table 11-8: Two Astroids for the Price of One

1. Draw circle AB with center at A and passing through point B	10. Let D be the unlabeled intersection of line A_1C_4 and circle A_1C
2. Let C be a random point on the circumference of circle AB	11. Draw line AD
3. Let A_1 be the image when point A is dilated about C by ¼	12. Draw line CC_4
4. Draw circle A_1C with center at A_1 and passing through C	13. Let point E be the intersection of line CC_4 and line AD
5. Let C_1 be the image when C is rotated about A_1 by $\angle$CAB	14. Draw circle EC_4 with center at E and passing through C_4
6. Let C_2 be the image when C_1 is rotated about A_1 by $\angle$CAB	15. Draw line segment EC_4
7. Let C_3 be the image when C_2 is rotated about A_1 by $\angle$CAB	16. Trace line segment EC_4 and change its color
8. Let C_4 be the image when C_3 is rotated about A_1 by $\angle$CAB	17. Animate point C around circle AB
9. Draw line A_1C_4	

In this construction, the trace of point C_4 will produce the Astroid for which we then construct its evolute. One can see this as the envelope of the C_4-end of line segment EC_4, which produces the inner Astroid as seen in Figure 11-4. Point E is, of course, the center of curvature for this Astroid, circle EC_4 is its osculating circle, and line CC_4 is its normal. Note that the evolute (the outer Astroid) is shifted 45° relative to the inner Astroid. Note also the relative size of the two Astroids. The evolute's area is twice that of the original Astroid.

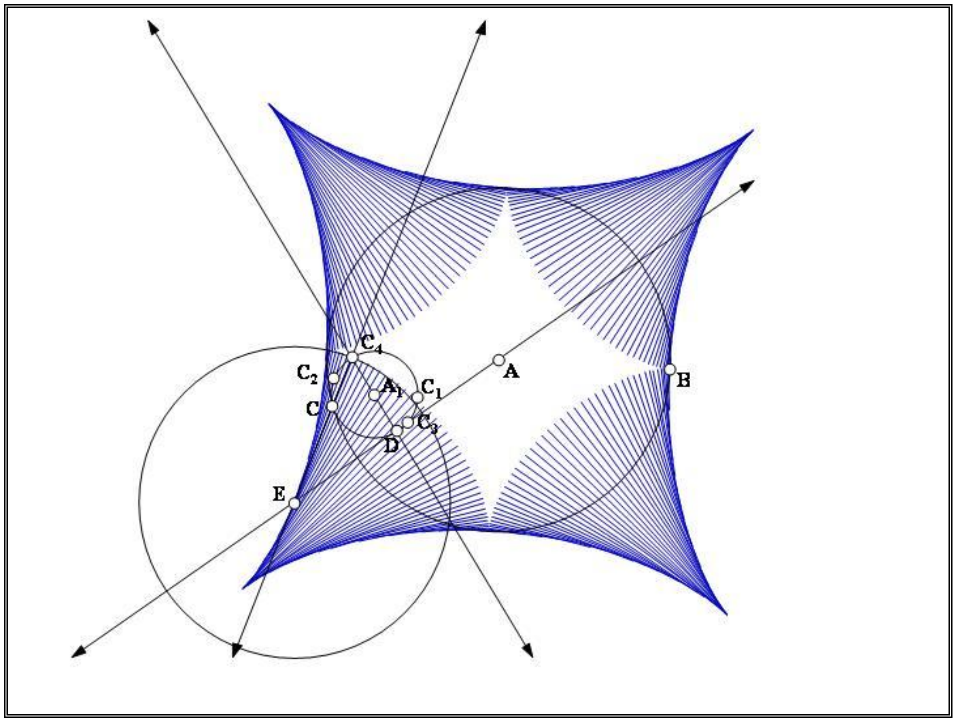

Figure 11-4: Two Astroids for the Price of One

11.5.8 An Astroid, a Deltoid, and a Common Tangent

The construction below illustrates a coupling between an Astroid and a Deltoid. Additionally, a common tangent to both curves is included (see Table 11-9).

Table 11-9: An Astroid, a Deltoid, and a Common Tangent

1. Draw circle AB with center at A and passing through point B	14. Let A_4 be the image when A is rotated about A_2 by $\angle BA_2E$
2. Let A_1 be the image when A is dilated about point B by 4	15. Let C_3 be the image when C_1 is rotated about A_3 by $\angle DA_1B$
3. Let A_2 be the image when point A is dilated about B by 3	16. Let C_4 be the image when C_2 is rotated about A_4 by $\angle EA_2B$
4. Draw circle A_1B with center at A_1 and passing through B	17. Let C_5 be the image when C_3 is rotated about A_3 by $\angle DA_1B$
5. Draw circle A_2B with center at A_2 and passing through B	18. Let C_6 be the image when C_4 is rotated about A_4 by $\angle EA_2B$
6. Let C be a random point on the circumference of circle AB	19. Let C_7 be the image when C_5 is rotated about A_3 by $\angle DA_1B$
7. Let D be a random point on the circumference of circle A_1B	20. Let C_8 be the image when C_6 is rotated about A_4 by $\angle EA_2B$
8. Let E be a random point on the circumference of circle A_2B	21. Let C_9 be the image when C_7 is rotated about A_3 by $\angle DA_1B$
9. Draw line segment CB	22. Draw circle A_4E with center at A_4 and passing through E
10. Construct $P_1 \perp$ to line segment CB through point C	23. Draw circle A_3D with center at A_3 and passing through D
11. Let A_3 be the image when A is rotated about A_1 by $\angle BA_1D$	24. Construct the locus of C_8 while point E traverses circle A_2B
12. Let C_1 be the image when C is rotated about A_1 by $\angle BA_1D$	25. Construct the locus of point C_9 while D traverses circle A_1B
13. Let C_2 be the image when C is rotated about A_2 by $\angle BA_2E$	26. Animate point C around circle AB

With this construction, one can also simultaneously animate point C around circle AB, point D around circle A_1B, and point E around circle A_2B. Points C_1 through C_9 then execute some interesting curves which can be seen by tracing them, although it is best to trace only one at a time, otherwise they overwrite one another and are difficult to distinguish.

11.5.9 The Astroid as an Envelope of Ellipses

The Astroid can also be generated as an envelope of co-axial ellipses wherein the sum of the major and minor axes is constant. A construction that gives verification of this property can be created with GSP as shown below in Table 11-10.

Table 11-10: The Astroid as an Envelope of Ellipses

1. Draw horizontal line AB	10. Draw line segment EF
2. Construct $P_1 \perp$ to line AB through point A	11. Let G be a random point on line segment EF
3. Let C be a random point on line AB	12. Construct $P_4 \perp$ to line segment EF through point D
4. Draw circle AC with center at A and passing through point C	13. Construct the locus of G while point D traverses circle AC
5. Let D be a random point on the circumference of circle AC	14. Let point H be the intersection of line segment EF and P_4
6. Construct $P_2 \perp$ to line AB through point D	15. Construct the locus of H while point D traverses circle AC
7. Construct P_3 to $\perp P_1$ through point D	16. Trace the locus and change its color
8. Let point E be the intersection pf perpendiculars P_1 and P_3	17. Animate point G along line segment EF
9. Let point F be the intersection of line AB and P_2	

11.5.10 The Astroid and a Four-Cusped Epicycloid

Here we show an interconnection between the Astroid (i.e., the four-cusped Hypocycloid) and the four-cusped Epicycloid. Table 11-11 contains this construction, while Figure 11-5 presents a snapshot of the animation.

Table 11-11: The Astroid and a Four-Cusped Epicycloid

1. Draw circle AB with center at point A and passing through B	8. Let C_2 be the image when C_1 is rotated about A' by $\angle$CAB
2. Let C be a random point on the circumference of circle AB	9. Let C_3 be the image when C_2 is rotated about A' by $\angle$CAB
3. Let A' be the image when point A is dilated about C by ¼	10. Let C_4 be the image when C_3 is rotated about A' by $\angle$CAB
4. Draw circle A'C with center at A' and passing through point C	11. Let C_5 be the image when C_4 is reflected in perpendicular P_1
5. Draw line segment AC	12. Draw circle CC_5 with center at C and passing through C_5
6. Construct $P_1 \perp$ to line segment AC through point C	13. Trace circle CC_5 and change its color
7. Let C_1 be the image when C is rotated about A' by $\angle$CAB	14. Animate point C around circle AB

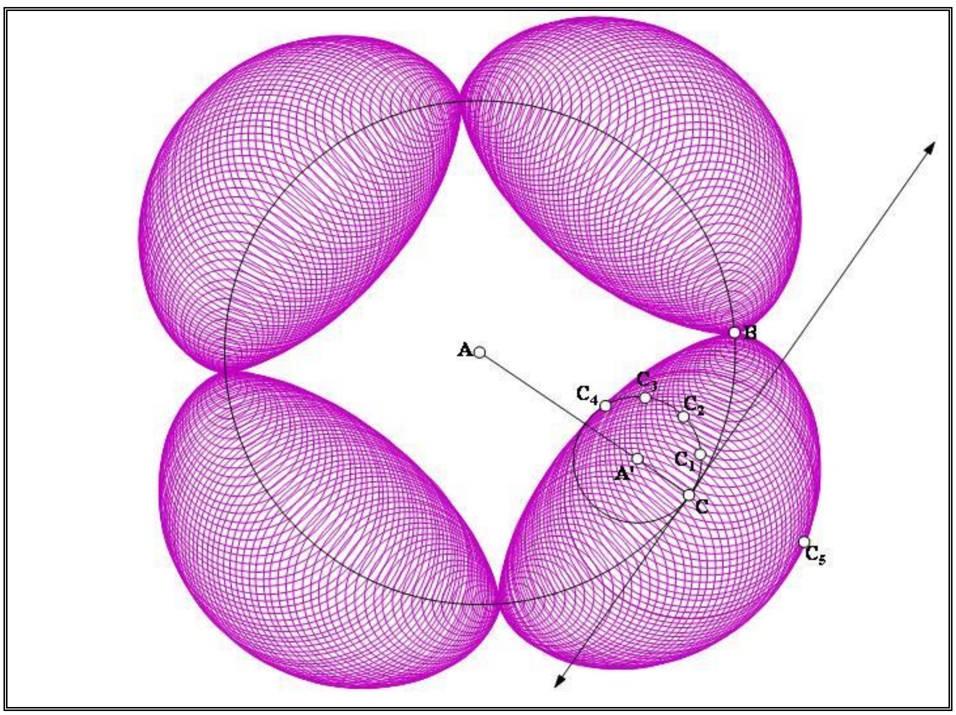

Figure 11-5: An Astroid Enveloped by a Four-Cusped Epicycloid

11.5.11 The Astroid as an Envelope of Lines

In section 11.5.3 we have a construction of the Astroid as an envelope of line segments. Table 11-12 presents an alternate construction that uses lines, not line segments.

Table 11-12: The Astroid as an Envelope of Lines

1. Draw circle AB with center at A and passing through point B	9. Construct $P_3 \perp$ to P_2 through point A
2. Draw line AB	10. Let point C" be the image when point C' is reflected across P_3
3. Let C be a random point on the circumference of circle AB	11. Let E and F be two random points on perpendicular P_1
4. Construct $\perp P_1$ to line AB through point C	12. Let point E' be the image when point E is reflected across P_2
5. Let point D be on circle AB diametrically opposed to point B	13. Let point F' be the image when point F is reflected across P_2
6. Draw line CD	14. Draw line E'F'
7. Let C' be the image when point C is reflected across line AB	15. Trace line E'F' and change its color
8. Construct $P_2 \perp$ to line CD through point C'	16. Animate point C around circle AB

11.5.12 The Astroid as a Hypocycloid – Revisited

We learned in section 11.5.1 that the Astroid is generated as the trace of a point on the circumference of a circle of radius $a/4$ rolling around the inside of a stationary circle of radius a. However, the Astroid can also be generated if the radius of the rolling circle is $3a/4$, as can be seen in Table 11-13 (i.e., the Double Generation theorem of Bernoulli strikes again).

Table 11-13: The Astroid as a Hypocycloid – Revisited

1. Draw circle AB with center at A and passing through point B	7. Let C_4 be the image when C_3 is rotated about A_1 by $\angle$CAB
2. Let C be a random point on the circumference of circle AB	8. Let A_2 be the image when A is translated by vector $A_1 \rightarrow C_4$
3. Let point A_1 be the image when A is dilated about C by ¼	9. Draw circle A_2C_4 with center at A_2 and passing through C_4
4. Let C_1 be the image when C is rotated about A_1 by $\angle$CAB	10. Trace point C_4 and change its color
5. Let C_2 be the image when C_1 is rotated about A_1 by $\angle$CAB	11. Animate point C around circle AB
6. Let C_3 be the image when C_2 is rotated about A_1 by $\angle$CAB	

It may not be obvious that circle A_2C_4 has a radius that is three fourths of the stationary circle (circle AB). However, with GSP this is very easy to check out. Simply measure the two radii of the respective circles and then form the ratio of the radius of circle A_2C_4 to that of the radius of circle AB. One will find that the result is 0.75.

11.5.13 A Compass-Only Construction for the Astroid

Here is another spectacular compass-only (GSP version thereof) construction from which it is relatively easy to also construct the Astroid's osculating circle (also compass-only). Refer to Table 11-14.

Table 11-14: A Compass-Only Astroid

1. Draw circle AB with center at A and passing through point B	14. Draw circle FC with center at F and passing through point C
2. Let C be a random point on the circumference of circle AB	15. Draw circle CF with center at C and passing through point F
3. Draw line segment AB	16. Let G and H be the two intersections of circles CF and FC
4. Let C' be the image as C is reflected across line segment AB	17. Draw circle GH with center at G and passing through point H
5. Draw circle BC with center at B and passing through point C	18. Let I be the unlabeled intersection of circle GH and circle FC
6. Draw circle C'C with center at C' and passing through point C	19. Draw circle IC with center at I and passing through point C
7. Draw line segment AC'	20. Let J and K be the two intersections of circles IC and CC"
8. Let C" be the image as C is reflected across line segment AC'	21. Draw circle JC with center at J and passing through point C
9. Draw circle CC" with center at C and passing through C"	22. Draw circle KC with center at K and passing through point C
10. Draw circle C"C with center at C" and passing through C	23. Let point L be the unlabeled intersection of circles KC and JC
11. Let D and E be the two intersections of circles CC" and C"C	24. Trace point L and change its color
12. Draw circle ED with center at E and passing through point D	25. Animate point C around circle AB
13. Let F be the unlabeled intersection of circles ED and C"C	

For the Astroid's osculating circle, change step 24 to "Construct the locus of point L as point C traverses circle AB," and add the following steps.

Table 11-14 (Continued): A Compass-Only Astroid

26. Draw circle CL with center at C and passing through point L	31. Draw circle OC with center at O and passing through point C
27. Draw circle LC with center at L and passing through point C	32. Let P and Q be the two intersections of circles OC and CL
28. Let M and N be the two intersections of circles CL and LC	33. Draw circle PQ with center at P and passing through point Q
29. Draw circle MN with center at M and passing through N	34. Let R be the unlabeled intersection of circles PQ and OC
30. Let O be the unlabeled intersection of circles MN and CL	35. Draw circle RL with center at R and passing through point L

Now make circle RL, which is the osculating circle, thick and of a different color and rerun the animation. Wow!

11.5.14 The Osculating Circle of the Astroid

In the previous subsection (A Compass-Only Astroid), we have added the compass-only steps to produce the Astroid's osculating circle. What follows in Table 11-15 is an alternate construction for the Astroid's osculating circle. It is not a compass-only construction as the previous construction was. Why do we include it since we already have a construction for the Astroid's osculating circle? Because it is an interesting construction, although rather complex. Further, we use a construction for the Astroid itself that has not been used up to this point. In addition, we also construct the Astroid's tangent line along the way. So this construction embodies three goodies: the Astroid, the tangent line, and the osculating circle all wrapped up in one nice, neat package. A good way to end this chapter!

Table 11-15: The Astroid, Its Tangent, and Its Osculating Circle

1. Draw horizontal line AB	28. Draw line AK
2. Draw circle AB centered at A and passing through point B	29. Construct P_{12} ⊥ to line AK through point H
3. Let C be a random point on the circumference of circle AB	30. Make P_{12} thick and change its color
4. Draw line AC	31. Let point K' be the image when K is rotated about A by 90°
5. Construct P_1 ⊥ to line AB through point A	32. Let point I_2 be the image when point I is dilated about A by 6
6. Construct P_2 ⊥ to line AB through point C	33. Let point L be the intersection of line AB with P_4
7. Let D be the intersection of line AB and perpendicular P_2	34. Let point L' be the image when L is dilated about A by 3
8. Construct P_3 ⊥ to line AC through point D	35. Draw line segment AL'
9. Let E be the intersection of line AC and perpendicular P_3	36. Let M be the midpoint of line segment AL'
10. Construct P_4 ⊥ to line AB through point E	37. Let M' be the image when M is translated by vector $I_2 \rightarrow M$
11. Construct P_5 ⊥ to P_4 through point C	38. Let A' be the image when A is translated by vector M' → A
12. Let point F be the intersection of perpendiculars P_1 and P_5	39. Construct P_{13} ⊥ to line AB through point A'
13. Construct P_6 ⊥ to line AC through point F	40. Let J_2 be the image when J is dilated about A by 6
14. Let point G be the intersection of line AC and P_6	41. Let point N be the intersection of perpendiculars P_1 and P_7
15. Construct P_7 ⊥ to P_1 through point G	42. Let point N' be the image when N is dilated about A by 3
16. Let point H be the intersection of perpendiculars P_4 and P_7	43. Let J_3 be the image when J_1 is translated by vector $N' \rightarrow J_1$
17. Construct the locus of point H as point C traverses circle AB	44. Construct P_{14} ⊥ to P_1 through point J_3
18. Make the locus thick and change its color	45. Let point O be the intersection of perpendiculars P_{13} and P_{14}
19. Construct P_8 ⊥ to line AB through point G	46. Construct P_{15} ⊥ to line AK through point O
20. Let point I be the intersection of line AB and P_8	47. Draw line segment KK'
21. Let point I_1 be the image when I is dilated about point A by 3	48. Construct P_{16} ⊥ to line segment KK' through point K'
22. Construct P_9 ⊥ to line AB through point I_1	49. Let point P be the intersection of line AK and P_{16}
23. Construct P_{10} ⊥ to P_1 through point E	50. Let H' be the image when H is translated by vector P → A
24. Let point J be the intersection of perpendiculars P_1 and P_{10}	51. Draw circle H'H centered at H' and passing through point H
25. Let point J_1 be the image when J is dilated about point A by 3	52. Make circle H'H thick and change its color
26. Construct P_{11} ⊥ to P_1 through point J_1	53. Animate point C around circle AB
27. Let point K be the intersection of perpendiculars P_9 and P_{11}	

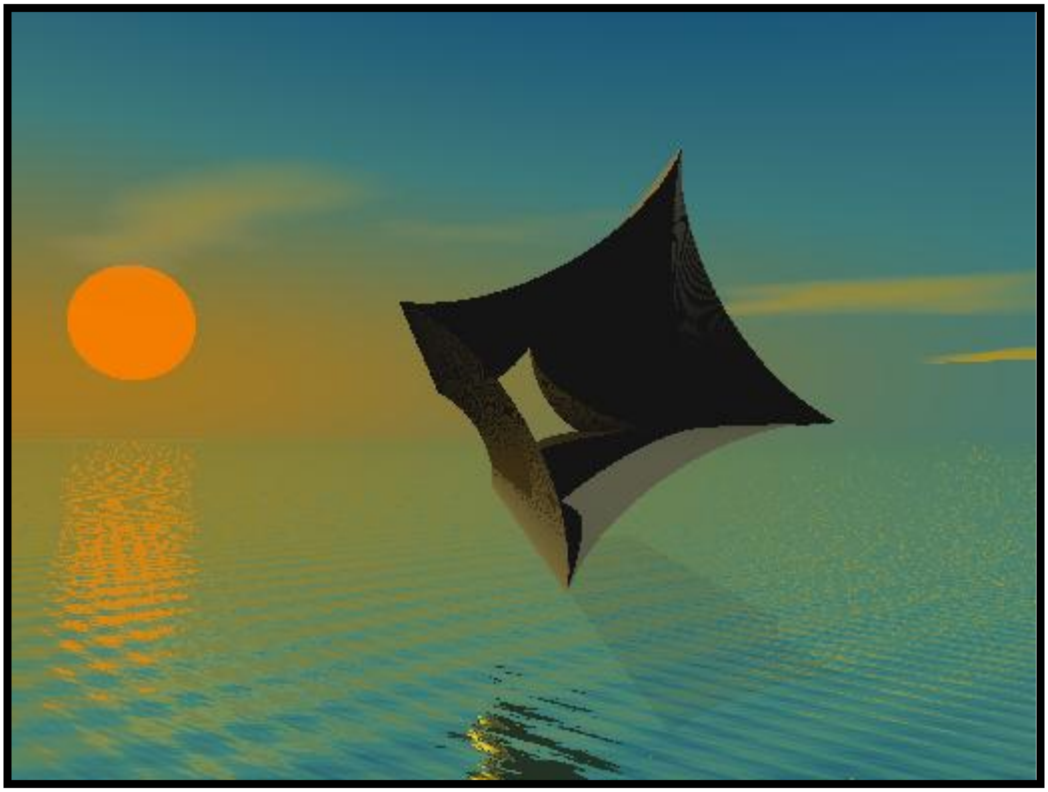

Figure 11-6: A Three-Dimensional Version of the Astroid

The Astroid is extruded into the third dimension and placed in a watery setting with a newly rising sun. Light sources are placed not only so as to cast the object's shadow onto the water, but also to reflect the object in the water.

Chapter 12 – The Hypocycloid

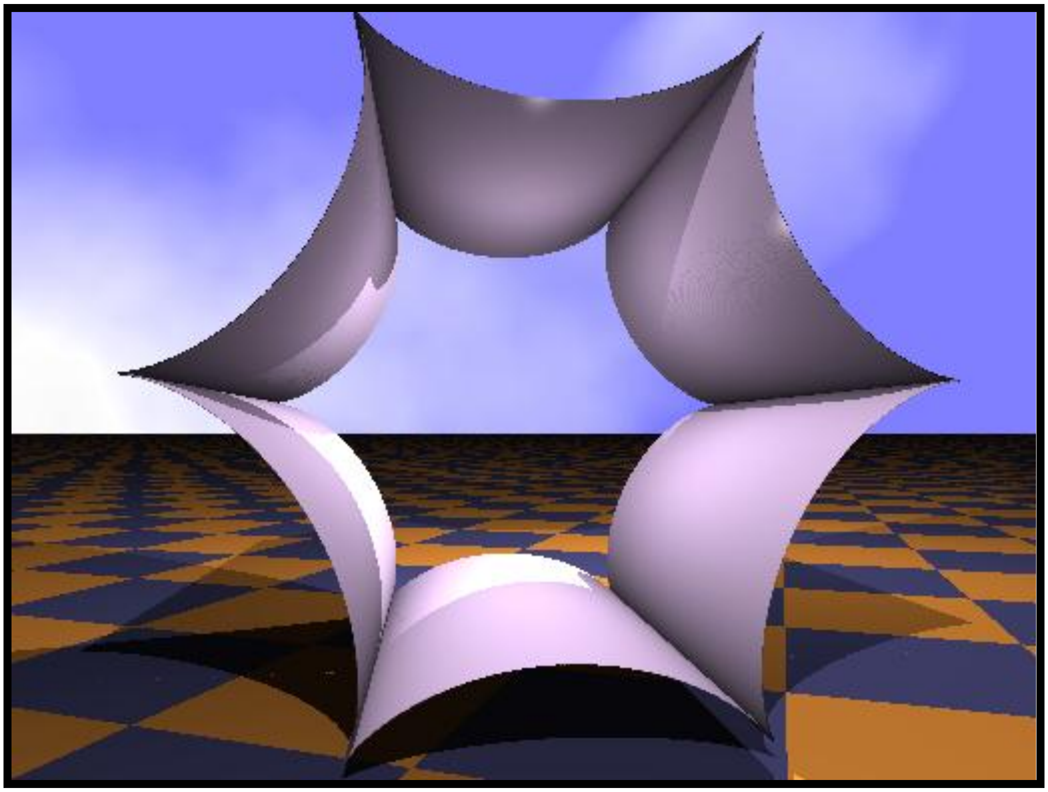

Figure 12-1: A Six-Cusped Hypocycloid in Three Dimensions

The cross-section of the object in the above figure is a six-cusped Hypocycloid. It is floating over a blue and gold checkered plane. The extruded object has been given a shiny light-purple finish. Light sources are placed so as to partially shadow the plane and to reflect off the interior sides of the extruded figure, thereby creating the dark and bright spots. The plane meets a bright-blue sky at the horizon.

12.1 Introduction

Chapter 10 briefly alluded to the concept of the Hypocycloid, which was defined to be the trace of a point on the circumference of a circle that is rolling (without slipping) around the inside of a second, fixed circle. Further, it stated that when the ratio of the radii of the fixed circle to the rolling circle is 3 to 1, the resulting traced curve is called a Deltoid. Then, Chapter 11 stated that when the ratio is 4 to 1, the resulting trace is called an Astroid. We now take up the Hypocycloid in general.

Although Cycloidal curves were first conceived by Roemer in 1674 while he was studying the best form for gear teeth, both Galileo and Mersenne had already discovered the ordinary Cycloid 75 years earlier, in 1599. As already mentioned, the beautiful Double Generation theorem of these curves was first noticed by Daniel Bernoulli in 1725. Astronomers find forms of the Cycloidal curves in various coronas. They also occur as Caustics. Rectification was first given by Newton in his *Principia*.

12.2 Equations and Graph of the Hypocycloid

Just as we derived the equations for the Deltoid in Chapter 10, if we let the radius of the fixed circle be a and the radius of the rolling circle be b, we find that the parametric equations for the general Hypocycloid are

$$(x, y) = \left[(a-b)\cos t + b\cos\left(\tfrac{a-b}{b}t\right), (a-b)\sin t - b\sin\left(\tfrac{a-b}{b}t\right)\right], \quad -\pi < t < \pi \qquad \textbf{Equation 12-1}$$

As a slight digression, consider the special case when the fixed circle has twice the radius of the rolling one, $a/2 = b$ and we find from Equation 12-1 that $x = a \cdot \cos t$ and $y = 0$ and the Hypocycloid degenerates into the diameter of the fixed circle, described back and forth. The interesting feature of this digression is that it provides a mechanical solution to the problem of drawing a straight line by using purely circular motions. Enough digression! Back to the Hypocycloid!

If the pedal point is situated at the center of the hypocycloid, then the pedal equation is

$$p^2 = \frac{(a-2b)^2}{4b(b-a)}\left(r^2 - a^2\right) \qquad \textbf{Equation 12-2}$$

The Whewell equation is

$$s = a \sin b\varphi, \quad b > 1 \qquad \textbf{Equation 12-3}$$

An n-cusped, non-self-intersecting Hypocycloid has $a/b = n$. As we have already alluded to, a two-cusped Hypocycloid is a line segment, a three-cusped Hypocycloid is called a Deltoid or Tricuspoid, and a four-cusped Hypocycloid is called an Astroid. If a/b is rational, the curve closes on itself and has b cusps. If a/b is irrational, the curve never closes and fills the entire interior of the fixed circle. Figure 12-2 portrays the graph of four different hypocycloids, namely three-, four-, five-, and six-cusped Hypocycloids in red, green, blue, and violet, respectively. Figures 12-3 through 12-6 depict a variety of different Hypocycloids for various selected values of the two radii of the associated circles (the fixed circle with radius a and the rolling circle with radius b).

The equation of the tangent line at the point $t = q$ is

$$y \cdot \left(\sin \tfrac{a-b}{b} q + \sin q\right) = \left(\cos \tfrac{a-b}{b} q - \cos q\right) \cdot x + 2(a-2b)\sin^2 \tfrac{aq}{2b}$$ **Equation 12-4**

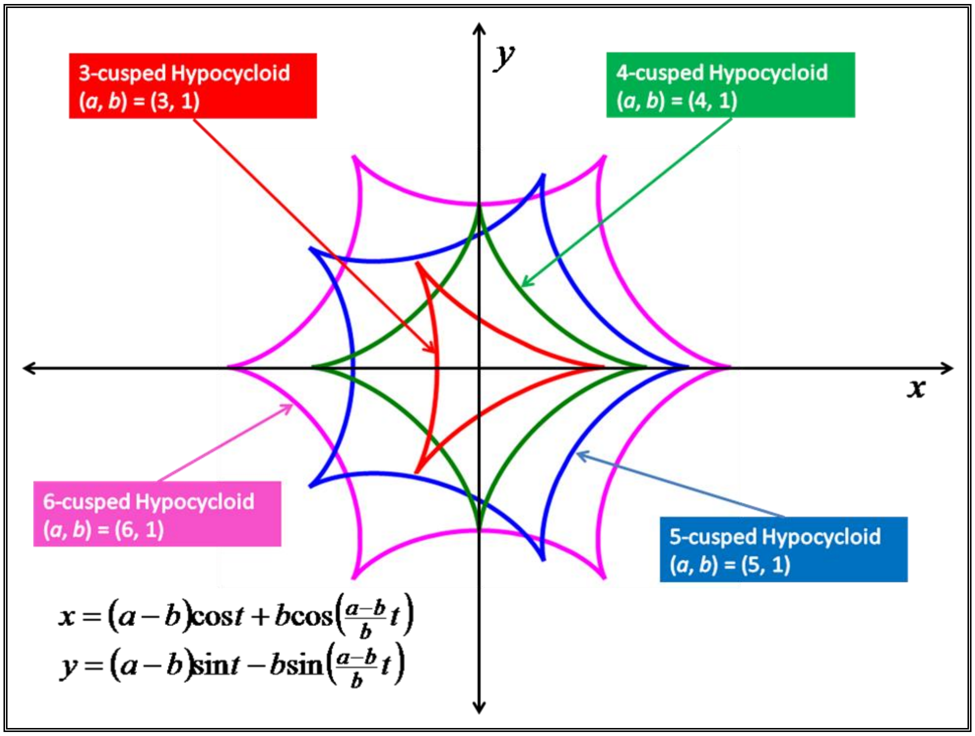

Figure 12-2: Graph of Four Different Hypocycloids

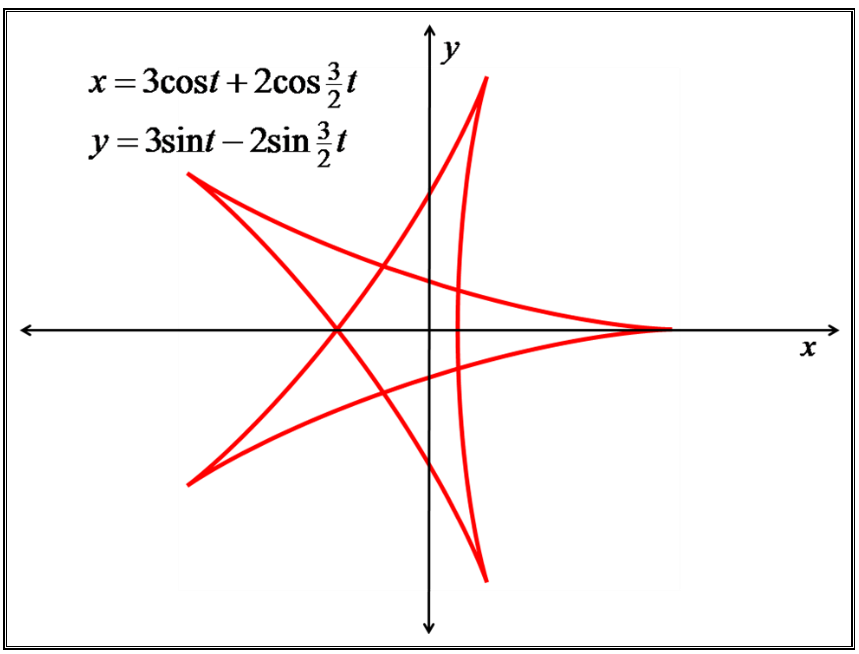

Figure 12-3: Hypocycloid with *a* = 5 and *b* = 2

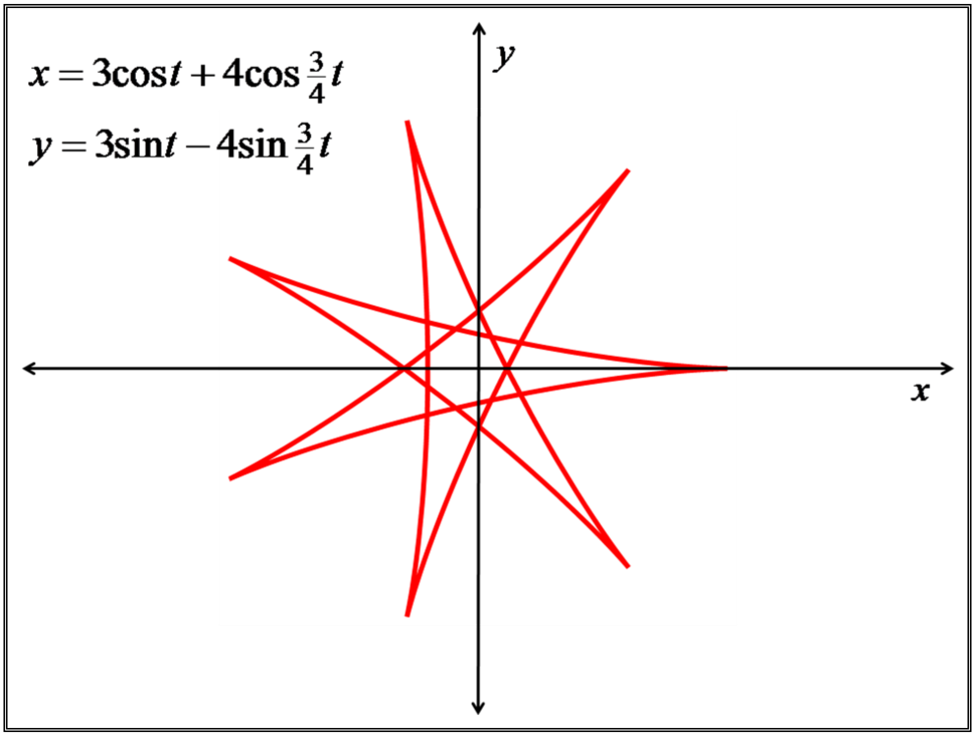

Figure 12-4: Hypocycloid with *a* = 7 and *b* = 4

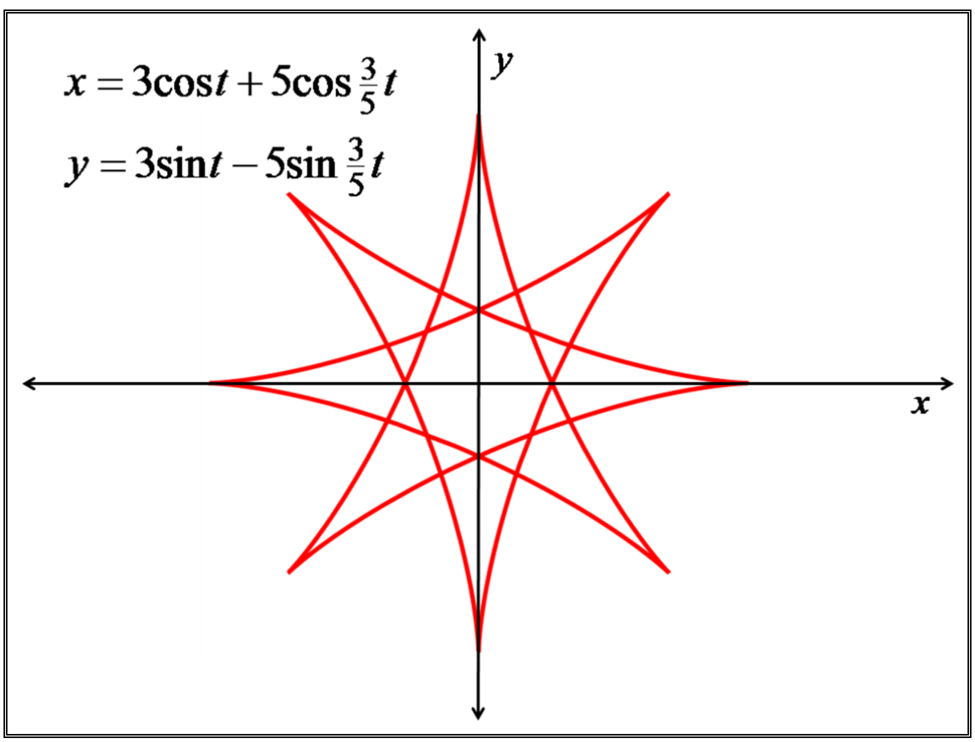

Figure 12-5: Hypocycloid with *a* = 8 and *b* = 5

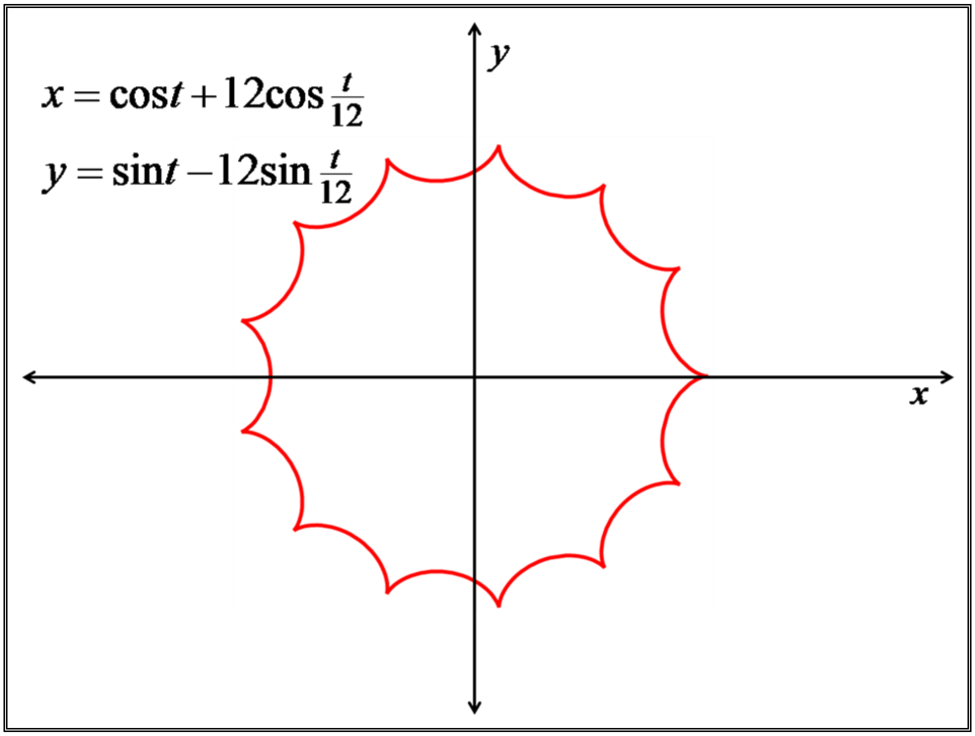

Figure 12-6: Hypocycloid with *a* = 13 and *b* = 12

12.3 Analytical and Physical Properties of the Hypocycloid

Based on the Hypocycloid's parametric representation found in Equation 12-1, that is, $x = (a-b)\cos t + b\cos\frac{a-b}{b}t$ and $y = (a-b)\sin t - b\sin\frac{a-b}{b}t$, the following subsections contain an analysis of the general Hypocycloid.

12.3.1 Derivatives of the Hypocycloid

- $\dot{x} = -(a-b)\left(\sin t + \sin\frac{a-b}{b}t\right)$.
- $\ddot{x} = -\dfrac{a-b}{b}\left[b\cos t + (a-b)\cos\frac{a-b}{b}t\right]$.
- $\dot{y} = (a-b)\left(\cos t - \cos\frac{a-b}{b}t\right)$.
- $\ddot{y} = -\dfrac{a-b}{b}\left[b\sin t - (a-b)\sin\frac{a-b}{b}t\right]$.
- $y' = \dfrac{\cos\frac{a-b}{b}t - \cos t}{\sin\frac{a-b}{b}t + \sin t}$.
- $y'' = \dfrac{2(2b-a)\sin^2\frac{at}{2b}}{b(b-a)\left(\sin\frac{a-b}{b}t + \sin t\right)^3}$.

12.3.2 Metric Properties of the Hypocycloid

To calculate the length of the Hypocycloid, we first calculate *dx*/*dt* and *dy*/*dt*, i.e.,

$$\frac{dx}{dt} = -(a-b)\left[\sin t + \sin \tfrac{a-b}{b}t\right] \quad \text{and} \quad \frac{dy}{dt} = (a-b)\left[\cos t - \cos \tfrac{a-b}{b}t\right].$$

Then,

$$\left(\frac{dx}{dt}\right)^2 = (a-b)^2\left[\sin^2 t + 2\sin t \sin \tfrac{a-b}{b}t + \sin^2 \tfrac{a-b}{b}t\right]$$

and

$$\left(\frac{dy}{dt}\right)^2 = (a-b)^2\left[\cos^2 t - 2\cos t \cos \tfrac{a-b}{b}t + \cos^2 \tfrac{a-b}{b}t\right].$$

Therefore, the sum of these last two expressions is

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2(a-b)^2\left[1 - \cos \tfrac{at}{b}\right].$$

However,

$$1 - \cos \tfrac{at}{b} = 2\sin^2 \tfrac{at}{2b}.$$

Hence, we have

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2(a-b)\sin \tfrac{at}{2b}.$$

To obtain the length of a single cusp, we then integrate this last expression from 0 to $2\pi b/a$. That is,

$$\text{Length of single cusp} = 2(a-b)\int_0^{2\pi b/a} \sin \tfrac{at}{2b}\, dt = -\frac{4b(a-b)}{a}\int_0^{2\pi b/a} d\left(\cos \tfrac{at}{2b}\right) = \frac{8b(a-b)}{a}.$$

As alluded to earlier, if $a/b = n$ is rational, then the curve closes on itself without intersection after n cusps. Therefore eliminating b in the expression for the length of a single cusp we arrive at the length of a single cusp in terms of the radius of the fixed circle, i.e.,

$$\frac{8a(n-1)}{n^2}.$$

Finally, multiplying this expression by the number of cusps, namely n, we derive the total length, s, of the n-cusped hypocycloid as,

$$\frac{8a(n-1)}{n}.$$

To calculate the area of the Hypocycloid, we first calculate $x\cdot dy/dt$ and then $y\cdot dx/dt$, that is,

$$x\frac{dy}{dt}=(a-b)^2\cos^2 t+\left(3ab-2b^2-a^2\right)\cos t\cos\tfrac{a-b}{b}t+\left(b^2-ab\right)\cos^2\tfrac{a-b}{b}t$$

and

$$y\frac{dx}{dt}=-(a-b)^2\sin^2 t+\left(3ab-2b^2-a^2\right)\sin t\sin\tfrac{a-b}{b}t+\left(ab-b^2\right)\sin^2\tfrac{a-b}{b}t\,.$$

Now, subtracting these two expressions to form $x\cdot dy/dt - y\cdot dx/dt$, we have after much manipulation the following expression

$$x\frac{dy}{dt}-y\frac{dx}{dt}=\left(3ab-2b^2-a^2\right)\left(\cos\tfrac{at}{b}-1\right).$$

But, the area of one cusp is

$$A=\frac{1}{2}\int_0^{2\pi b/a}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt=\frac{1}{2}\int_0^{2\pi b/a}\left(3ab-2b^2-a^2\right)\left(\cos\tfrac{at}{b}-1\right)dt.$$

This expression integrates as

$$A=\frac{3ab-2b^2-a^2}{2}\int_0^{2\pi b/a}\cos\tfrac{at}{b}\,dt-\frac{3ab-2b^2-a^2}{2}\int_0^{2\pi b/a}dt\,.$$

The first integral is zero and the second integral has the value

$$\frac{(n-1)(n-2)\pi a^2}{n^3}\text{, where } a = b\cdot n.$$

Now, if n is rational, after n cusps, the area is

$$A=\frac{(n-1)(n-2)\pi a^2}{n^2}.$$

If p represents the distance from the origin to the tangent of the Hypocycloid, then

$$p=(2b-a)\sin\tfrac{at}{2b}.$$

If r denotes the distance from the origin to the curve, then

$$r=\sqrt{(a-b)^2+b^2+2b(a-b)\cos\tfrac{a}{b}t}\,.$$

12.3.3 Curvature of the Hypocycloid

If ρ is the radius of curvature of the Hypocycloid, then

$$\rho = \frac{4b(a-b)}{2b-a}\sin\tfrac{at}{2b}.$$

If (α, β) are the coordinates of the center of curvature for the Hypocycloid, then

$$\alpha = \frac{a}{a-2b}\left[(a-b)\cos t - b\cos\tfrac{a-b}{b}t\right] \quad \text{and} \quad \beta = \frac{a}{a-2b}\left[b\sin\tfrac{a-b}{b}t + (a-b)\sin t\right].$$

12.3.4 Angles for the Hypocycloid

If ψ is the tangential-radial angle of the Hypocycloid, then

$$\tan\psi = \frac{2b-a}{a}\cdot\frac{1-\cos\frac{at}{b}}{\sin\frac{at}{b}}.$$

If ϕ denotes the tangential angle for the Hypocycloid, then

$$\tan\phi = \frac{\cos\frac{a-b}{b}t - \cos t}{\sin\frac{a-b}{b}t + \sin t}.$$

If θ denotes the radial angle for the Hypocycloid, then

$$\tan\theta = \frac{(a-b)\sin t - b\sin\frac{a-b}{b}t}{(a-b)\cos t + \cos\frac{a-b}{b}t}.$$

12.4 Geometry of the Hypocycloid

There are $\frac{a-b}{b}$ + 1 cusps if $\frac{a-b}{b}$ is an integer. The curve is symmetric about the *x*-axis, and is symmetric about the *y*-axis if $\frac{a-b}{b}$ is an odd integer. The curve is completely contained within a circle defined by $|r| \le a$.

12.5 Dynamic Geometry of the Hypocycloid

The next few subsections contain constructions concerning the Hypocycloid.

12.5.1 A Five-Cusped Hypocycloid

The construction below in Table 12-1 is for a five-cusped Hypocycloid. After presenting the construction, we will show how to modify the construction to make it for an *n*-cusped Hypocycloid, where *n* is any integer.

Table 12-1: A Five-Cusped Hypocycloid

1. Draw circle AB with center at A and passing through point B	7. Let C_3 be the image when C_2 is rotated about A' by $\angle$CAB
2. Let C be a random point on the circumference of circle AB	8. Let C_4 be the image when C_3 is rotated about A' by $\angle$CAB
3. Let A' be the image when A is dilated about point C by 1 / 5	9. Let C_5 be the image when C_4 is rotated about A' by $\angle$CAB
4. Draw circle A'C with center at A' and passing through point C	10. Trace point C_5 and change its color
5. Let C_1 be the image when C is rotated about A' by $\angle$CAB	11. Animate point C around circle AB
6. Let C_2 be the image when C_1 is rotated about A' by $\angle$CAB	

To modify this construction for that of an n-cusped Hypocycloid, change step 3 to read

Let point A' be the image when point A is dilated about point C by a factor of $1/n$.

Then, replace steps 6 through 9 with the steps that result from executing the following pseudo-language loop:

begin loop
for $i = 2$ to n
$4 + i$. Let C_i be the image when point C_{i-1} is rotated about point A' by $\angle$CAB.
end loop

Finally, change what is now step 10 (and will be step $5 + n$) to read

Trace point C_n and change its color.

12.5.2 An Adjustable Hypocycloid

Using the graphing capability of GSP, one can create a Hypocycloid which can be adjusted to trace any number of cusps. This is not a true geometric construction, but nevertheless it is interesting. Refer to Table 12-2.

Table 12-2: An Adjustable Hypocycloid

1. Create x-y axes with origin A and unit point B = (1, 0)	9. Calculate the y-coordinate of point E and relabel it as a
2. Draw circle BC centered at B and passing through point C	10. Calculate the y-coordinate of point F and relabel it as b
3. Let D be a random point on the circumference of circle BC	11. Calculate $(a - b) / b$
4. Draw line segment BD	12. Calculate $x = (a - b)\cos t + b\cos((a - b)/b)t$
5. Measure $\angle$CBD (in radians)	13. Calculate $y = (a - b)\sin t - b\sin((a - b)/b)t$
6. Relabel the measure of $\angle$CBD as t	14. Let point G be the result of plotting x and y, i.e., G(x, y)
7. Let E and F be two random points on the y-axis	15. Trace point G and change its color
8. Measure the coordinates of point E	16. Animate point D around circle AB

Well, this is quite a remarkable and interesting construction. By dragging point E or point F (or both) up and down the y-axis, one can adjust the value of the quantity $(a - b) / b$. Adjusting it to be an integer, say n, gives us closed Hypocycloids where the number of cusps is, as we have stated earlier, equal to $n + 1$. Making its value the integer 1 gives us a straight line as discussed in the digression at the beginning of this chapter; making its value 2 gives us a three-cusped Hypocycloid, or the Deltoid; making its value 3 gives us a four-cusped, or the Astroid; etc. If you have trouble adjusting it to be an integer, try changing the precision of the angle and distance measurements to "tenths" in the object preference window of GSP. Finally, by keeping point A above the x-axis (i.e., on the positive y-axis), dragging point B below the x-axis (i.e., on the negative y-axis), and re-adjusting to continue having the quantity $(a - b) / b$ as an integer, one can obtain the sister curves to the Hypocycloids, namely, the Epicycloids.

12.5.3 Hypocycloid Gears

Maybe if Roemer had had a Geometer's Sketchpad to aid in his study of gears, his job would have been much easier. Table 12-3 contains this construction.

Table 12-3: Hypocycloid Gears

1. Draw circle AB with center at A and passing through point B	14. Let D_4 be the image when D_3 is rotated about A_1 by $\angle DAB$
2. Let B_1 be the image of point B rotated about point A by 180º	15. Let D_5 be the image when D_4 is rotated about A by $\angle BAB_4$
3. Draw circle B_1B with center at B_1 and passing through point B	16. Construct the locus of D_5 while point D traverses circle AB
4. Let C be a random point on the circumference of circle B_1B	17. Let E_1 be the image when E is rotated about B_3 by $\angle EB_1C$
5. Let D be a random point on the circumference of circle AB	18. Let E_2 be the image when E_1 is rotated about B_3 by $\angle EB_1C$
6. Let A_1 be the image when A is dilated about point D by ¼	19. Let E_3 be the image when E_2 is rotated about B_3 by $\angle EB_1C$
7. Let E be a random point on the circumference of circle B_1B	20. Let E_4 be the image when E_3 is rotated about B_3 by $\angle EB_1C$
8. Let B_2 be the image when B is rotated about A by $\angle BB_1C$	21. Let E_5 be the image when E_4 is rotated about B_3 by $\angle EB_1C$
9. Let B_3 be the image when B_1 is dilated about point E by ⅛	22. Let E_6 be the image when E_5 is rotated about B_3 by $\angle EB_1C$
10. Let B_4 be the image when B_2 is rotated about A be $\angle BB_1C$	23. Let E_7 be the image when E_6 is rotated about B_3 by $\angle EB_1C$
11. Let D_1 be the image when D is rotated about A_1 by $\angle DAB$	24. Let E_8 be the image when E_7 is rotated about B_3 by $\angle EB_1C$
12. Let D_2 be the image when D_1 is rotated about A_1 by $\angle DAB$	25. Construct the locus of E_8 while point E traverses circle B_1B
13. Let D_3 be the image when D_2 is rotated about A_1 by $\angle DAB$	26. Animate point C around circle B_1B

12.5.4 A Five-Pointed Star

This Hypocycloid is one that crosses itself before closing and repeating its trace. Check it out in Table 12-4.

Table 12-4: A Five-Pointed Star

1. Draw circle AB with center at A and passing through point B	13. Let C_7 be the image when C_1 is rotated about A_1 by $\angle C_6AB$
2. Let C be a random point on the circumference of circle AB	14. Trace point C_7 and change its color
3. Let C_1 be the image when C is rotated about A by $\angle BAC$	15. Let C_8 be the image when C_7 is rotated about A by –120º
4. Let C_2 be the image when C_1 is rotated about A by $\angle BAC$	16. Let C_9 be the image when C_8 is rotated about A_2 by +120º
5. Let A_1 be the image when A is dilated about point C_1 by 2/5	17. Let C_{10} be the image when C_9 is rotated about A by –120º
6. Let C_3 be the image when C is rotated about point A by –120º	18. Let C_{11} be the image when C_{10} is rotated about A_3 by +120º
7. Let C_4 be the image when C_2 is rotated about A by $\angle BAC$	19. Draw line segment C_7C_9
8. Draw circle A_1C_1 with center at A_1 and passing through C_1	20. Draw line segment C_9C_{11}
9. Let A_2 be the image when A_1 is rotated about A by –120º	21. Draw line segment C_7C_{11}
10. Let C_5 be the image when C_3 is rotated about A by –120º	22. Construct polygon $C_7C_9C_{11}$
11. Let C_6 be the image when C_4 is rotated about A by $\angle BAC$	23. Animate point C around circle AB
12. Let A_3 be the image when A_2 is rotated about A by –120º	

We not only have this marvelous Hypocycloid, but we have an equilateral triangle that has each of its vertices on the Hypocycloid and each vertex traces the curve and the equilateral triangle's sides stay constant. Fascinating!

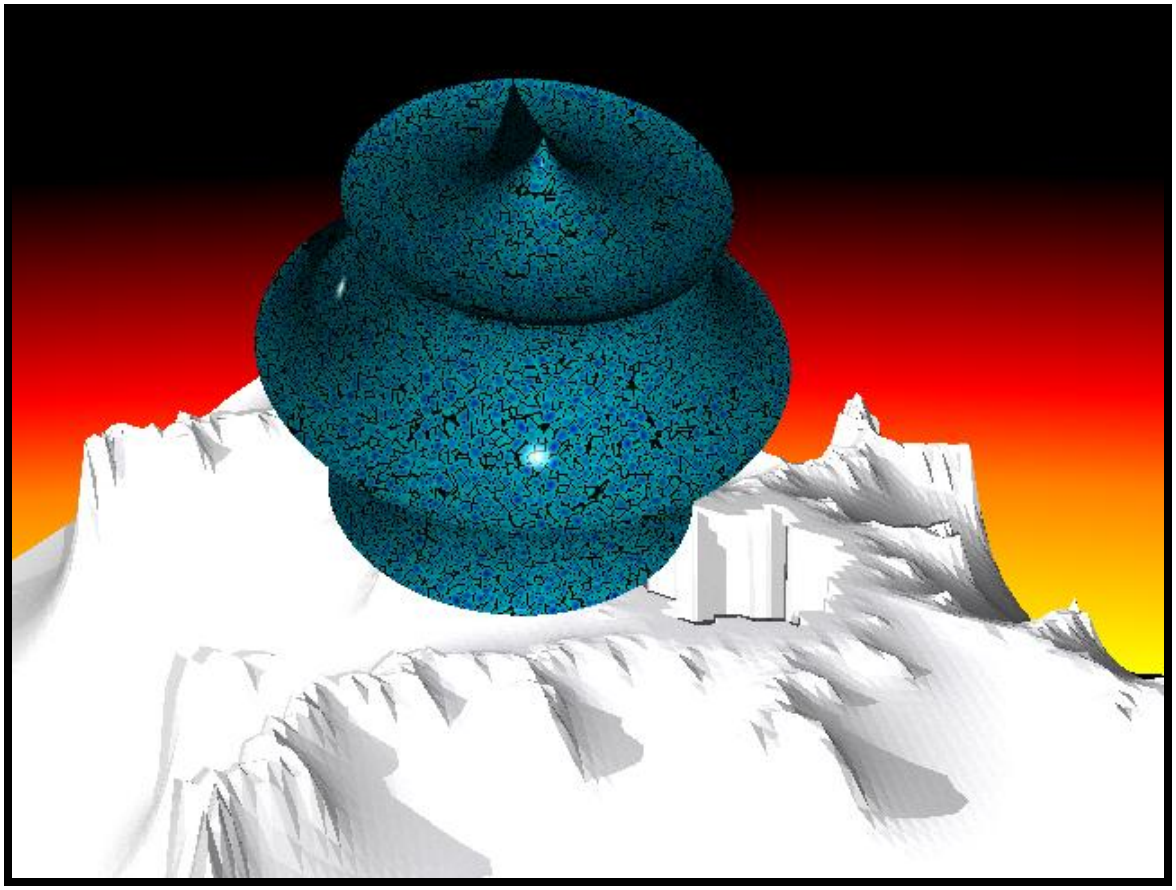

Figure 12-7: The Solid of Revolution Formed from an Eight-Cusped Hypocycloid

The object in the figure above is the solid of revolution that is formed when the curve represented by the parametric equations $x = 7\cos t + \cos 7t$ and $y = 7\sin t - \sin 7t$ is revolved about the y-axis. The resulting solid has then been placed so as to appear to be floating over snow covered mountains in the background. The solid has been given a bluish, crackled finish. Light sources have been located so as to cast a shadow of the uppermost cusp upon the solid itself.

Chapter 13 – The Hypotrochoid

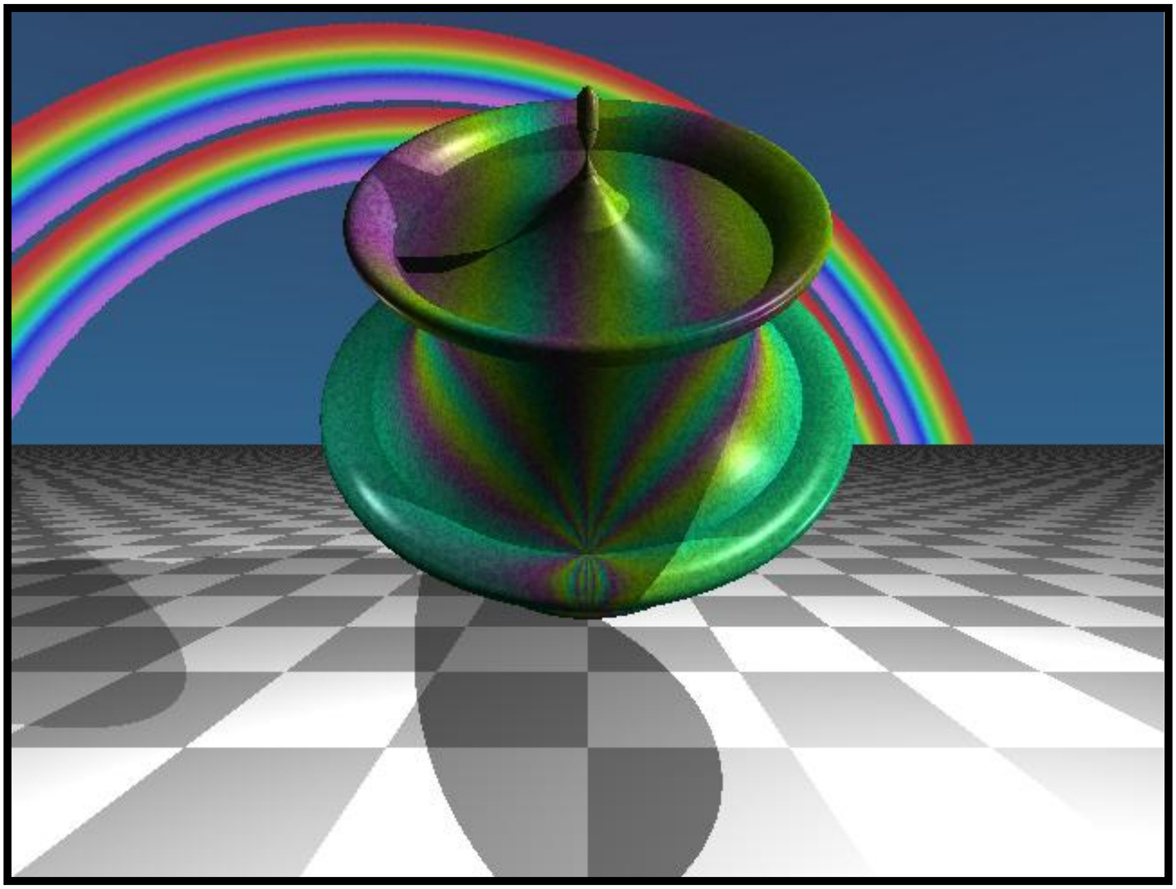

Figure 13-1: A Hypotrochoid Solid of Revolution

The object in the figure above is the solid of revolution that is formed when the Hypotrochoid represented by the parametric equations $x = 14\cos(t) + 3\cdot\cos(7t)$ *and* $y = 14\sin(t) - 3\sin(7t)$ *is revolved about the y-axis. The resulting solid has then been placed so as to appear to be floating over an infinite gray and white checkered plane which meets a blue, cloudless sky and rainbow at the horizon. Light sources have been placed so as to cast shadows on the plane and on the solid itself. The finish of the solid simulates reflection of the rainbow.*

13.1 Introduction

A Hypotrochoid is defined as the roulette traced by a point, P, attached to a circle rolling about the inside of a fixed circle. Of course, this sound very much like a Hypocycloid; what's the difference? In the case of a Hypocycloid, the point P is restricted to the circumference of the rolling circle. This is not the case for a Hypotrochoid; the point P may be interior to the rolling circle or it may be exterior to the rolling circle (i.e., on an extended radius of the rolling circle). Since the point P can be anywhere on the radius (or extended radius) of the rolling circle, Hypotrochoid really refers to a family of curves as opposed to one specific curve. If the traced point is outside the circumference of the rolling circle, the Hypotrochoid is sometimes referred to as a Prolate Hypocycloid; on the other hand, if the traced point is inside the circumference of the rolling circle, the Hypotrochoid is often referred to as a Curtate Hypocycloid.

Mathematicians first became fascinated with this curve in the early 16th century. The initial interest seems to have stemmed from a paper written in 1501 by Charles Bouvelles in an effort to solve the problem of squaring the circle. Giles Persone de Roberval, who played an integral role in finding the area for these curves, is given credit for the name "trochoid." Blaise Pascal, who referred to these curves as "roulettes," actually offered cash prizes for anyone able to solve the problems of finding their area and center of gravity. Galileo Galilei, who referred to these shaped as "cycloids," revered these curves for their graceful beauty and their architectural potential. Ultimately, for 20th century dwellers, it was Hasbro's release of the Spirograph (a child's toy that was very popular some years ago) that put the Hypotrochoid into mainstream awareness.

13.2 Equations and Graph of the Hypotrochoid

Now let us derive the parametric equations from this definition. Refer to Figures 13-2 and 13-3, which depict the initial position of the point P (i.e., at $t = 0$) and its position after the rolling circle has carried point P through an angle $t > 0$. Let a be the radius of the fixed circle while b is the radius of the rolling circle. Let h be the distance of the point P from the center of the rolling circle and let t be the angle between the horizontal and the line segment connecting the center of the fixed circle to the center of the rolling circle. Finally, let θ be the angle between the horizontal and the line segment connecting the center of the rolling circle to the point P. Note that as the small circle rolls around the circumference of the fixed circle, the center of the rolling circle travels on the circumference of a circle centered at the origin with radius $a - b$.

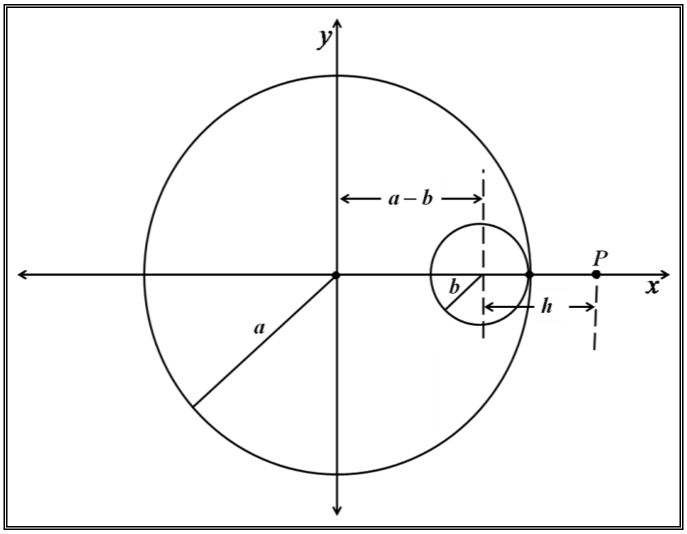

Figure 13-2: Initial Position of Rolling Circle and Point *P*

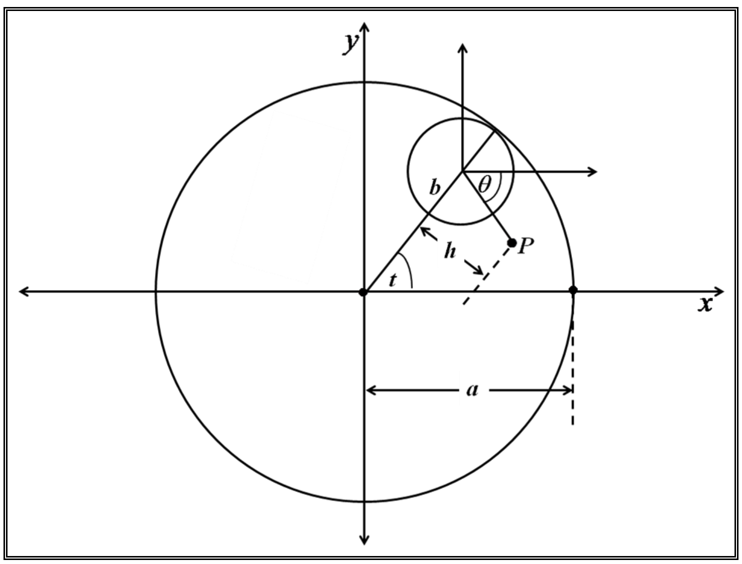

Figure 13-3: Position of Rolling Circle and Point *P* at Time *t* > 0

Since the circle is rolling without slippage, the length of the arc traveled by the smaller circle must be equal to the length of the arc traveled so far by the circle that is its path (the circle of radius $a - b$). In other words,

$$(a-b)t = b\theta,$$

or

$$\theta = \frac{a-b}{b} \cdot t .$$

The coordinates of the center of the rolling circle are $(a - b) \cdot \cos(t)$, $(a - b) \cdot \sin(t)$.

Now consider the origin to be the center of the rolling circle. With this new origin, the coordinates of the point P are $h \cdot \cos(-\theta)$, $h \cdot \sin(-\theta)$. However,

$$\cos(-\theta) = \cos(\theta) = \cos \tfrac{a-b}{b} t \quad \text{and} \quad \sin(-\theta) = -\sin(\theta) = -\sin \tfrac{a-b}{b} t.$$

Therefore, from the original origin, we can combine the two sets of coordinates to find the parametric equations of the Hypotrochoid, namely,

$$x = (a-b)\cos t + h\cos \tfrac{a-b}{b} t \quad \text{and} \quad y = (a-b)\sin t - h\sin \tfrac{a-b}{b} t. \qquad \textbf{Equation 13-1}$$

The equation of the tangent to the Hypotrochoid at the point $t = q$ is

$$y = \frac{h\cos \frac{a-b}{b} q - b\cos q}{b\sin q + h\sin \frac{a-b}{b} q} \cdot x + \frac{b(a-b) - h^2 - h(a-2b)\cos \frac{a}{b} q}{b\sin q + h\sin \frac{a-b}{b} q} \qquad \textbf{Equation 13-2}$$

Just as is the case with Epitrochoids, the triplet (a, b, h) completely specifies a particular Hypotrochoid. Figure 13-4 shows the graph of two different Hypotrochoids.

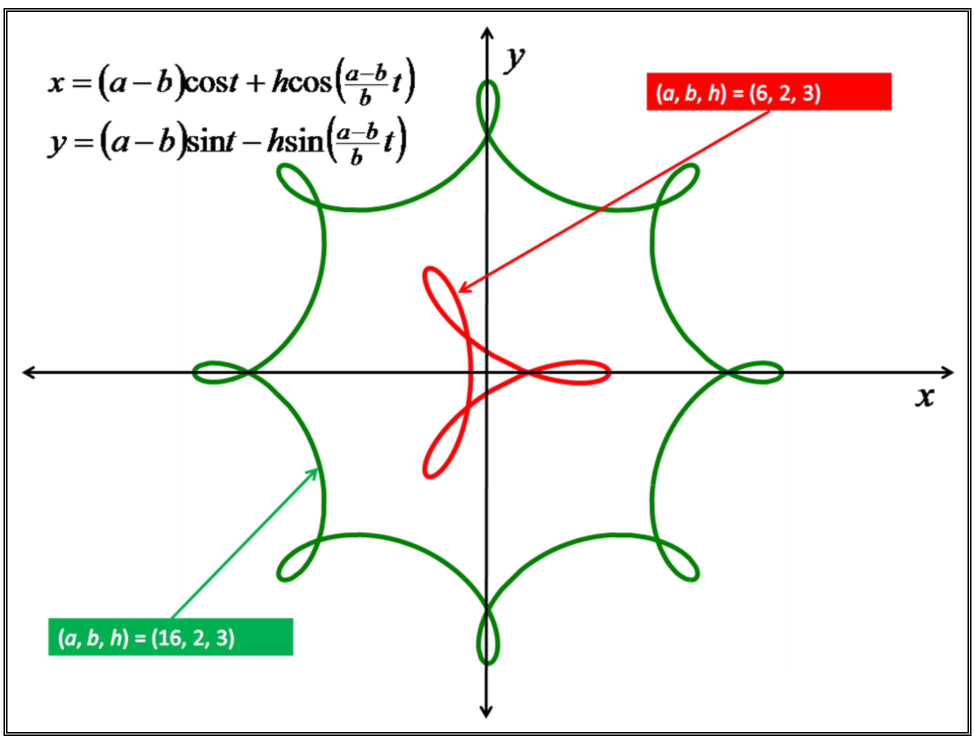

Figure 13-4: Graph of Two Distinct Hypotrochoids

13.3 Analytical and Physical Properties of the Hypotrochoid

Based on the Hypotrochoid's parametric representation found in Equation 13-1, that is, $x = (a-b)\cos t + h\cos \frac{a-b}{b} t$ and $y = (a-b)\sin t - h\sin \frac{a-b}{b} t$, the following subsections contain an analysis of the Hypotrochoid.

13.3.1 Derivatives of the Hypotrochoid

- $\dot{x} = \dfrac{b-a}{a}\left(b\sin t + h\sin\tfrac{a-b}{b}t\right).$

- $\ddot{x} = -(a-b)\cos t - \dfrac{h(a-b)^2}{b^2}\cos\tfrac{a-b}{b}t.$

- $\dot{y} = (a-b)\left(\cos t - \dfrac{h}{b}\cos\tfrac{a-b}{b}t\right).$

- $\ddot{y} = \dfrac{h(a-b)^2}{b^2}\sin\tfrac{a-b}{b}t - (a-b)\sin t.$

- $y' = \dfrac{h\cos\tfrac{a-b}{b}t - b\cos t}{b\sin t + h\sin\tfrac{a-b}{b}t}.$

- $y'' = \dfrac{h^2(a-b) - b^3 - bh(a-2b)\cos\tfrac{a}{b}t}{(a-b)\left(b\sin t + h\sin\tfrac{a-b}{b}t\right)^3}.$

13.3.2 Metric Properties of the Hypotrochoid

If p is the distance from the origin to the tangent of the Hypotrochoid, then

$$p = \frac{h^2 - b(a-b) + h(a-2b)\cos\tfrac{a}{b}t}{\sqrt{b^2 + h^2 - 2bh\cos\tfrac{a}{b}t}}.$$

If r denotes the distance from the origin to the Hypotrochoid, then

$$r = \sqrt{(a-b)^2 + h^2 + 2h(a-b)\cos\tfrac{a}{b}t}.$$

13.3.3 Curvature of the Hypotrochoid

If ρ is the radius of curvature for the Hypotrochoid, then,

$$\rho = \frac{(a-b)\left(b^2 + h^2 - 2bh\cos\tfrac{a}{b}t\right)^{3/2}}{b^3 - h^2(a-b) + bh(a-2b)\cos\tfrac{a}{b}t}.$$

If (α, β) denotes the coordinates of the center of curvature for the Hypotrochoid, then

$$\alpha = \frac{ah\left[b(a-b)\cos t\cos\tfrac{a}{b}t - h(a-b)\cos t + b^2\cos\tfrac{a-b}{b}t - bh\cos\tfrac{a-b}{b}t\cos\tfrac{a}{b}t\right]}{b^3 - h^2(a-b) + bh(a-2b)\cos\tfrac{a}{b}t} \quad \text{and}$$

$$\beta = \frac{ah\left[b(a-b)\sin t\cos\frac{a}{b}t - h(a-b)\sin t - b^2\sin\frac{a-b}{b}t + bh\sin\frac{a-b}{b}t\cos\frac{a}{b}t\right]}{b^3 - h^2(a-b) + bh(a-2b)\cos\frac{a}{b}t}$$

13.3.4 Angles for the Hypotrochoid

If ψ is the tangential-radial angle for the Hypotrochoid, then

$$\tan\psi = \frac{h^2 - b(a-b) + h(a-2b)\cos\frac{a}{b}t}{ah\sin\frac{a}{b}t}.$$

If ϕ denotes the tangential angle of the Hypotrochoid, then

$$\tan\phi = \frac{h\cos\frac{a-b}{b}t - b\cos t}{b\sin t + h\sin\frac{a-b}{b}t}.$$

If θ denotes the radial angle of the Hypotrochoid, then

$$\tan\theta = \frac{(a-b)\sin t - h\sin\frac{a-b}{b}t}{(a-b)\cos t + h\cos\frac{a-b}{b}t}.$$

13.4 Geometric Properties of the Hypotrochoid

The Hypotrochoid consists of $1 + (a-b)/b$ outer loops if $(a-b)/b$ is an integer. The curve is symmetric about the y-axis if $(a-b)/b$ is an odd integer. The curve is completely contained within a circle defined by $|\,r\,| \le a - b + h$.

13.5 Dynamic Geometry of the Hypotrochoid

The following seven subsections delineate constructions germane to the Hypotrochoid.

13.5.1 An Adjustable Hypotrochoid

For the complex (but very interesting and elegant) construction found in Table 13-1, assume that your computer screen is divided into four equal quadrants.

Table 13-1: An Adjustable Hypotrochoid

1. Draw horizontal line segment AB across top of screen	22. In upper-right quadrant draw vertical line segment LM
2. Let C be a random point on line segment AB	23. Somewhere to the left of line segment LM, place point N
3. Draw line segment AC and hide line segment AB and point B	24. Let N_1 be the image as N is reflected across line segment LM
4. Label line segment AC as *b*	25. Let C_1 be the circle centered at N with radius = *b*
5. Measure the length of line segment *b*	26. Let C_1' be the reflection of circle C_1 across line segment LM
6. Draw horizontal line segment DE across top of screen below *b*	27. Let C_2 be the circle centered at point N with radius = *h*
7. Let F be a random point on line segment DE	28. Let C_2' be the reflection of circle C_2 across line segment LM
8. Draw line segment DF and hide line segment DE and point E	29. Let O be a random point on the circumference of circle C_1
9. Label line segment DF as *a*	30. Let P be a random point on the circumference of circle C_2
10. Measure the length of line segment *a*	31. Let N_2 be the image when N is translated by *h* at $\angle$PNO
11. Calculate $a - b$	32. Let N_3 be the image as N_2 is reflected across line segment LM
12. Draw horizontal line segment GH across screen top below *a*	33. Draw line segments NN_2 and N_1N_3
13. Let I be a random point on line segment GH	34. Let Q be a point in the middle of the lower-left quadrant
14. Draw line segment GI and hide line segment GH and point H	35. Let circle C_3 be centered at Q with radius = *a*
15. Label line segment GI as *h*	36. Let C_4 be the circle centered at Q with radius = $a - b$
16. Measure the length of line segment *h*	37. Let R be a random point on the circumference of circle C_4
17. Draw horizontal line segment JK across screen top below *h*	38. Translate line segment N_1N_3 be vector $N_1 \rightarrow R$
18. Let J' be the image when J is translated by $a - b$ at $\angle$ 0º	39. Let N_4 be the image when N_3 is translated by vector $N_1 \rightarrow R$
19. Draw line segment JJ' and hide line segment JK and point K	40. Trace N_4 and change its color
20. Label line segment JJ' as $a - b$	41. Translate circle C_1' by vector $N_1 \rightarrow R$
21. Calculate $(a - b)/ b$	42. Simultaneously animate O about circle C_1 and R about C_4

By adjusting the length of line segments *a*, *b*, and/or *h* (that is, dragging points C, F, and/or I) one can generate different members of the Hypotrochoid family. Specifically, try and adjust segments *a* and/or *b* so that the quantity $(a - b) / b$ becomes an integer. Then the generated Hypotrochoid will be a closed curve (at least within the tolerances of GSP). Also, hide the construction that was placed in the upper-right quadrant and now move the construction that was placed in the lower-left quadrant into the middle of the screen. Doing this gives you more room to change the radii of the circles. Finally, hide circle C_4; doing so makes it clear that we have a circle of radius *b* rolling around the inside of a circle of radius *a* and that the point being traced is on a line segment radiating from the center of the rolling circle and is adjustable by manipulating the line segment labeled *h*. Have fun!

13.5.2 Variable Gears

This remarkable construction allows one to adjust (within certain limits) the shape of the gears. Refer to Table 13-2.

Table 13-2: Variable Gears

1. Draw circle AB with center at A and passing through point B	19. Draw line A_2D_4
2. Let A_1 be the image when A is rotated about point B by 180º	20. Let E be a random point on line A_2D_4
3. Let C be a random point on the circumference of circle AB	21. Let E_1 be the image when E is rotated about A by ∠BAC
4. Draw circle A_1B with center at A_1 and passing through B	22. Draw line segment A_2E
5. Let D be a random point on the circumference of circle AB	23. Let E_2 be the image when E_1 is rotated about A by $\angle D_1AB$
6. Let D_1 be the image when D is rotated about A by ∠BAC	24. Construct the locus of E_1 while point D traverses circle AB
7. Let B_1 be the image when B is rotated about A_1 by ∠DAB	25. Measure the length of line segment A_2E
8. Let point A_2 be the image when A is dilated about D by ⅓	26. Measure the length of line segment A_2D_4
9. Let A_3 be the image when A_2 is rotated about A by ∠BAC	27. Calculate scaling factor SF= (length of A_2E)/(length of A_2D_4)
10. Let B_2 be the image when B_1 is rotated about A_1 by ∠CAB	28. Let B_6 be the image when B_5 is dilated about B_3 by SF
11. Let B_3 be the image when B_1 is dilated about point A_1 by 4/3	29. Let E_3 be the image when E_2 is rotated about A_4 by $\angle BAD_1$
12. Let D_2 be the image when D is rotated about A_2 by ∠DAB	30. Let B_7 be the image when B_6 is rotated about A_1 by ∠CAB
13. Let A_4 be the image when A_3 is rotated about A by $\angle D_1AB$	31. Let E_4 be the image when E_3 is rotated about A_4 by $\angle BAD_1$
14. Let B_4 be the image when B_3 is rotated about A_1 by ∠CAB	32. Construct the locus of B_7 while point D traverses circle AB
15. Let D_3 be the image when D_2 is rotated about A_2 by ∠DAB	33. Let E_5 be the image when E_4 is rotated about A_4 by $\angle BAD_1$
16. Let D_4 be the image when D_3 is rotated about A_2 by ∠DAB	34. Draw line segment BE_5
17. Draw line segment A_2D_4	35. Construct $P_1 \perp$ to line segment BE_5 through point E_5
18. Let B_5 be the image when B_1 is rotated about B_3 by $\angle DA_2D_4$	36. Animate point C around circle AB

Well, what have we got here? First of all, note that point E is a random point on line A_2D_4 (step 20). As such, it can be dragged along line A_2D_4, and as one does so, the associated loci change shape. If one drags point E so that it stays between point A_2 and point D_4 (in other words remains on segment A_2D_4), the loci will remain as gears meshed with one another. When point E coincides with point A_2 (meaning the numerator of the scaling factor in step 27 is zero and of course the scaling factor itself is therefore zero), the two loci become circles that are tangent at point E_5. At the other extreme, that is when point E coincides with point D_4 (the scaling factor is one), one locus becomes a Deltoid and the other locus becomes a three-cusped Epicycloid. In between these two extremes, we have a Hypotrochoid gear meshing with an Epitrochoid gear. Pretty neat! Incidentally, perpendicular P_1 is always tangent to the Hypotrochoid and when point E is between points A_2 and D_4, it is also tangent to the Epitrochoid.

13.5.3 Another Adjustable Hypotrochoid

Just for variety's sake, here is another construction in which the relevant parameters can be adjusted to display other members of the Hypotrochoid family. In this construction found in Table 13-3, dragging point G changes the distance between the center of the rolling circle and the point being traced, i.e., the parameter that we have called *h* in the Hypotrochoid equations. In other words, dragging point G changes the radius of the rolling circle (*b* in the equations). For best results and a clear picture of what is going on, hide the following construction elements: circles C_2, C_3, C_4, perpendiculars P_1, P_2, P_3, P_4, and ray CD.

Table 13-3: Another Adjustable Hypotrochoid

1. Draw horizontal line AB	17. Let point J be the intersection of perpendiculars P_3 and P_4
2. Let C be a random point on line AB	18. Let C_2 be the circle centered at J and passing through point I
3. Let D be a second random point on line AB	19. Draw line segment JI
4. Let circle C_1 be centered at point D and pass through point C	20. Let C_3 be the circle centered at F with radius = segment JI
5. Draw ray CD starting at C and through D, then hide line AB	21. Let K be a random point on the circumference of circle C_2
6. Let E be a random point on ray CD	22. Let L be the intersection of circle C_3 and line segment FD
7. Let F be a random point on the circumference of circle C_1	23. Draw line segment JK
8. Construct $P_1 \perp$ to ray CD through point C	24. Let C_4 be the circle centered at L with radius = segment CE
9. Construct $P_2 \perp$ to ray CD through point D	25. Let F ' be the image when F is rotated about point L by $\angle$KJI
10. Let G be a third random point on ray CD	26. Draw line segment LF
11. Draw line segment CE	27. Let C_5 be the circle centered at L and passing through point F
12. Draw line segment FD	28. Draw ray LF ' starting at L and passing through point F'
13. Construct $P_3 \perp$ to ray CD through point G	29. Rotate segment LF about point L by $\angle$KJI
14. Let H be a random point on perpendicular P_2	30. Let point M be the intersection of circle C_4 and ray LF '
15. Construct $P_4 \perp$ to P_2 through point H	31. Trace point M and change its color
16. Let point I be the intersection of perpendiculars P_1 and P_4	32. Simultaneously animate K on circle C_2 and F on circle C_1

Of course, if point G is dragged so that it coincides with point C, the animation cannot be executed and further, if point G is dragged so that it either coincides with point D or is on the opposite side of point D from point C, the tracing point vanishes and no trace is drawn. To execute Hypotrochoids, point G must be confined to the space between points C and D.

13.5.4 A Three-Cornered/Cusped/Looped Hypotrochoid

Table 13-4 presents still another adjustable Hypotrochoid; however, in this case only the parameter h is adjustable. The quantity $(a - b) / b$ is fixed at 2. This means that when h is adjusted for a curtate configuration the constructed locus will have three rounded corners; when h is adjusted equal to b, the locus will have three cusps (i.e., a Deltoid); and finally, if h is adjusted for a prolate configuration the locus will have three loops. Now maybe the title of this subsection makes a little bit more sense. Oh, by the way, we will also create the tangent in this construction.

Table 13-4: A Three-Cornered/Cusped/Looped Hypotrochoid and Tangent

1. Draw circle AB with center at A and passing through point B	12. Draw line AC
2. Let C be a random point on the circumference of circle AB	13. Draw line AB'
3. Draw circle BC with center at B and passing through point C	14. Construct line L_1 parallel to line AB' through point D
4. Draw line segment AB	15. Construct line L_2 parallel to line AC through point D
5. Let C' be the image of C reflected across line segment AB	16. Let point E be the intersection of lines AB' and L_2
6. Draw circle C'B with center at C' and passing through point B	17. Let E' be the image as E is translated by vector A → E
7. Draw line segment AC'	18. Let point F be the intersection of lines AC and L_1
8. Let B' be the image of B reflected across line segment AC'	19. Draw line segment E'F
9. Draw line B'C	20. Construct $P_1 \perp$ to line segment E'F through point D
10. Let D be a random point on line B'C	21. Animate point C around circle AB
11. Construct the locus of D as point C traverses circle AB	

Drag point D along line B'C and watch the locus change configurations!

13.5.5 A Four-Cornered/Cusped/Looped Hypotrochoid with Tangent

Table 13-5 presents a construction analogous to the construction of the previous subsection except this time the quantity $(a - b) / b$ is fixed at 3, thereby giving an Astroid when $h = b$. Again, drag point D, this time along line CC" and watch the locus change.

Table 13-5: A Four-Cornered/Cusped/Looped Hypotrochoid with Tangent

1. Draw circle AB with center at A and passing through point B	12. Draw line AC
2. Let C be a random point on the circumference of circle AB	13. Draw line AC"
3. Draw circle BC centered at B and passing through point C	14. Construct line L_1 parallel to line AC through point D
4. Draw line segment AB	15. Construct line L_2 parallel to line AC" through point D
5. Let C' be the image of C reflected across line segment AB	16. Let point E be the intersection of lines AC" and L_1
6. Draw circle C'C centered at C' and passing through point C	17. Let E' be the image when E is translated by vector A → E
7. Draw line segment AC'	18. Let E" be the image when E' is translated by vector E → E'
8. Let C" be the image of C reflected across line segment AC'	19. Let point F be the intersection of lines AC and L_2
9. Draw line CC"	20. Draw line segment E"F
10. Let D be a random point on line CC"	21. Construct $P_1 \perp$ to line segment E"F through point D
11. Construct the locus of point D as C traverses circle AB	22. Animate point C around circle AB

13.5.6 Two Rotating Circles for One Hypotrochoid

To demonstrate that the Double Generation theorem of Bernoulli holds for Hypotrochoids, take a look at the construction in Table 13-6. We have even given spokes to the rotating circles to make it easier to observe what is happening.

Table 13-6: Double Generation of a Hypotrochoid

1. Draw circle AB centered at A and passing through point B	18. Draw line CE
2. Let C be a random point on the circumference of circle AB	19. Let point F be the intersection of lines AA" and CE
3. Let A' be the image when point A is dilated about C by ⅓	20. Draw circle AF centered at point A and passing through F
4. Draw circle A'C centered at A' and passing through point C	21. Draw circle A"F centered at A" and passing through point F
5. Let D be a 2nd random point on the circumference of circle AB	22. Draw line A"E
6. Draw circle DC	23. Let H and G be the intersections of line A"E with circle A"F
7. Draw line segment AD	24. Draw line segment HG
8. Let C' be the image of C reflected across line segment AD	25. Rotate line segment HG about point A" by 60°
9. Draw circle C'D centered at point C' and passing through D	26. Rotate line segment HG about point A" by – 60°
10. Draw line segment AC'	27. Let I be the unlabeled intersection of line A'D" and circle A'C
11. Let D' be the image of D reflected across line segment AC'	28. Draw line segment ID"
12. Let point D" be the image of point D' dilated about C by ⅓	29. Rotate line segment ID" about point A' by 60°
13. Draw line A'D"	30. Rotate line segment ID" about point A' by – 60°
14. Let E be a random point on line A'D"	31. Draw line segment A'E
15. Construct the locus of E as point C traverses circle AB	32. Draw line segment A"E
16. Let A" be the image when A is translated by vector A' → E	33. Animate point C around circle AB
17. Draw line AA"	

Well, this doesn’t really demonstrate that the Double Generation theorem holds for *all* Hypotrochoids, but it does demonstrate that it holds for all Hypotrochoids that have three corners/cusps/loops; the reader can extrapolate from here. It’s a nice construction and one can see that the small circle, circle A'C rotates about circle AB while the larger circle, circle A"F, rotates about circle AF. Further, each rotating circle’s radius has been extended to intersect point E, the point tracing the Hypotrochoid. Drag point E along line A'D" to change the shape of the Hypotrochoid.

13.5.7 The Osculating Circle for the Astroidal Type of Hypotrochoid

As a final construction for this chapter, consider the construction delineated in Table 13-7. By way of explanation, Astroidal type means an adjustable Hypotrochoid where the quantity $(a - b) / b$ is fixed at 3, but the parameter h can take on any value, and when $h = b$, we get an Astroid. A snapshot of the construction appears in Figure 13-5.

Table 13-7: The Osculating Circle for the Astroidal Type of Hypotrochoid

1. Draw circle AB centered at A and passing through point B	10. Construct the locus of point D as point C traverses circle AB
2. Let C be a random point on the circumference of circle AB	11. Draw line CD
3. Let point A' be the image when A is dilated about C by ¼	12. Construct $P_1 \perp$ to line CD through point C
4. Let C_1 be the image when C is rotated about A' by ∠CAB	13. Let point E be the intersection of line A'C_4 and P_1
5. Let C_2 be the image when C_1 is rotated about A' by ∠CAB	14. Draw line AE
6. Let C_3 be the image when C_2 is rotated about A' by ∠CAB	15. Let point F be the intersection of lines CD and AE
7. Let C_4 be the image when C_3 is rotated about A' by ∠CAB	16. Draw circle FD centered at point F and passing through D
8. Draw line A'C_4	17. Construct the interior of circle FD
9. Let D be a random point on line A'C_4	18. Animate point C around circle AB

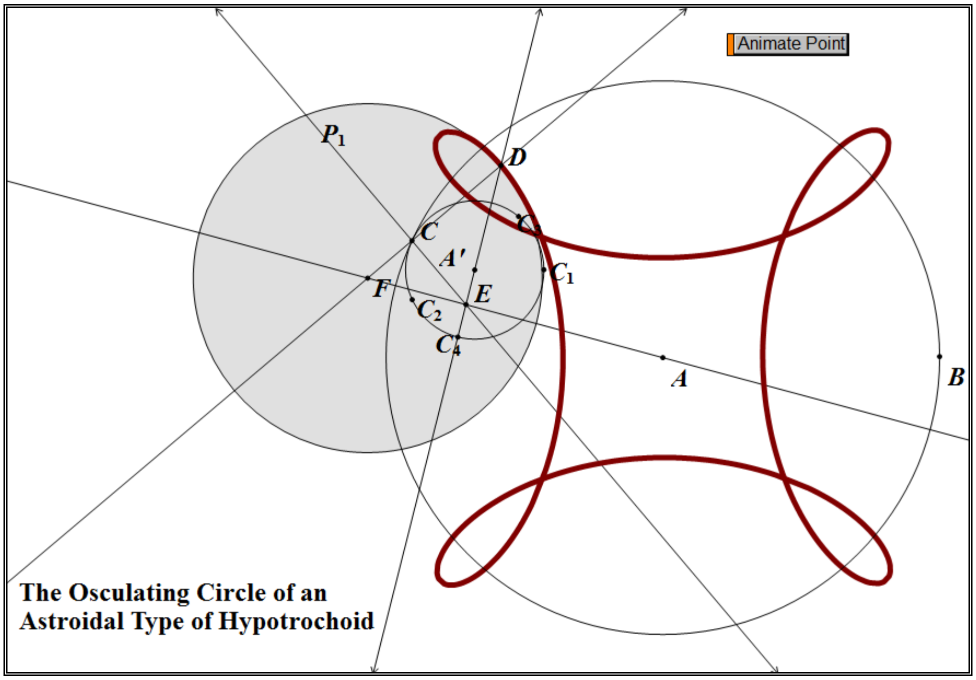

Figure 13-5: Osculating Circle for the Astroidal Type of Hypotrochoid

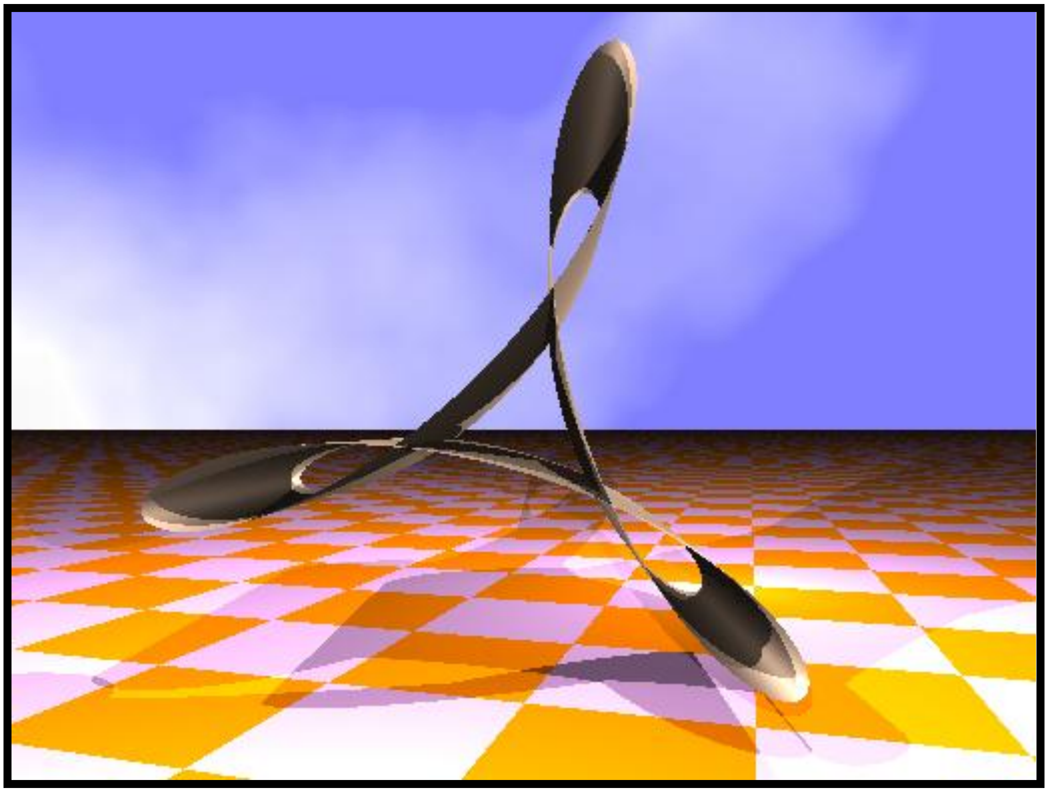

Figure 13-6: A Three-Dimensional Version of a Hypotrochoid

The Hypotrochoid with parameters (a, b, h) = (6, 2, 3) has been extruded into the third dimension to render the object in the figure above. It has been given a light sand colored finish and placed over the orange and plum colored plane. Multiple light sources have been used to illuminate the object, which causes the shadows to fall partially on the object itself and create the strange configuration of shadows seen on the plane. (This specific Hypotrochoid is used as the logo for the software Adobe Reader.)

Chapter 14 – The Conic Sections

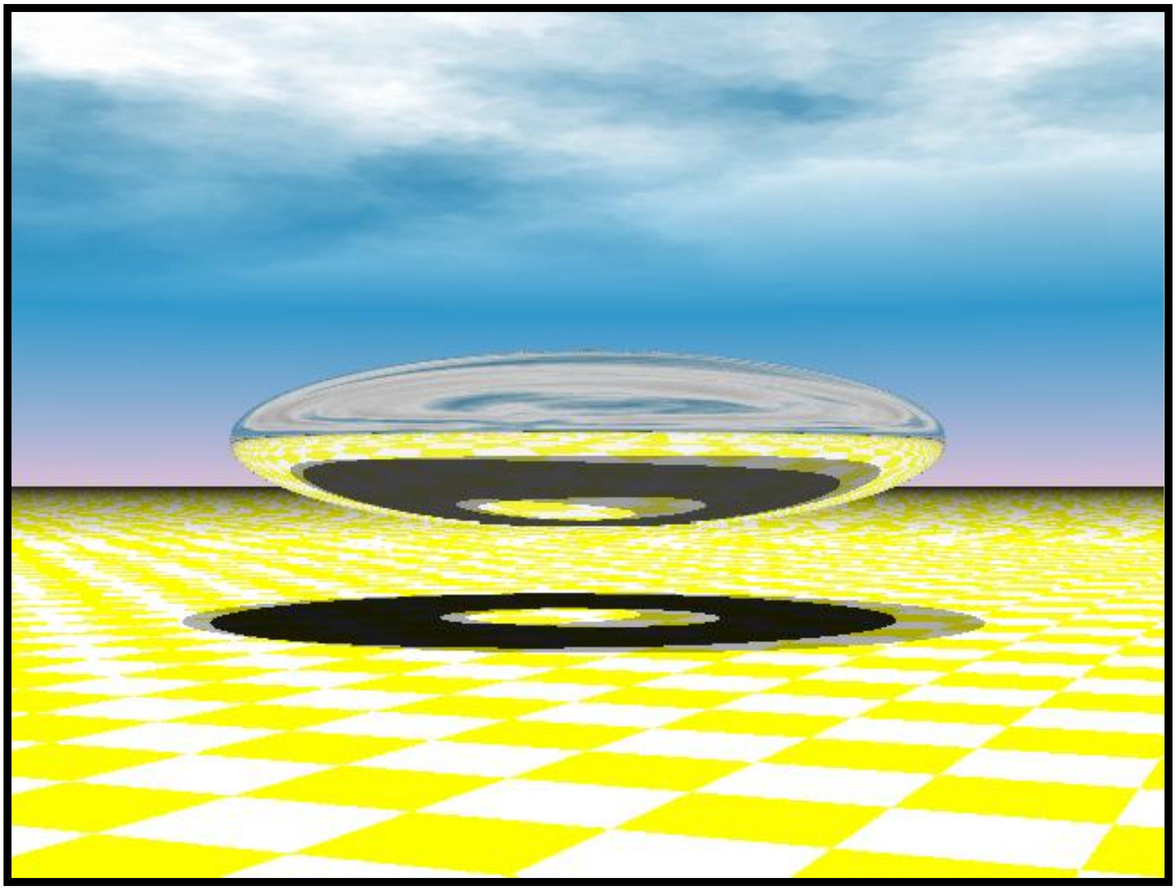

Figure 14-1: The Solid of Revolution Formed by an Ellipse

The ellipse with simple equation $x^2/16 + y^2 = 1$ has been revolved around the y-axis to form the object shown above. It has been given a mirrored reflective finish and placed over a white and yellow checkered plane with a partially cloudy sky at the horizon. The lower half of the object, of course, reflects the plane while the upper half reflects the sky. Light sources have been placed so as to cast a shadow of the object on the plane (thereby revealing its elliptical nature). Note how the object's shadow is also reflected in the finish.

14.1 Introduction

Simply put, a conic section is merely the intersection of a plane and a right-circular double cone (two single cones placed apex to apex). By changing the angle and the location of the intersection, one can produce a circle, an ellipse, a parabola, or a hyperbola. (Of course, if the plane intersects the cone and passes through the cone's vertex, a point, line, or two intersecting lines are produced; these are degenerate conic sections and will not be considered here.) If the plane is perpendicular to the axis of the cone, a circle is produced (see Figure 14-2). For a plane not perpendicular to the cone's axis, not parallel to the cone's generator line, and intersecting only the upper cone (or only the lower cone), an ellipse is produced. For a plane parallel to a generator line of the cone, a parabola is produced. And finally, for a plane intersecting both upper and lower cones, a hyperbola is produced.

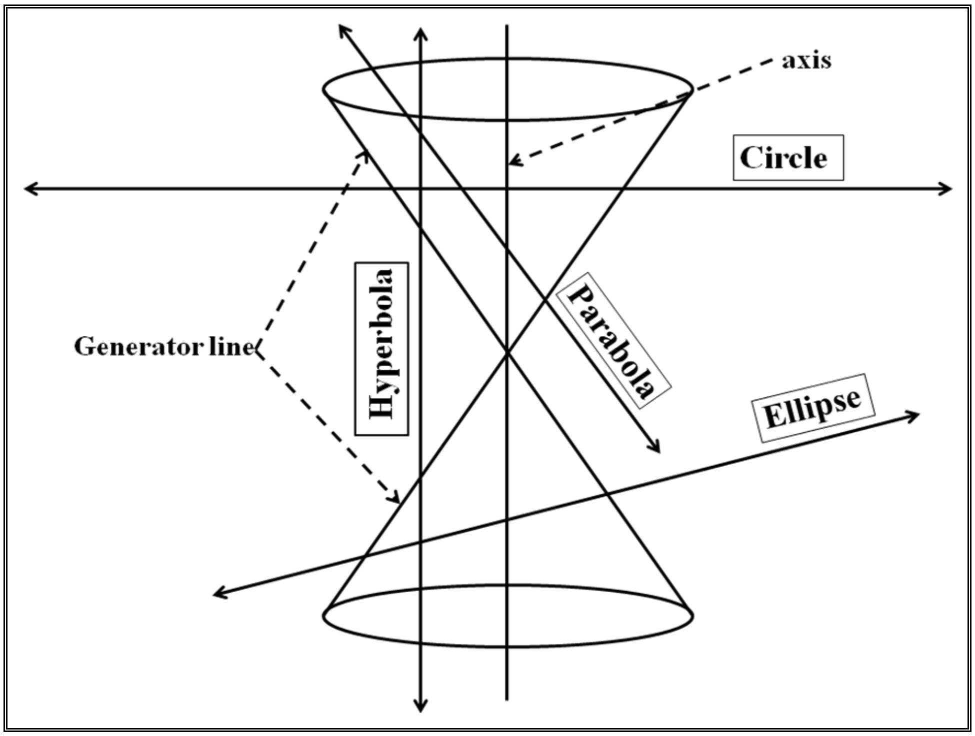

Figure 14-2: The Conic Sections

For an alternate and more rigorous definition of a conic section, consider a point F (called the *focus*), a line L that does not contain F (called the *directrix*), and a positive number e (called the *eccentricity*). A conic section is then defined as the locus of all points whose distance to F equals e times their distance to L. For $0 < e < 1$ we obtain an ellipse, for $e = 1$, we obtain a parabola, and for $e > 1$, we get a hyperbola. The case of a circle needs special treatment; one takes $e = 0$ and imagines the directrix as infinitely removed from the focus. The eccentricity of a conic section is therefore a measure of how far it deviates from being circular.

Conic sections have some interesting reflective properties that have very important real-world applications. Parabolic mirrors (mirrors made in the shape of parabaloids, that is, surfaces formed from rotating parabolas about their central axis) are used in reflecting telescopes because parabolic mirrors reflect all rays that are parallel to the mirror's axis to the focal point of the mirror, thereby forming a sharp image at the focus. Reverse this process (that is, put the light source at the focus) and all rays are reflected from the mirror parallel to one another. This is how an automobile's headlights operate. Parabolas of revolution are also used as signal receptors—a satellite dish is a good example of this property. If a mirror were in the form of an ellipsoid (an ellipse of revolution—see Figure 14-1 at the beginning of this chapter), the rays emitted from one focus are reflected toward the other focus. Therefore, in a room with an elliptical ceiling, sound emitted from one focus can be clearly heard at the other focus. This is called the whispering gallery effect, and a good example of this is found in Rome at St. Paul's Cathedral. Other applications of conic sections are found in planetary motion and more recently in space craft trajectories or astronavigation. Johannes Kepler discovered that the planetary orbits are ellipses with the sun at one of the foci. Newton was then able to derive the shape of orbits mathematically, under the assumption that gravitational force varies as the inverse square of distance. Depending on the energy of the orbiting body, orbit shapes which are any of the four types of conic sections are possible. Conic sections also play a role in projectile motion; a projectile will travel in the path of a parabola (if we neglect air resistance), a fact that is used for many military purposes.

14.2 Equations of the Conic Sections

A conic section with directrix at $x = 0$ (i.e., the y-axis), focus at the point $(p, 0)$, and eccentricity $e > 0$ (see Figure 14-3) has, by definition, the Cartesian equation

$$\sqrt{(x-p)^2 + y^2} = ex$$

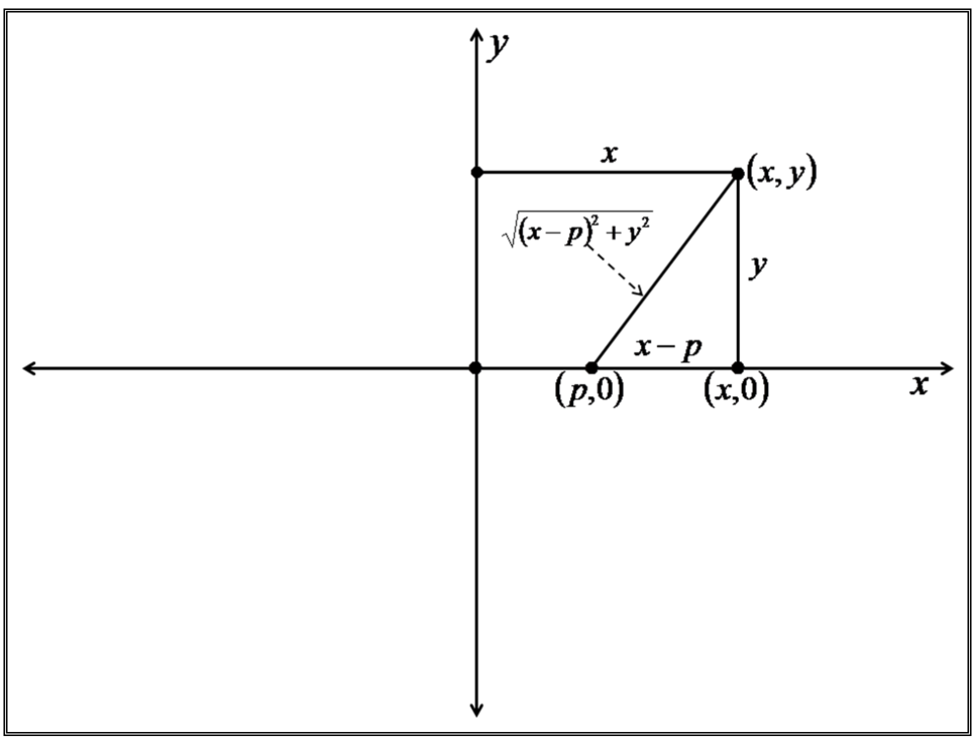

Figure 14-3: Distance to the Focus

Squaring and rearranging terms, we have

$$y^2 + (1 - e^2)x^2 - 2px + p^2 = 0 \quad \textbf{Equation 14-1}$$

The polar equation for a conic section is

$$r = \frac{ep}{1 + e\cos\theta} \quad \textbf{Equation 14-2}$$

In order to obtain a convenient form for the equation of a specific conic, equation 14-1 is usually subjected to a coordinate transformation. We will take up each of these coordinate transformations in turn.

14.2.1 Equations and Graph of the Parabola

In the case of a parabola, i.e., $e = 1$, the coordinate transformation is $(x, y) = (u + p/2, v)$. Substituting these values in Equation 14-1 for x and y yields

$$v^2 - 2pu = 0.$$

Since the variables can be named whatever we want, u and v may be renamed and we can revert to our usual x,y notation. Also, let $p/2 = a$ and we have the more common form for the Cartesian equation of a parabola. That is,

$$y^2 = 4ax \quad \textbf{Equation 14-3}$$

A parametric equation for the parabola may be derived by setting $t = 2\cot\theta$, where θ has the usual meaning. Therefore,

$$t = 2\cot\theta = \frac{2x}{y} = \frac{2}{y}\cdot\frac{y^2}{4a} = \frac{y}{2a}.$$

Hence, $y = 2at$, and x can easily be found by putting this value of y into Equation 14-3 and solving for x, that is, $x = at^2$. So we have a parametric form for the parabola,

$$(x, y) = at(t,2) \quad -\infty < t < +\infty \quad \textbf{Equation 14-4}$$

A polar form for the parabola can also be found. That is,

$$r^2 = x^2 + y^2 = x^2 + 4ax = r^2\cos^2\theta + 4ar\cos\theta.$$

Canceling r from both sides leaves $r = r\cos^2\theta + 4a\cos\theta$. Transposing so that all of the r-terms are on one side of the equation and then solving for r, we have

$$r = 4a\cot\theta\csc\theta \quad \textbf{Equation 14-5}$$

The equation of the tangent line to the parabola at the point $t = q$ is

$$qy = x + aq^2 \quad \textbf{Equation 14-6}$$

Figure 14-4 contains a graph of the parabola.

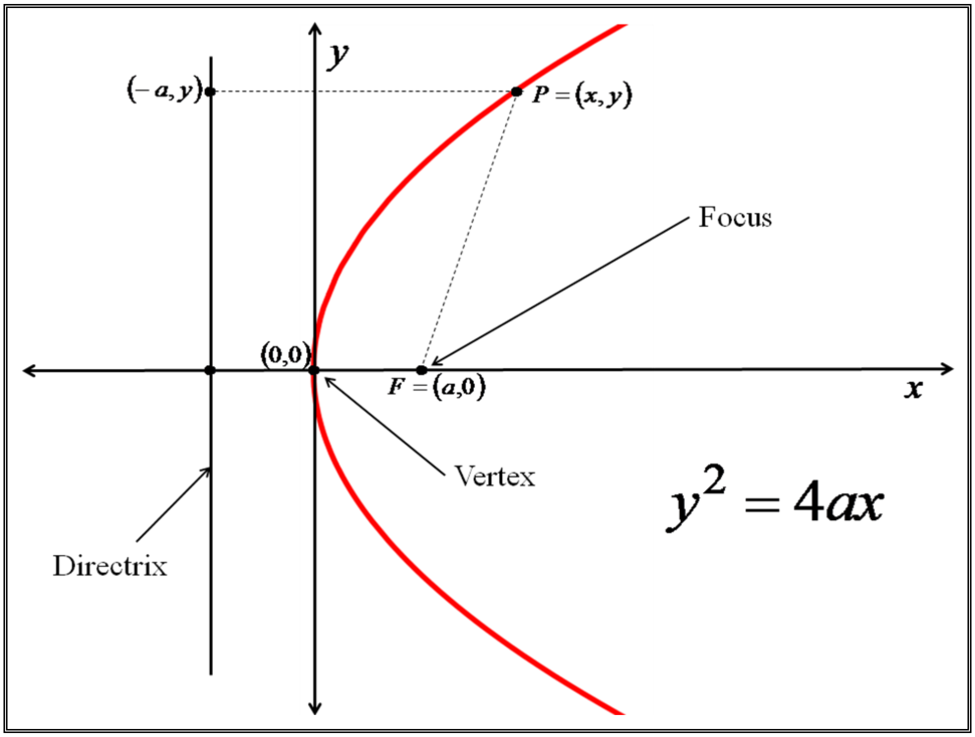

Figure 14-4: Graph of the Parabola

14.2.2 Analytical and Physical Properties of the Parabola

Based on the parabola's parametric representation found in Equation 14-4, that is, $x = at^2$ and $y = 2at$, the following subsections contain an analysis of the parabola.

14.2.2.1 Derivatives of the Parabola

- $\dot{x} = 2at$
- $\ddot{x} = 2a$
- $\dot{y} = 2a$
- $\ddot{y} = 0$
- $y' = \dfrac{1}{t}$
- $y'' = -\dfrac{1}{2at^3}$

14.2.2.2 Metric Properties of the Parabola

If p is the distance from the origin to the tangent of the parabola, then

$$p = \frac{at^2}{\sqrt{1+t^2}}.$$

If r denotes the distance from the origin to the parabola, then

$$r = at\sqrt{t^2 + 4}\,.$$

14.2.2.3 Curvature of the Parabola

If ρ is the radius of curvature for the parabola, then,

$$\rho = -2a\left(1 + t^2\right)^{3/2}.$$

If (α, β) are the coordinates of the center of curvature for the parabola, then

$$\alpha = 2a + 3at^2 \quad \text{and} \quad \beta = -2at^3.$$

14.2.2.4 Angles for the Parabola

If θ is the radial angle, then

$$\cot\theta = \frac{t}{2}\,.$$

If ψ is the tangential-radial angle for the parabola, then

$$\tan\psi = -\frac{t}{t^2 + 2}\,.$$

If ϕ denotes the tangential angle, then

$$\tan\phi = \frac{1}{t}\,.$$

14.2.3 Geometry of the Parabola

- Intercepts: (0, 0)
- Extrema: (0, 0)
- Extent: $0 \le x < \infty$; $-\infty < y < +\infty$
- Symmetries: Parabola $y^2 = 4ax$ is symmetric about the x-axis.

14.2.4 Equations and Graph of the Ellipse

In the case of an ellipse, i.e., $0 < e < 1$, then the transformation of coordinates is

$$(x, y) = \left(u + \frac{p}{1 - e^2}, v\right).$$

Substituting these values for x and y in Equation 14-1 yields

$$v^2 + u^2\left(1-e^2\right) - \frac{e^2 p^2}{1-e^2} = 0.$$

As before, reverting to x, y notation, and letting $a = ep / (1 - e^2)$ and $b^2 = a^2 (1 - e^2)$ we have the more common equation for an ellipse. That is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \textbf{Equation 14-7}$$

where $F(-ae, 0)$ is the location of the focus and the directrix has equation $x + a/e = 0$. Note, that by symmetry, $(ae, 0)$ and $x - a/e = 0$ are also a focus and directrix. A parametric representation may be derived by letting

$$\tan t = \frac{a}{b}\tan\theta = \frac{a}{b}\cdot\frac{y}{x}.$$

However, from Equation 14-7

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Therefore

$$\tan t = \frac{a}{b}\cdot\frac{\frac{b}{a}\cdot\sqrt{a^2-x^2}}{x} = \sqrt{\frac{a^2-x^2}{x^2}}.$$

From this we can see that

$$\sec^2 t = \tan^2 t + 1 = \frac{a^2}{x^2}.$$

Hence, $x = a \cos t$ and of course, substituting this expression back into Equation 14-7 we can see that $y = b \sin t$. So, our parametric representation for the ellipse is

$$(x, y) = (a\cos t, b\sin t) \quad -\pi \le t \le \pi \quad \textbf{Equation 14-8}$$

The polar equation for the ellipse is a direct result of plugging the common polar transformations $x = r \cos \theta$ and $y = r \sin \theta$ into Equation 14-7, that is,

$$r = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \quad \textbf{Equation 14-9}$$

The equation of the tangent line to the ellipse at the point $t = q$ is

$$a\cdot y + b\cot q\cdot x = ab\csc q \quad \textbf{Equation 14-10}$$

See Figure 14-5 for a graph of the ellipse.

14.2.5 Analytical and Physical Properties of the Ellipse

Based on the ellipse's parametric representation found in Equation 14-8, that is, $x = a \cos t$ and $y = b \sin t$, the following subsections contain an analysis of the ellipse.

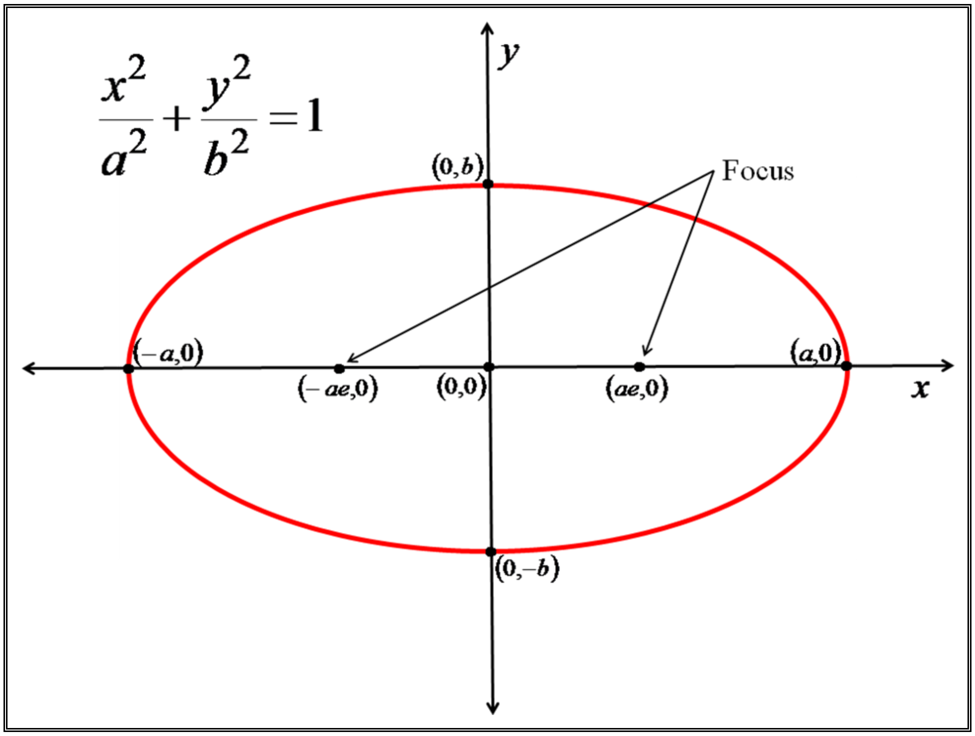

Figure 14-5: Graph of the Ellipse

14.2.5.1 Derivatives of the Ellipse

- $\dot{x} = -a\sin t$
- $\ddot{x} = -a\cos t$
- $\dot{y} = b\cos t$
- $\ddot{y} = -b\sin t$
- $y' = -\frac{b}{a}\cot t$
- $y'' = -\frac{b}{a^2}\csc^3 t$

14.2.5.2 Metric Properties of the Ellipse

In order to calculate the area of the ellipse, consider an incremental rectangle of height y and width dx in the first quadrant of the ellipse; its area is, of course, $y \cdot dx$. If we integrate that quantity from 0 to a, we obviously have the area of the portion of the ellipse that lies in the first quadrant. If we then multiply that result by 4 (due to symmetry), we have the total area of the ellipse. Hence,

$$A = 4\int_0^a y dx.$$

However, from Equation 14-7 we know that

$$y = \frac{b}{a}\sqrt{a^2 - x^2}\,.$$

Therefore,

$$A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx\,.$$

This integral can easily be evaluated by making the substitution $x = a \sin u$. Under this transformation, the integral becomes

$$A = \frac{4b}{a}\int_0^{\pi/2} a^2 \cos^2 u \cdot du = 4ab\int_0^{\pi/2} \cos^2 u \cdot du = 4ab\int_0^{\pi/2}\left(\frac{1}{2} + \frac{1}{2}\cdot\cos 2u\right)du\,.$$

Hence, $A = \pi ab$.

In a similar manner, when the ellipse is rotated about the *x*-axis to form a solid of revolution (i.e., an ellipsoid), its volume can be calculated. Consider an incremental circular disk (in the first and fourth quadrants of the ellipse) whose radius is *y* and whose thickness is *dx*; its volume is, of course, $\pi y^2 \cdot dx$. If we integrate that quantity from 0 to *a*, we have half of the required volume. Multiplying that result by 2 (the symmetry argument again), we have the total volume of the ellipsoid. That is,

$$V = 2\pi\int_0^a y^2 dx = \frac{2\pi b^2}{a^2}\int_0^a \left(a^2 - x^2\right)dx = \frac{4}{3}\pi ab^2.$$

If *p* is the distance from the origin to the tangent of the ellipse, then

$$p = \frac{-ab}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}\,.$$

If *r* is the distance from the origin to the ellipse, then

$$r = \sqrt{a^2 \cos^2 t + b^2 \sin^2 t}\,.$$

14.2.5.3 Curvature of the Ellipse

If ρ is the radius of curvature for the ellipse, then,

$$\rho = \frac{1}{ab}\left(a^2 \sin^2 t + b^2 \cos^2 t\right)^{3/2}.$$

If (α, β) are the coordinates of the center of curvature for the ellipse, then

$$\alpha = \frac{a^2 - b^2}{a}\cos^3 \mathrm{t} \quad \text{and} \quad \beta = \frac{b^2 - a^2}{b}\sin^3 t\,.$$

14.2.5.4 Angles for the Ellipse

If θ is the radial angle, then

$$\tan\theta = \frac{b}{a}\tan t.$$

If ψ is the tangential-radial angle for the ellipse, then

$$\tan\psi = \frac{ab}{b^2 - a^2}\cdot \sec t \csc t\,.$$

If ϕ denotes the tangential angle for the ellipse, then

$$\tan\phi = -\frac{b}{a}\cot t\,.$$

14.2.6 Geometry of the Ellipse

- Intercepts: $(a, 0)$; $(–a, 0)$; $(0, b)$; $(0, –b)$.
- Extrema: $(a, 0)$; $(–a, 0)$; $(0, b)$; $(0, –b)$.
- Symmetry: The ellipse is symmetric about the x-axis, the y-axis, and the origin.

14.2.7 Equations and Graph of the Hyperbola

In the case of a hyperbola, i.e., $e > 1$, then the transformation of coordinates is $(x, y) = \left(u - \frac{p}{e^2 - 1}, v\right)$. Substituting these values for x and y in Equation 14-1 yields

$$v^2 - (e^2 - 1)u^2 + \frac{e^2 p^2}{e^2 - 1} = 0\,.$$

Again, reverting to x,y notation and letting $a = ep/(e^2 – 1)$ and $b^2 = a^2(e^2 – 1)$ we have the common equation for a hyperbola. That is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \textbf{Equation 14-11}$$

where $F(–ae, 0)$ and $F(ae, 0)$ are the two foci.

In order to derive a parametric representation for the hyperbola, consider the following: let $\tan\theta = (b/a)\sin t$. From Equation 14-11 we have

$$y = \frac{b}{a}\sqrt{x^2 - a^2}\,.$$

Therefore,

$$\tan\theta = \frac{y}{x} = \frac{b}{ax}\sqrt{x^2 - a^2} = \frac{b}{a}\sin t$$

or

$$\sin^2 t = \frac{x^2 - a^2}{x^2}.$$

Solving this expression for *x*, we find that $x = a \sec t$. Plugging this value of *x* back into Equation 14-11 and solving for *y*, we find that $y = b \tan t$. Hence, a parametric representation for the hyperbola is

$$(x, y) = (a\sec t, b\tan t) \quad -\pi \le t \le \pi \qquad \textbf{Equation 14-12}$$

The polar equation for the hyperbola is a direct result of plugging the common polar transformations $x = r \cos \theta$ and $y = r \sin \theta$ into Equation 14-11, that is,

$$r = \frac{ab}{\sqrt{b^2\cos^2\theta - a^2\sin^2\theta}} \qquad \textbf{Equation 14-13}$$

The equation of the tangent line to the hyperbola at the point $t = q$ is

$$a \cdot y = b\csc q \cdot x - ab\cot q \qquad \textbf{Equation 14-14}$$

See Figure 14-6 for a graph of the hyperbola.

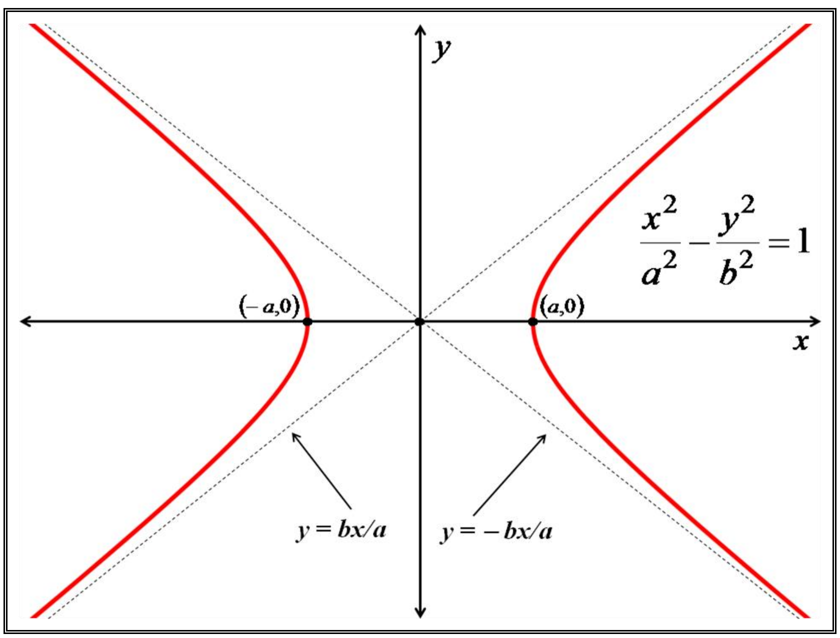

Figure 14-6: Graph of the Hyperbola

14.2.8 Analytical and Physical Properties of the Hyperbola

Based on the hyperbola's parametric representation found in Equation 14-12, that is, $x = a \sec t$ and $y = b \tan t$, the following subsections contain an analysis of the hyperbola.

14.2.8.1 Derivatives of the Hyperbola

- $\dot{x} = a \tan t \sec t = a \sin t \sec^2 t$
- $\ddot{x} = a \sec t\left(1 + 2\tan^2 t\right) = a \sec^3 t\left(1 + \sin^2 t\right)$
- $\dot{y} = b \sec^2 t$
- $\ddot{y} = 2b \sin t \sec^3 t$
- $y' = \dfrac{b}{a} \csc t$
- $y'' = -\dfrac{b}{a^2} \cot^3 t$

14.2.8.2 Metric Properties of the Hyperbola

If p is the distance from the origin to the tangent of the hyperbola, then

$$p = \frac{-ab\cos t}{\sqrt{a^2 \sin^2 t + b^2}}.$$

If r denotes the radial distance to the hyperbola, then

$$r = \sqrt{\left(a^2 + b^2\right)\tan^2 t + a^2}.$$

14.2.8.3 Curvature of the Hyperbola

If ρ is the radius of curvature for the hyperbola, then

$$\rho = -\frac{\sec^3 t\left(a^2 \sin^2 t + b^2\right)^{3/2}}{ab}.$$

If (α, β) are the coordinates of the center of curvature for the hyperbola, then

$$\alpha = \frac{a^2 + b^2}{a\cos^3 t} \quad \text{and} \quad \beta = -\frac{\sin^3 t\left(a^2 + b^2\right)}{b\cos^3 t}.$$

14.2.8.4 Angles for the Hyperbola

If θ is the radial angle, then

$$\tan\theta = \frac{b}{a}\sin t.$$

If ψ is the tangential-radial angle for the hyperbola, then

$$\tan\psi = \frac{ab\cos^2 t}{\sin t\left(a^2 + b^2\right)}.$$

If ϕ denotes the tangential angle for the hyperbola, then

$$\tan\phi = \frac{b}{a}\csc t\,.$$

14.2.9 Geometry of the Hyperbola

- Intercepts: (a, 0); ($-a$, 0).
- Extrema: (a, 0); ($-a$, 0).
- Extent: $-\infty < x < +\infty$; $-\infty < y < +\infty$.
- Symmetry: The hyperbola is symmetric about the x-axis, the y-axis, and the origin.
- Asymptotes: $y = (b/a)\,x$; $y = -\,(b/a)\,x$.

14.2.10 Equations and Graph of the Circle

The preceding discussion that started in section 14.2 regarding the equations of the various conic sections does not include the circle, since the eccentricity, e, is, by definition, nonzero. However, the circle may be considered as a limiting case of the ellipse where $a = b$. As a result, from Equation 14-7 we have for the equation of the circle

$$x^2 + y^2 = a^2$$ **Equation 14-15**

where a is the radius of the circle. Similarly, from Equation 14-8 we have a parametric representation of the circle, i.e.,

$$(x, y) = a\,(\cos t, \sin t) \qquad -\pi \le t \le \pi$$ **Equation 14-16**

Of course, the polar equation for a circle is trivial, it being

$$r = a$$ **Equation 14-17**

The pedal, Whewell, and Cesáro equations are also quite simple. Respectively, they are

$$pa = r^2$$ **Equation 14-18**

$$s = a\varphi$$ **Equation 14-19**

$$\rho = a$$ **Equation 14-20**

Finally, the equation of the circle's tangent at the point $t = q$ is

$$y = -\cot q \cdot x + a\csc q$$ **Equation 14-21**

Figure 14-7 depicts a graph of the circle.

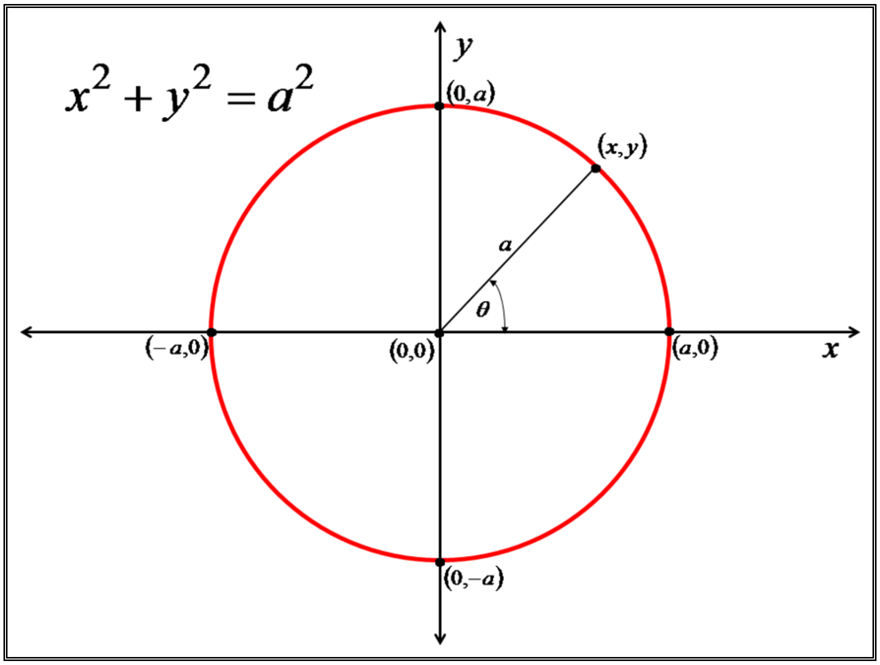

Figure 14-7: Graph of the Circle

14.2.11 Analytical and Physical Properties of the Circle

Based on the circle's parametric representation found in Equation 14-16, that is, $x = a \cos t$ and $y = a \sin t$, the following subsections contain an analysis of the circle.

14.2.11.1 Derivatives of the Circle

- $\dot{x} = -a \sin t$
- $\ddot{x} = -a \cos t$
- $\dot{y} = a \cos t$
- $\ddot{y} = -a \sin t$
- $y' = -\cot t$
- $y'' = -\frac{1}{a} \csc^3 t$

14.2.11.2 Metric Properties of the Circle

The familiar formula for the area of a circle can easily be derived by considering the area of an incremental rectangle of height y and width dx in the first quadrant of the circle, i.e., $dA = y \cdot dx$. Upon integration of this expression from 0 to a we obtain the area of the circle in the first quadrant. Multiplying by 4 due to symmetry then gives the total area of the circle. That is,

$$A = 4\int_0^a y \cdot dx = 4\int_0^a \sqrt{a^2 - x^2}\,dx .$$

The transformation $x = a \sin u$ reduces this integral to

$$A = 4a^2 \int_0^{\pi/2} \cos^2 u \cdot du = 2a^2 \int_0^{\pi/2} (1 + \cos 2u)du = \pi a^2 .$$

The familiar formula for the volume of a sphere can also easily be derived by rotating the circle about the *x*-axis and considering the solid of revolution so produced. Within this solid of revolution, assume a thin disk of radius *y* and thickness *dx*. The volume of this disk will be $dV = \pi y^2 \cdot dx$. Integrating from 0 to *a* and multiplying by 2 will then give the total volume of the sphere. That is,

$$V = 2\pi \int_0^a y^2 \cdot dx = 2\pi \int_0^a (a^2 - x^2)dx = 2\pi a^2 \Big|_0^a - \frac{2\pi}{3} x^3 \Big|_0^a = 2\pi a^3 - \frac{2}{3}\pi a^3 = \frac{4}{3}\pi a^3 .$$

Of course, the length of the circle (usually called its circumference) is also easily computed by considering the integral

$$s = \int_0^{2\pi} \sqrt{(dx)^2 + (dy)^2}\,dt = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t}\,dt = \int_0^{2\pi} a \cdot dt = 2\pi a.$$

The surface area of the sphere formed as a solid of revolution from the circle can also be calculated by considering the integral

$$S = 2\pi \int_0^{\pi} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 2\pi \int_0^{\pi} a \sin t \sqrt{a^2 \sin^2 t + a^2 \cos^2 t}\,dt = 4\pi a^2.$$

Finally, although it may be obvious and trivial, if *p* is the distance from the origin to the circle's tangent and r is the radial distance, then

$$p = -a$$

and

$$r = \sqrt{x^2 + y^2} .$$

14.2.12 Geometry of the Circle

- Intercepts: $(a, 0)$; $(-a, 0)$; $(0, a)$; $(0, -a)$.
- Extrema: $(a, 0)$; $(-a, 0)$; $(0, a)$; $(0, -a)$.
- Extent: $-a \le x \le a$; $-a \le y \le a$.
- Symmetry: The circle is symmetric about the *x*-axis, the *y*-axis, and the origin.

14.3 Dynamic Geometry of the Conic Sections

The following subsections present various constructions involving the conic sections.

14.3.1 A Selectable Conic Section

This construction allows you, by dragging a specific point, to verify some of the theory presented in the introduction to this chapter. Namely, if the eccentricity is between zero and one the conic section is an ellipse; if it's equal to one, the conic section is a parabola; and if it's greater than one, a hyperbola. See Table 14-1 to check it out!

Table 14-1: A Selectable Conic Section

1. Draw horizontal line segment AB	9. Let m_2 be a measure of the distance CB
2. Let C be a random point on line segment AB	10. Let $m_3 = m_1 / m_2$
3. Let D be the midpoint of line segment AB	11. Let m_4 be a measure of the distance AB
4. Draw circle AE with center at A and passing through point E	12. Let m_5 be the measure of ∠BAF
5. Let F be a random point on the circumference of circle AE	13. Let $m_6 = (m_3 \cdot m_4) / [1 + m_3 \cdot \cos(m_5)]$
6. Draw ray AF starting at point A and passing through point F	14. Let A' be the image when A is translated by m_6 at $\angle m_5$
7. Construct $P_1 \perp$ to line segment AB through point B	15. Trace point A' and change its color
8. Let m_1 be a measure of distance AC	16. Animate point F around circle AE

Drag point C along line segment AB. If you drag it so that it lies between points A and D, the trace you see when the animation is run will be that of an ellipse. If you place point C between points D and B, your trace will be that of a hyperbola. And, if you place it to coincide with point D, your trace will be a parabola. Why? Well, first of all, note that point A is the focus, perpendicular P_1 is the directrix, ∠BAF is the radial angle, and the ratio of AC to CB (or in the notation of the steps above, m_1 to m_2) is the eccentricity. Therefore, what is calculated in step 13 (that is, m_6) is the polar equation for a conic section (see Equation 14-2). So point A' is simply the radial distance from the pole point to the curve. That is why point A' traces the conic section. Now, the ratio of m_1 to m_2 is equal to 1 when point C coincides with the midpoint of the line segment, that is, point D (and we know what happens when the eccentricity is 1—we get a parabola). When the ratio is greater than 1, that is, when point C lies between points D and B, we get the hyperbola, and when the ratio is less than 1 (when C lies between A and D) we get the ellipse. Note that we cannot get a circle with this construction; however, as the eccentricity approaches 0 (in other words as point C approaches point A), the ellipse gets closer and closer to a circle. You can see this by dragging point C very close to point A and then running the animation.

14.3.2 Another Selectable Conic Section

It just so happens that the envelope of the locus of perpendicular bisectors of the line segments joining the focus to any point on a circle is a conic section. The construction presented in Table 14-2 is based on this fact.

Table 14-2: Another Selectable Conic Section

1. Draw circle AB with center at A and passing through point B	5. Let E be a random point on line segment CD
2. Let C be a random point on the circumference of circle AB	6. Construct $P_1 \perp$ to line segment CD through point E
3. Let D be any random point in the plane	7. Trace P_1 and change its color
4. Draw line segment CD	8. Animate point C around circle AB

Drag point D, the focus, so that it is outside of circle AB, and the envelope is a hyperbola. Drag point D so that it is inside of circle AB and the envelope is an ellipse. Drag point D so that it is coincident with the center of circle AB (point A) and the envelope is a circle. Pretty slick! Drag point E along line segment CD and rerun the animation for different members of the family of ellipses or hyperbolas (depending on the location of point D).

14.3.3 An Elegant Hyperbola

This simple construction (Table 14-3) is one way of generating a hyperbola.

Table 14-3: An Elegant Hyperbola

1. Draw horizontal line segment AB	8. Draw line segment AE
2. Let C be a random point on line segment AB	9. Construct $P_2 \perp$ to line BE through point E
3. Draw circle BC with center at B and passing through point C	10. Let F be the midpoint of line segment AE
4. Let D be the midpoint of line segment AB	11. Construct $P_3 \perp$ to line segment AE through point F
5. Let E be a random point on the circumference of circle BC	12. Let G be the intersection of line BE and perpendicular P_3
6. Construct $P_1 \perp$ to line segment AB through point D	13. Trace point G and change its color
7. Draw line BE	14. Animate point E around circle BC

In this construction, the two focal points are A and B. Perpendicular P_3 is the tangent to the hyperbola. Note how it is tangent to one branch and then smoothly switches to be tangent to the other branch. Very neat!

14.3.4 An Elegant Ellipse

And this simple construction (Table 14-4) is one way of generating an ellipse.

Table 14-4: An Elegant Ellipse

1. Draw circle AB with center at A and passing through point B	8. Let F be the intersection of line segment AC and P_1*
2. Let C be a random point on the circumference of circle AB	9. Trace point F and change its color
3. Draw line segment AC	10. Draw line segment DF
4. Let D be a random point any where in the plane	11. Measure line segments AF and FD
5. Draw line segment DC	12. Calculate AF + FD
6. Let E be the midpoint of line segment DC	13. Animate point C around circle AB
7. Construct $P_1 \perp$ to line segment DC through point E	

*If AC and P_1 do not intersect, drag point D until they do.

Note that as the animation is run, the lengths of line segments AF and FD change but their sum does not. Points A and D are the two foci of the ellipse and the distance between the two foci by way of the ellipse is always constant. This is a property of ellipses and can be used as a basis for construction. Also note that P_1 is tangent to the ellipse. Drag point D and rerun the animation for other ellipses in the family. Make the two foci coincident, and see the resulting circle (as you would suspect)!

14.3.5 An Ellipse from Two Intersecting Circles

In the previous construction we stated that the distance betwee the two foci of the ellipse by way of the circumference is constant. This property is often used as the definition of an ellipse and can be formally stated as: An ellipse is the locus of points P such that the sum of the distances from two fixed points F_1 and F_2 (called foci) are constant. In the simple construction of Table 14-5, note how this property is utilized.

In this construction, points D and E are the two foci. The sum of the distances from a point on the ellipse to the two foci is simply the sum of the two radii of the circles. Although these radii change as the animation is executed, their sum does not because their sum is constrained to be equal to the length of line segment AB.

Table 14-5: An Ellipse from Two Intersecting Circles

1. Draw horizontal line segment AB	7. Let E be a 2nd random point in the plane
2. Let C be a random point on line segment AB	8. Construct circle C_2 centered at E with radius = to segment CB
3. Draw line segment AC	9. Let F and G be the intersections of circles C_1 and C_2*
4. Draw line segment CB	10. Trace points F and G and change their color
5. Let D be a random point in the plane	11. Animate point C along line segment AB
6. Construct circle C_1 centered at D with radius = to segment AC	

*If they do not intersect, drag either point D or point E (or both) until they do.

14.3.6 An Elegant Parabola

The parabola can be defined as the locus of a set of points equidistant from a fixed point, the focus, and a fixed line, the directrix. The construction found in Table 14-6 is based upon that definition.

Table 14-6: An Elegant Parabola

1. Draw horizontal line AB	8. Construct $P_2 \perp$ to line AB through point G
2. Let C be a point in the plane <u>not</u> on line AB	9. Draw line segment CG
3. Draw circle DE centered at D and passing through point E	10. Let H be the midpoint of line segment CG
4. Let F be a random point on the circumference of circle DE	11. Construct $P_3 \perp$ to line segment CG through point H
5. Draw line DF	12. Let point I be the intersection of perpendiculars P_2 and P_3
6. Let point G be the intersection of line DF and line AB	13. Trace point I and change its color
7. Construct $P_1 \perp$ to line AB through point C	14. Animate point F around circle DE

Since point H is constructed to be the midpoint of line segment CG, the lengths of line segments GH and HC are always equal. However, GH is the distance to the directrix, the directrix being line AB, and HC is the distance to the focus, the focus being point C.

14.3.7 A Family of Ellipses

Table 14-7 presents a clever little construction for an ellipse that allows one to trace different members of a family of ellipses with each different execution of the animation.

Table 14-7: A Family of Ellipses

1. Draw horizontal line segment AB	10. Let E be either intersection of circle BA with P_1
2. Let measurement m_1 be the distance from point A to point B	11. Construct $P_2 \perp$ to P_1 through point D
3. Draw circle BA with center at B and passing through point A	12. Draw circle CC' with center at C and passing through point C'
4. Let C be a random point on line segment AB	13. Let F be the intersection of circle BC and line segment BD
5. Draw circle BC with center at B and passing through point C	14. Construct $P_3 \perp$ to line segment AB through point F
6. Let D be a random point on the circumference of circle BA	15. Let point G be the intersection of perpendiculars P_2 and P_3
7. Draw line segment BD	16. Trace point G and change its color
8. Construct $P_1 \perp$ to line segment AB through point B	17. Animate point D around circle BA
9. Let C' be the image when C is translated distance m_1 at $\angle 0°$	

By dragging point C along line segment AB, different ellipses of a family can be observed. What family are we talking about? The family whose semi-major axis is length BE and whose semi-minor axis varies between length 0 and length AB.

14.3.8 A Simple Ellipse

The obvious construction presented in Table 14-8 is suggested by the parametric representation found in Equation 14-8, that is, $(x, y) = (a \cos t, b \sin t)$.

Table 14-8: A Simple Ellipse

1. Draw horizontal line segment AB	7. Let E be the intersection of circle AC and line segment AD
2. Draw circle AB with center at A and passing through point B	8. Construct $P_1 \perp$ to line segment AB through point E
3. Let C be a random point on line segment AB	9. Construct $P_2 \perp$ to P_1 through point D
4. Let D be a random point on the circumference of circle AB	10. Let point F be the intersection of perpendiculars P_1 and P_2
5. Draw circle AC with center at A and passing through point C	11. Trace point F and change its color
6. Draw line segment AD	12. Animate point D around circle AB

Note how dragging point C back and forth on line segment AB changes the value of the semi-minor axis of the ellipse. Of course, dragging point B so that line segment AB becomes either longer or shorter, changes the semi-major axis.

14.3.9 One Parabola from Another

If two of the normals which can be drawn to a parabola through a given point intersect at right angles to one another, then the locus of that intersection point is another parabola. The construction outlined in Table 14-9 illustrates this remarkable property.

Table 14-9: One Parabola from Another

1. Draw horizontal line AB	14. Trace point J and change its color
2. Let C be a point in the plane <u>not</u> on line AB	15. Let C' be the image when C is dilated about I by a factor of ½
3. Draw circle DE with center at D and passing through point E	16. Construct $P_4 \perp$ to line AB through point I
4. Let F be a random point on the circumference of circle DE	17. Construct $P_5 \perp$ to P_3 through point J
5. Draw line DF	18. Let perpendicular P_5 be a thick line of different color
6. Let point G be the intersection of lines AB and DF	19. Construct $P_6 \perp$ to P_2 through point C'
7. Construct $P_1 \perp$ to line AB through point G	20. Let point K be the intersection of perpendiculars P_4 and P_6
8. Draw line segment CG	21. Construct $P_7 \perp$ to P_6 through point K
9. Construct $P_2 \perp$ to line segment CG through point C	22. Let perpendicular P_7 be a thick line of different color
10. Let H be the midpoint of line segment CG	23. Let point L be the intersection of perpendiculars P_5 and P_7
11. Construct $P_3 \perp$ to line segment CG through point H	24. Trace point L and change its color
12. Let point I be the intersection of line AB and P_2	25. Animate point F around circle DE
13. Let point J be the intersection of perpendiculars P_1 and P_3	

Point L is the given point under consideration, while perpendiculars P_5 and P_7 are, of course, the two normals to the parabola traced by point J. As you can see, P_5 and P_7 are at right angles to one another and indeed, point L also traces a parabola. Quite a remarkable property!

14.3.10 A Compass-Only Construction for the Hyperbola

Refer to Chapter 6, section 6.5.15, for a discussion that addresses what is meant by a GSP-version of a compass-only construction. Refer to Table 14-10.

Table 14-10: The Hyperbola by Compass-Only

1. Draw circle AB with center at A and passing through point B	10. Draw circle EF with center at E and passing through point F
2. Let C be a random point on the circumference of circle AB	11. Let G be the unlabeled intersection of circles EF and CA
3. Draw circle CA with center at C and passing through point A	12. Draw circle GA with center at G and passing through point A
4. Let D be any point outside of circle AB but inside circle CA	13. Let H and I be the two intersections of circles GA and AB
5. Draw circle DA with center at D and passing through point A	14. Draw circle HA with center at H and passing through point A
6. Draw line segment CD	15. Draw circle IA with center at I and passing through point A
7. Let A' be the image as A is reflected across line segment CD	16. Let point J be the unlabeled intersection of circles HA and IA
8. Draw circle A'C with center at A' and passing through point C	17. Trace point J and change its color
9. Let E and F be the two intersections of circles A'C and CA	18. Animate point C around circle AB

14.3.11 The Orthogonal Normals of an Ellipse

The construction following in Table 14-11 is rather complex and crowded, but well worth reproducing and understanding. We will construct two normals to an ellipse that intersect at right angles. The intersection point of those two normals will trace, as you will see, a quite beautiful curve.

Table 14-11: The Orthogonal Normals of an Ellipse

1. Draw horizontal line AB	18. Draw circle AB' with center at A and passing through B'
2. Draw circle AB with center at A and passing through point B	19. Let J be a random point on the circumference of circle AB'
3. Let C be a random point on line AB but inside of circle AB	20. Let I' be the image when I is rotated about point B by 180°
4. Let D be a random point on the circumference of circle AB	21. Draw circle JI with center at J and passing through point I
5. Construct $P_1 \perp$ to line AB through point A	22. Draw circle HI' with center at H and passing through point I'
6. Draw circle AC with center at A and passing through point C	23. Let K and L be the intersections of circle JI with circle HI'
7. Draw line segment AD	24. Draw line segments IL, IK, HL, and HK
8. Construct $P_2 \perp$ to line AB through point D	25. Construct $P_4 \perp$ to line segment IL through point J
9. Let E be the intersection of line segment AD and circle AC	26. Construct $P_5 \perp$ to line segment IK through point J
10. Let F be either intersection of perpendicular P_1 and circle AC	27. Let M be the intersection of P_4 and line segment HL
11. Construct $P_3 \perp$ to P_2 through point E	28. Let N be the intersection of P_5 and line segment HK
12. Let point G be the intersection of perpendiculars P_2 and P_3	29. Construct $P_6 \perp$ to P_4 through point M
13. Construct the locus of G while point D traverses circle AB	30. Construct $P_7 \perp$ to P_5 through point N
14. Make the locus a thick line and change its color (say green)	31. Make P_6 and P_7 thick lines and change their color (say cyan)
15. Let circle C_1 be the translation of circle AB by vector A → F	32. Let point O be the intersection of perpendiculars P_6 and P_7
16. Let B' be the image when B is translated by vector A → F	33. Trace point O and change its color (say yellow)
17. Let H and I be the intersections of line AB and circle C_1	34. Animate point J around circle AB'

To readily see what is going on and for a cleaner animation, it is suggested that you hide the following construction elements: All of the circles (i.e., circles AB, AC, AB', JI, HI', and the translated circle [C_1]); all of the perpendiculars except P_6 and P_7; line AB; line segments AD, IL, IK, HL, and HK; and, finally, points A, B, D, E, F, B', G, H, I, J, and I'. By dragging point C along line AB, you can change the eccentricity of the ellipse, thereby affecting the size of the curve that point O traces as well as whether the curve is contained inside the ellipse or intersects the ellipse.

14.3.12 A Parabola Arising from Two Lines and a Point

Given two non-parallel lines and a point not on either line, it is possible to draw a parabola tangent to the two given lines with the given point as the focus. Table 14-12 contains a construction that shows how.

Table 14-12: A Parabola from Two Lines and a Point

1. Draw horizontal line AB	12. Construct $P_2 \perp$ to line E_1E_2 through point E_1
2. Draw line CD not parallel to line AB	13. Draw line segment EI
3. Let E be a random point neither on line AB nor on line CD	14. Construct $P_3 \perp$ to line E_1E_2 through point I
4. Let E_1 be the image when point E is reflected across line AB	15. Let J be the intersection of line AB and perpendicular P_1
5. Let E_2 be the image when point E is reflected across line CD	16. Let K be the intersection of line CD and perpendicular P_2
6. Draw line E_1E_2	17. Let L be the midpoint of line segment EI
7. Draw circle FG with center at F and passing through point G	18. Construct $P_4 \perp$ to line segment EI through point L
8. Let H be a random point on the circumference of circle FG	19. Let point M be the intersection of perpendiculars P_3 and P_4
9. Draw line FH	20. Trace point M and change its color
10. Let point I be the intersection of lines E_1E_2 and FH	21. Make lines AB and CD thick and change their color
11. Construct $P_1 \perp$ to line E_1E_2 through point E_2	22. Animate point H around circle FG

Lines AB and CD are, of course, the two tangent lines while point E is the focus. You can play with this construction by changing the orientation of either line AB, CD, or both (by dragging point A, B, C, or D) and by dragging point E to change the location of the focus.

14.3.13 Hyperbolas and Parallelogram

The following construction is quite long and complex, but well worth the effort. The result is a beautiful animation. Be careful to label each line as suggested in the steps below as the plethora of lines can get very confusing. Refer to Table 14-13.

Table 14-13: Hyperbolas and Parallelogram

1. Draw horizontal line AB	24. Let point L be the intersection of lines AB and FJ
2. Draw circle AB with center at A and passing through point B	25. Draw circle AL with center at A and passing through point L
3. Let C be a random point on line AB	26. Construct $P_8 \perp$ to line AB through point L
4. Let D be a random point on the circumference of circle AB	27. Draw line HK
5. Construct $P_1 \perp$ to line AB through point A	28. Let M be the intersection of perpendicular P_8 and line AD
6. Let L_1 be the image when line AB is rotated about A by 45°	29. Let point N be the intersection of perpendiculars P_5 and P_8
7. Draw circle AC with center at A and Passing through point C	30. Let L_2 be the image when line HK is rotated about A by 180°
8. Draw line AD	31. Construct $P_9 \perp$ to P_2 through point M
9. Construct $P_2 \perp$ to line AB through point C	32. Draw line AN
10. Let E and F be the two intersections of P_1 with circle AB	33. Let point O be the intersection of perpendiculars P_7 and P_9
11. Construct $P_3 \perp$ to line AB through point D	34. Let L_3 be the image when line AN is reflected across line AB
12. Construct $P_4 \perp$ to line AD through point D	35. Let O' be the image when point O is reflected across line L_1
13. Let G be the intersection of line AD and perpendicular P_2	36. Trace point O' and change its color (say green)
14. Draw line CE	37. Draw line HO
15. Construct $P_5 \perp$ to P_1 through point E	38. Let L_4 be the image when line HO is reflected across line L_1
16. Let H be the intersection of line AB and perpendicular P_3	39. Let L_5 be the image when line L_4 is rotated about A by 180°
17. Let I be the intersection of line AB and perpendicular P_4	40. Let point P be the intersection of lines L_4 and HK
18. Construct $P_6 \perp$ to P_2 through point G	41. Let point Q be the intersection of lines L_2 and L_4
19. Let J be the unlabeled intersection of line CE and circle AB	42. Let point R be the intersection of lines L_5 and HK
20. Construct $P_7 \perp$ to line AB through point I	43. Let point S be the intersection of lines L_2 and L_5
21. Draw line FJ	44. Construct polygon SQPR and color it (say yellow)
22. Let point K be the intersection of perpendiculars P_6 and P_7	45. Measure the area and perimeter of polygon SQPR
23. Trace point K and change its color (say blue)	46. Animate point D around circle AB

For the best looking animation, hide all lines and all points except point C, the two tracing points (K and O'), and the four points making up the parallelogram (P, Q, R, and S). Drag point C to change the eccentricity of the two hyperbolas. Incidentally, the two hyperbolas are conjugates. That is, if one hyperbola has equation $x^2/a^2 - y^2/b^2 = 1$, then its conjugate has equation $y^2/b^2 - x^2/a^2 = 1$. They share the same asymptotes and the same axes. If the first hyperbola's eccentricity is e and the conjugate's eccentricity is e', then the two eccentricities are related by $1/e^2 + 1/e'^2 = 1$. Note how the area of the parallelogram remains constant (except for a small glitch from GSP when the length of the parallelogram becomes infinite and its width becomes zero) even though its perimeter constantly changes as the animation executes. A really wonderful construction!

14.3.14 An Ellipse as Derived from a Compass-Only Construction

See Chapter 6, section 6.5.15, for a discussion of the GSP-version of a compass-only construction. Refer to Table 4-14.

Table 14-14: An Ellipse as Derived from a Compass-Only Construction

1. Draw circle AB with center at A and passing through point B	9. Draw circle EA with center at E and passing through point A
2. Let C be a random point on the circumference of circle AB	10. Draw circle FA with center at F and passing through point A
3. Draw line segment AB	11. Let G be the unlabeled intersection of circles FA and EA
4. Let C' be the image as C is reflected across line segment AB	12. Draw line segment CC'
5. Let D be a random point inside circle AB	13. Let G' be the reflection of G across line segment CC' *
6. Draw circle AD with center at A and passing through point D	14. Trace point G' and change its color
7. Draw circle CA with center at C and passing through point A	15. Animate point C around circle AB
8. Let E and F be the two intersections of circles AD and CA	

*Note that point G' would also be the intersection of circles CG and C'G if these two circles had been constructed. The GSP limitation discussed in Chapter 6, section 6.5.15 precludes us from creating G' in that manner.

14.3.15 A Triangle that Draws Three Ellipses

The construction that follows in Table 14-15 is a marvelous construction that is fun to play with. Execute it and see what you think.

Table 14-15: A Triangle that Draws Three Ellipses

1. Draw small ΔABC composed of line segments AB, BC & AC	12. Trace point J and change its color (say red)
2. Draw horizontal line DE	13. Let K be a second random point on parallel L_2
3. Let F be a random point on line DE	14. Trace point K and change its color (say green)
4. Draw circle EF with center at E and passing through point F	15. Let L_3 be the image when L_2 is rotated about K by ∠BAC
5. Let G and H be random points on circle EF's circumference	16. Let L_4 be the image when L_2 is rotated about J by ∠ABC
6. Draw lines GE and HE	17. Let point L be the intersection of lines L_3 and L_4
7. Construct line L_1 parallel to line DE through point H	18. Trace point L and change its color (say cyan)
8. Let point I be the intersection of line GE and parallel L_1	19. Construct polygon JKL and color it (say yellow)
9. Let E' be the image when E is translated by vector H → I	20. Measure the area and perimeter of polygon JKL
10. Construct line L_2 parallel to HE through point E'	21. Animate point H around circle EF
11. Let J be a random point on parallel L_2	

Hide all of the construction elements except triangle ABC, points J, K, L and polygon JKL. Step 1, the construction of triangle ABC, should be executed off to the side of the screen; use triangle ABC to change the eccentricity and orientation of the ellipses by manipulating angles BAC and ABC. Note how the vertices of triangle (polygon) JKL trace the three ellipses and also not that the area and perimeter of the so-called tracing triangle are constant. Neat!

14.3.16 A Parabola as Derived from a Compass-Only Construction

This GSP-version of a compass-only construction (Table 14-16) has only two steps that are non-compass steps, that is, steps 4 and 5. At the risk of sounding redundant, see Chapter 6, section 6.5.15 for a discussion of the GSP-version of a compass-only construction.

Table 14-16: A Parabola from a Compass-Only Construction

1. Draw circle AB with center at A and passing through point B	10. Draw circle BA with center at B and passing through point A
2. Let C be a random point on the circumference of circle AB	11. Draw circle FB with center at F and passing through point B
3. Draw circle CB with center at C and passing through point B	12. Let G and H be the two intersections of circles BA and FB
4. Draw line segment AC	13. Draw circle GB with center at G and passing through point B
5. Let B' be the image as B is reflected across line segment AC	14. Draw circle HB with center at H and passing through point B
6. Draw circle B'C with center at B' and passing through point C	15. Let I be the unlabeled intersection of circles GB and HB
7. Let E and D be the intersections of circle B'C and circle CB	16. Trace point I and change its color
8. Draw circle DE with center at D and passing through point E	17. Animate point C around circle AB
9. Let F be the unlabeled intersection of circle DE and circle CB	

14.3.17 A Conic Section by Straight Edge Alone

Here is a really weird construction that requires a lot of playing around to obtain all of the conics. Refer to Table 14-17.

Table 14-17: A Conic Section by Straight Edge Alone

1. Let A, B, C, and D be four random points in the plane	8. Draw line BI
2. Draw lines AB, BC, and AC	9. Draw line AI
3. Draw line DE such that it intersects lines AC and BC	10. Let point J be the intersection of lines AI and BC
4. Draw circle FG with center at F and passing through point G	11. Let point K be the intersection of lines BI and AC
5. Let H be a random point circle FG	12. Draw line JK
6. Draw line FH	13. Trace line JK and change its color
7. Let point I be the intersection of line FH and line DE	14. Animate point H around circle FG

Drag points A, B, C, and D to different positions and rerun the animation to obtain different conics. Why is this construction entitled "... by straight edge alone" when there is clearly a circle drawn (therefore a compass used) in step 4? Circle FG is merely the driving mechanism for the animation; in lieu of steps 4 – 7, we could just as easily have put a random point on line DE instead of point I and simply animated that random point along line DE. So the compass was not really needed!

14.3.18 The Ellipse and Its Tangent

Table 14-18 contains a simple construction for the tangent to the ellipse.

Table 14-18: The Ellipse and Its Tangent

1. Draw circle AB with center at A and passing through point B	10. Let point F be the intersection of perpendiculars P_1 and P_2
2. Let C be a random point outside of circle AB	11. Construct the locus of F while point D traverses circle AC
3. Draw circle AC with center at point A and passing through C	12. Construct $P_3 \perp$ to line AB through point E
4. Draw line AB	13. Construct $P_4 \perp$ to P_1 through point D
5. Let D be a random point on the circumference of circle AC	14. Let point G be the intersection of perpendiculars P_3 and P_4
6. Draw line AD	15. Draw line AG
7. Construct $P_1 \perp$ to line AB through point D	16. Construct $P_5 \perp$ to line AG through point F
8. Let E be either intersection of line AD with circle AB	17. Animate point D around circle AC
9. Construct $P_2 \perp$ to P_1 through point E	

14.3.19 The Hyperbola and Its Tangent

The construction of a hyperbola's tangent is also simple, but very elegant, as can be seen from the construction of Table 14-19.

Table 14-19: The Hyperbola and Its Tangent

1. Draw circle AB with center at A and passing through point B	12. Construct the locus of point F as point C traverses circle AB
2. Draw line AB	13. Let G be the intersection of perpendicular P_2 and line AC
3. Let C be a random point on the circumference of circle AB	14. Construct $P_5 \perp$ to P_2 through point G
4. Draw line AC	15. Construct $P_6 \perp$ to line AC through point E
5. Construct $P_1 \perp$ to line AC through point C	16. Let H be the intersection of perpendicular P_6 and line AB
6. Let point D be the intersection of P_1 and line AB	17. Construct $P_7 \perp$ to line AB through point H
7. Construct $P_2 \perp$ to line AB through point D	18. Let point I be the intersection of perpendiculars P_5 and P_7
8. Construct $P_3 \perp$ to line AB through point B	19. Let I' be the image when point I is reflected across line AB
9. Let point E be the intersection of P_3 and line AC	20. Draw line AI'
10. Construct $P_4 \perp$ to P_3 through point E	21. Construct P_8 to line AI' through point F
11. Let point F be the intersection of perpendiculars P_2 and P_4	22. Animate point C around circle AB

Perpendicular P_8 is obviously the tangent to the hyperbola. Note how the tangent switches from one branch of the hyperbola to the other as the animation is executed.

14.3.20 Dancing Ellipses

See Table 14-20 for the last construction of this chapter.

Table 14-20: A Dance of Ellipses

1. Draw horizontal line AB	11. Let G be the midpoint of line segment DE
2. Draw circle AB with center at A and passing through point B	12. Construct $P_1 \perp$ to line segment BD through point F
3. Let C be a random point inside circle AB <u>not</u> on line AB	13. Construct $P_2 \perp$ to line segment DE through point G
4. Draw circle AC with center at A and passing through point C	14. Let H be the intersection of line AB and perpendicular P_1
5. Let D be a random point on the circumference of circle AC	15. Let I be the intersection of line AE and perpendicular P_2
6. Let E be a random point on the circumference of circle AB	16. Construct the locus of point I as point E traverses circle AB
7. Draw line segment BD	17. Let I' be the image when I is reflected across perpendicular P_1
8. Draw line segment DE	18. Construct the locus of point I' as point E traverses circle AB
9. Draw line AE	19. Animate point D on circle AC
10. Let F be the midpoint of line segment BD	

Here are a few of the things that you can do to play with this construction. Trace either perpendicular P_1 or P_2 and rerun the animation to obtain the envelope of a hyperbola. Note how the eccentricities of the ellipses change as one drags point C closer and closer to point B. Drag point B so that distance AB becomes less than distance AC and observe how the two ellipses are transformed into two hyperbolas. Etc., etc., etc.

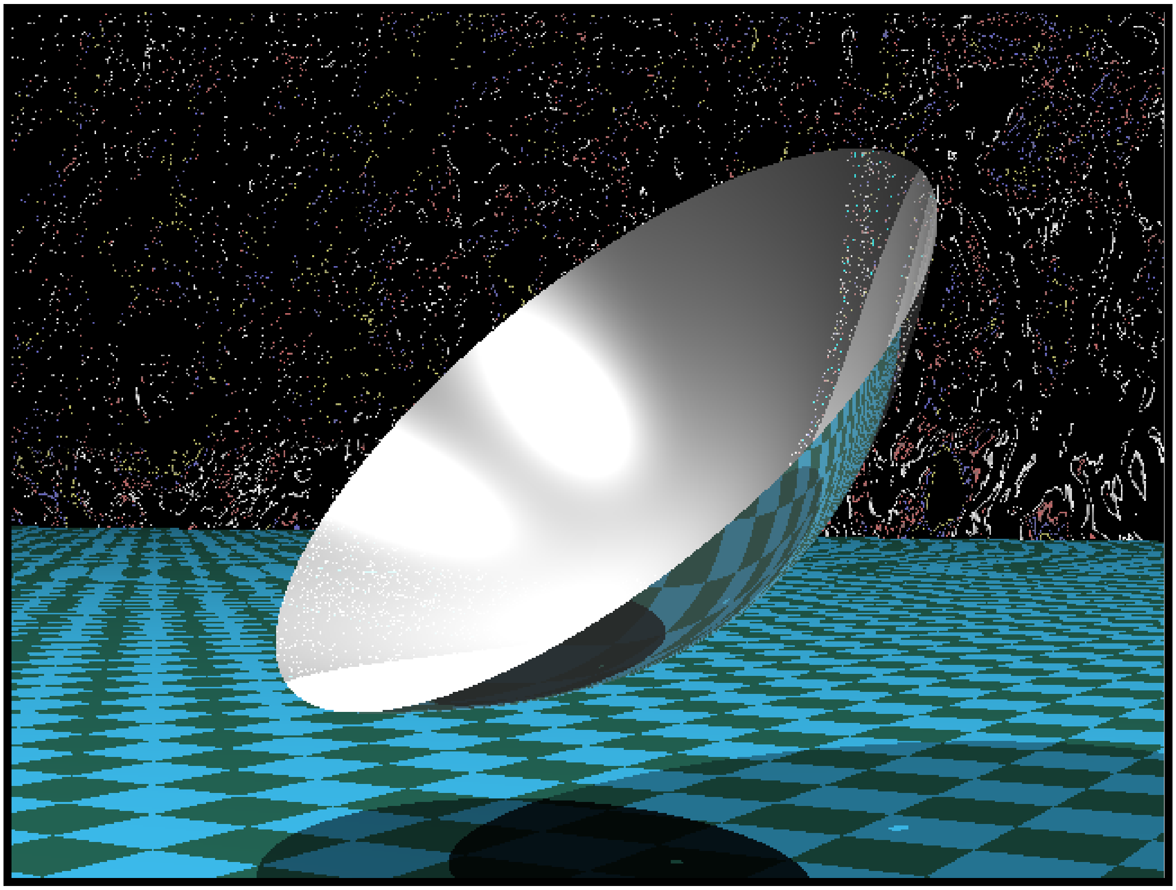

Figure 14-8: A Paraboloid

The solid of revolution formed when the parabola with equation $y^2 = 4x$ *is revolved about the x-axis, or, in other words, a paraboloid. The paraboloid has been give a shiny-metallic surface finish which is seen reflecting the background environment and the light sources that have been placed to light the scene. The background is a star-lit night sky meeting an infinite blue and green checkered plane at the horizon. Note how some of the stars are reflected on the inside of the paraboloid.*

Chapter 15 – The Lemniscate of Bernoulli

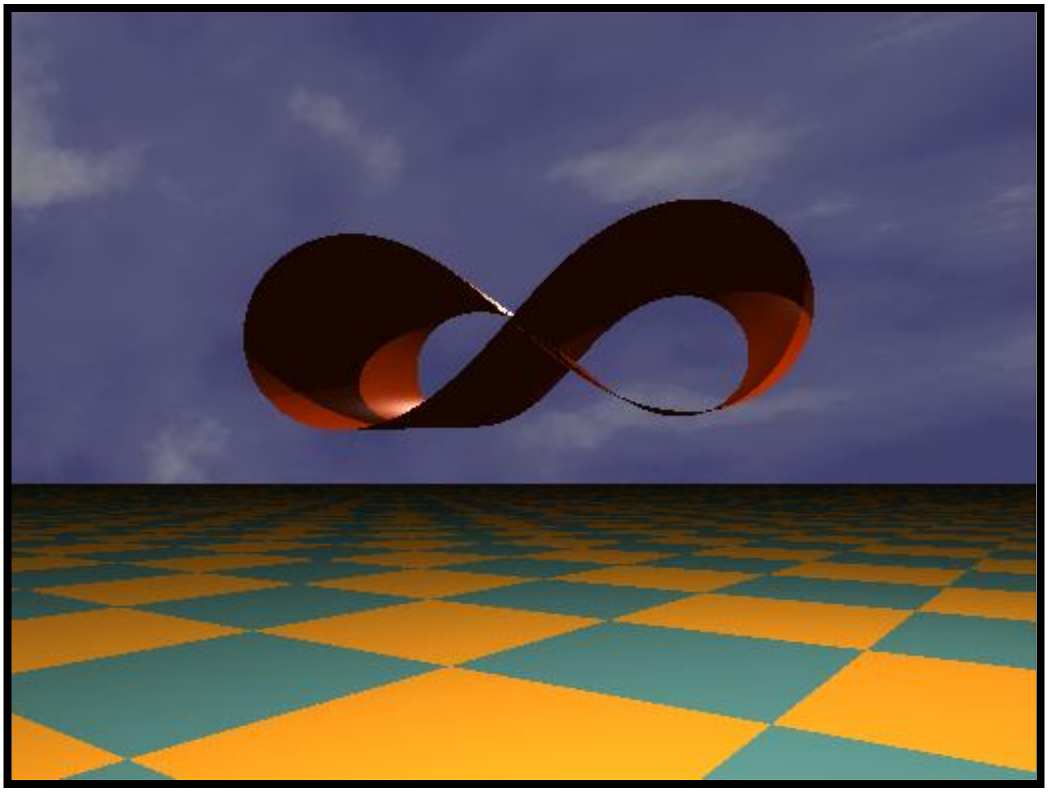

Figure 15-1: The Lemniscate of Bernoulli in Three Dimensions

The cross-section of the object above is the Lemniscate of Bernoulli. It has been extruded into the third dimension (normal to the plane of the paper) and rendered with a shiny-red surface. It was then placed over the orange and green checkered plane which meets a threatening, darkish sky at the horizon. Light sources have been placed so as to partially shadow the object.

15.1 Introduction

In 1694 Jakob Bernoulli published a curve in *Acta Eruditorum* that he described as being shaped like a figure eight, or a knot, or bow of a ribbon. Following the protocol of his day, he gave this curve the Latin name of *lemniscus*, which translates as a pendant ribbon to be fastened to a victor's garland. Bernoulli was not aware that the curve he described was a special case of a Cassini Oval which had already been described by Cassini some 14 years earlier. Today, we know the curve as the Lemniscate (pronounced lem·nis·cate) of Bernoulli. This curve is most commonly defined as the locus of a point which moves so that the product of its distances from two fixed points is constant and is equal to the square of half the distance between these points. The general properties of the Lemniscate were discovered by G. Fagnano (an Italian mathematician and ordained priest) in 1750. Later investigations of the arc length by Gauss and Euler ultimately led to the development of elliptic functions. In the last chapter, we examined curves that were defined as the sections obtained when a plane cut a cone, i.e., conic sections. Another way to define the Lemniscate is as a toric section, that is, when a plane cuts a torus (see Figure 15-2).

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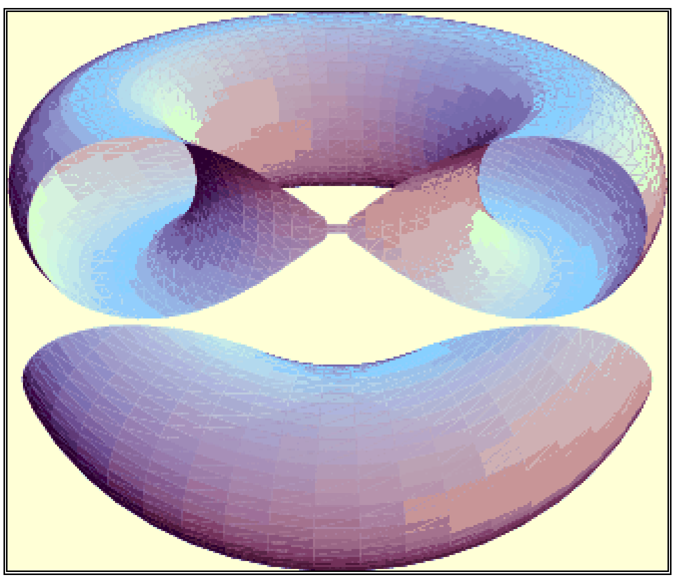


Figure 15-2: The Lemniscate of Bernoulli as a Toric Section

15.2 Equations and Graph of the Lemniscate of Bernoulli

Using the "locus" definition of the Lemniscate in the introduction above, and if the two fixed points are located at $\left(\pm\frac{a}{\sqrt{2}},0\right)$, then the Cartesian equation will be

$$\left[\left(x-\frac{a}{\sqrt{2}}\right)^2+y^2\right]\cdot\left[\left(x+\frac{a}{\sqrt{2}}\right)^2+y^2\right]=\frac{a^4}{4}.$$

By squaring both sides, performing the indicated multiplication, and then simplifying the result, one gets

$$\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2\right) \quad \textbf{Equation 15-1}$$

The polar equation is quite easily derived by making the usual substitutions of $x^2 + y^2 = r^2$ and $(x, y) = r(\cos\theta, \sin\theta)$. That is,

$$r^2 = a^2 \cos 2\theta \quad \textbf{Equation 15-2}$$

For a parametric representation, let $y = x \sin t$ then substitute this value for y in Equation 15-1 and solve for x. Doing this yields

$$x = \frac{a\cos t}{1+\sin^2 t}.$$

And, since $y = x \sin t$, we have that

$$y = \frac{a\sin t\cos t}{1+\sin^2 t}.$$

Therefore, our parametric equations are

$$(x,y) = \frac{a\cos t}{1+\sin^2 t}(1,\sin t), \quad -\pi \le t \le +\pi \quad \textbf{Equation 15-3}$$

Further, the pedal and bipolar equations are, respectively

$$r^3 = a^2 p \quad \textbf{Equation 15-4}$$

$$rr' = \frac{a^2}{2} \quad \textbf{Equation 15-5}$$

The equation of the tangent line to the Lemniscate at the point $t = q$ is

$$\sin q\left(\sin^2 q - 3\right)\cdot y = \left(1-3\sin^2 q\right)\cdot x - a\cos^3 q \quad \textbf{Equation 15-6}$$

Figure 15-3 depicts a graph of the Lemniscate of Bernoulli.

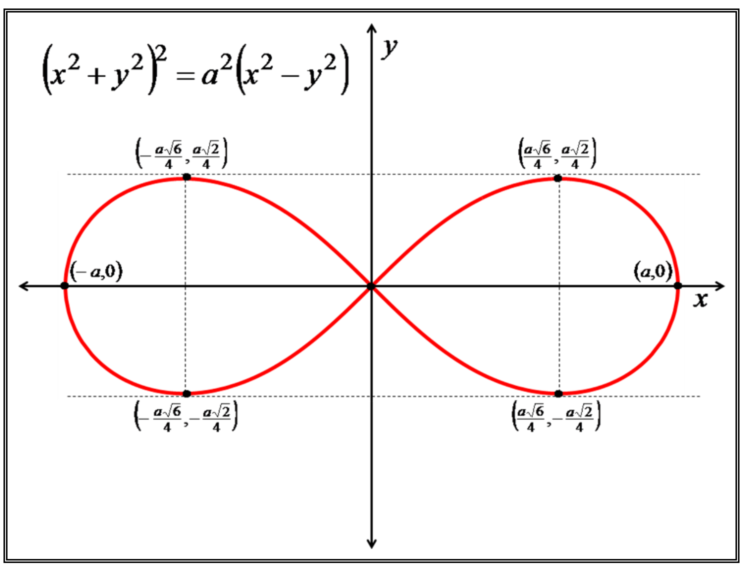


Figure 15-3: Graph of the Lemniscate of Bernoulli

15.3 Analytical and Physical Properties of the Lemniscate of Bernoulli

Based on the Lemniscate's parametric representation found in Equation 15-3, that is, $x = a\cos t / (1 + \sin^2 t)$ and $y = a\cos t \sin t / (1 + \sin^2 t)$, the following subsections contain an analysis of the Lemniscate.

15.3.1 Derivatives of the Lemniscate of Bernoulli

- $\dot{x} = \dfrac{a\sin t\left(\sin^2 t - 3\right)}{\left(1 + \sin^2 t\right)^2}$.
- $\ddot{x} = \dfrac{a\cos t\left(12\sin^2 t - \sin^4 t - 3\right)}{\left(1 + \sin^2 t\right)^3}$.
- $\dot{y} = \dfrac{a\left(1 - 3\sin^2 t\right)}{\left(1 + \sin^2 t\right)^2}$.
- $\ddot{y} = \dfrac{2a\sin t\cos t\left(3\sin^2 t - 5\right)}{\left(1 + \sin^2 t\right)^3}$.
- $y' = \dfrac{1 - 3\sin^2 t}{\sin t\left(\sin^2 t - 3\right)}$.
- $y'' = \dfrac{3\cos t\left(1 + \sin^2 t\right)^4}{a\sin^3 t\left(\sin^2 t - 3\right)^3}$.

15.3.2 Metric Properties of the Lemniscate of Bernoulli

For the area, we use the polar form for ease of calculation. That is,

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta .$$

For one loop of the Lemniscate we therefore have,

$$\frac{a^2}{2}\int_{-\pi/4}^{\pi/4} \cos 2\theta \cdot d\theta = \frac{a^2}{4}\int_{-\pi/4}^{\pi/4} d(\sin 2\theta) = \frac{a^2}{4}\left[\sin 2\theta\right]_{-\pi/4}^{\pi/4} = \frac{a^2}{2} .$$

Therefore, the area of the total Lemniscate (i.e., both loops) is $A = a^2$, quite an elegant result.

If p is the distance from the origin to the tangent of the Lemniscate, then

$$p = \frac{-a\cos^3 t}{\left(1+\sin^2 t\right)^{3/2}}.$$

If r represents the radial distance to the Lemniscate, then

$$r = \frac{a\cos t}{\sqrt{1+\sin^2 t}}.$$

15.3.3 Curvature of the Lemniscate of Bernoulli

If ρ represents the radius of curvature for the Lemniscate, then

$$\rho = \frac{a\sqrt{1+\sin^2 t}}{3\cos t}.$$

If (α, β) represents the coordinates of the center of curvature for the Lemniscate, then

$$(\alpha, \beta) = \frac{-2a}{3\cos t\left(1+\sin^2 t\right)}\left(-1, \sin^3 t\right).$$

15.3.4 Angles for the Lemniscate of Bernoulli

If ψ is the tangential-radial angle for the Lemniscate, then

$$\tan\psi = -\frac{\cos^2 t}{2\sin t}.$$

If θ is the radial angle of the Lemniscate, then

$$\tan\theta = \sin t.$$

If ϕ denotes the tangential angle of the Lemniscate, then

$$\tan\phi = \frac{1-3\sin^2 t}{\sin t\left(\sin^2 t - 3\right)}.$$

15.4 Geometric Properties of the Lemniscate of Bernoulli

- Intercepts: $(a, 0)$; $(0, 0)$; $(-a, 0)$.
- Extrema: $(a, 0)$ is the x-maximum, $(-a, 0)$ is the x-minimum.
- $\left(\pm\frac{a\sqrt{6}}{4}, \pm\frac{a\sqrt{2}}{4}\right)$ are the y-maximum and minimum, respectively.

- Point of inflection: (0, 0).

- Extent: $-a \le x \le a$; $\frac{-a\sqrt{2}}{4} \le y \le \frac{a\sqrt{2}}{4}$

- Symmetries: The Lemniscate is symmetric about the *x*-axis, the *y*-axis, and the origin.

- Loops: two loops, one for $t \le -\pi/2$ and $t \ge \pi/2$; the other for $-\pi/2 \le t \le \pi/2$.

15.5 Dynamic Geometry of the Lemniscate of Bernoulli

The following nine subsections delineate dynamic geometry constructions that either generate or illustrate some property or characteristic of the Lemniscate of Bernoulli.

15.5.1 The Lemniscate of Bernoulli from the Midpoint of a Line Segment

Here is a very simple construction that generates the Lemniscate of Bernoulli. Refer to Table 15-1.

Table 15-1: The Lemniscate of Bernoulli from the Midpoint of a Line Segment

1. Draw circle AB with center at A and passing through point B	7. Let C" be the image as C' is reflected across line segment A'C
2. Let C be a random point on the circumference of circle AB	8. Draw line segment A'C"
3. Let A' be the image when A is rotated about point B by – 90°	9. Draw line segment CC"
4. Let C' be the image when C is translated by vector A → A'	10. Let D be the midpoint of line segment CC"
5. Draw line segment A'C	11. Trace point D and change its color
6. Draw line segment AC	12. Animate point C around circle AB

15.5.2 The Lemniscate of Bernoulli as the Pedal Curve of the Hyperbola

If we are given a rectangular hyperbola and a point on that hyperbola and if we draw a tangent to the hyperbola through the given point and then construct a perpendicular to the tangent through the origin, the locus of the intersection point of the tangent and the perpendicular is the Lemniscate of Bernoulli. That's the long way of saying that the pedal curve of a rectangular hyperbola is a Lemniscate when the pedal point is the origin, and you can verify that fact with the construction of Table 15-2.

Table 15-2: The Lemniscate of Bernoulli as the Pedal Curve of the Hyperbola

1. Create an *x-y* axis with origin at A and unit point B = (1, 0)	16. Construct $P_5 \perp$ to the *x*-axis through point H
2. Draw circle AB with center at A and passing through point B	17. Draw circle AF' with center at A and passing through point F'
3. Let C be a random point on the *x*-axis	18. Let I be the intersection of perpendiculars P_4 and P_5
4. Let D be a random point on the circumference of circle AB	19. Trace point I and change its color (say green)
5. Construct $P_1 \perp$ to the *x*-axis through point C	20. Let J be either intersection of the x-axis with circle AF'
6. Draw line AD	21. Draw line GI
7. Draw circle AC with center at A and passing through point C	22. Draw line segment AI
8. Construct $P_2 \perp$ to the *x*-axis through point D	23. Construct $P_6 \perp$ to line GI through point A
9. Construct $P_3 \perp$ to line AD through point D	24. Let K be the midpoint of line segment AI
10. Let E be the intersection of Line AD and perpendicular P_1	25. Let L be the intersection of line GI and perpendicular P_6
11. Let F be either intersection of the *y*-axis with circle AC	26. Trace point L and change its color (say yellow)
12. Let G be the intersection of the *x*-axis and perpendicular P_2	27. Draw line segment KL
13. Let H be the intersection of the *x*-axis and perpendicular P_3	28. Construct $P_7 \perp$ to line segment KL through point L
14. Construct $P_4 \perp$ to P_1 through point E	29. Animate point D around circle AB
15. Let F' be the image when F is translated by vector A → B	

In this construction, point I is the point on the hyperbola and line GI is a tangent to the hyperbola. Further, as an added bonus, perpendicular P_7 is a tangent to the Lemniscate. For a really beautiful animation, hide all the lines except the *x*-axis, the *y*-axis, and the two tangents and make the tangents thick and of different colors. Hide all the circles, all the line segments, and all of the points except the two tracing points (I and L) and the point C. Drag point C to change the eccentricity of the hyperbola and thereby the shape of the Lemniscate loops.

15.5.3 The Lemniscate of Bernoulli as the Cissoid of Two Circles

The Lemniscate of Bernoulli can be generated if treated as the Cissoid of two circles with respect to a point that is located at a distance $R \cdot \sqrt{2}$ from the center of each circle, where R is the radius of each circle. A demonstration of this property is found in Table 15-3.

Table 15-3: The Lemniscate of Bernoulli as the Cissoid of Two Circles

1. Draw horizontal line AB	8. Let D be the unlabeled intersection of line A_1C and circle AB
2. Draw circle AB with center at A and passing through point B	9. Let point A_2 be the translation of point A_1 by vector C → D
3. Let m_1 be a measure of the distance AB	10. Trace point A_2 and change its color
4. Calculate $m_2 = m_1 \cdot \sqrt{2}$	11. Construct $P_1 \perp$ to line AB through point A_1
5. Let A_1 be the image when A is translated by distance m_2 at 0°	12. Let A_3 be the reflection of point A_2 across perpendicular P_1
6. Let C be a random point on the circumference of circle AB	13. Trace point A_3 and change its color
7. Draw line A_1C	14. Animate point C around circle AB

Lest you think that there is a circle missing here, think again. If you must see the second circle, simply reflect circle AB across perpendicular P_1—that's the second circle.

15.5.4 The Lemniscate of Bernoulli and a Circumscribing Circle

Table 5-4 delineates a rather bizarre construction that not only generates the Lemniscate of Bernoulli but also inscribes the Lemniscate inside a circle.

Table 15-4: The Lemniscate of Bernoulli and a Circumscribing Circle

1. Draw line AB	9. Let F' be the image when F is translated by m_1 at 0°
2. Let C and D be two random points on line AB	10. Draw circle FF' with center at F and passing through F'
3. Let E be a random point <u>not</u> on line AB	11. Let G and H be the intersections of circles CE and FF'
4. Draw circle CE with center at C and passing through point E	12. Draw line segments FG and FH
5. Draw line segment CE	13. Let I be the midpoint of line segment FG
6. Let C_1 be the circle centered at D and radius = to line segment CE	14. Let J be the midpoint of line segment FH
7. Let F be a random point on the circumference of circle C_1	15. Trace points I and J and change their color
8. Let m_1 be a measure of distance CD	16. Animate point F around circle C_1

With this construction, you might say that we are simultaneously generating a conic and toric section (a little mathematical humor—very little).

15.5.5 The Lemniscate of Bernoulli as an Envelope of Circles

It just so happens that if you take any point on a rectangular hyperbola as the center of a circle which passes through the point at the center of the hyperbola, then the envelope of the locus of all of those circles forms the Lemniscate of Bernoulli. The following construction can be used to verify this property. Refer to Table 15-5.

Dragging point D will change the eccentricity of the hyperbola and of the Lemniscate. However, if you drag point D so that it lies inside of circle AB, the hyperbola turns into an ellipse and the envelope becomes something other then a Lemniscate. Try it!

Table 15-5: The Lemniscate of Bernoulli as an Envelope of Circles

1. Draw circle AB with center at A and passing through point B	9. Construct P_1 ⊥ to line segment CD through point F
2. Let C be a random point on the circumference of circle AB	10. Let point G be the intersection of P_1 and line AC
3. Let D be any point in the plane outside of circle AB	11. Draw line segment DG
4. Draw line segment AD	12. Draw circle GE with center at G and passing through point E
5. Let E be the midpoint of line segment AD	13. Trace circle GE and change its color
6. Draw line segment CD	14. Construct the locus of G while point C traverses circle AB
7. Let point F be the midpoint of line segment CD	15. Animate point C around circle AB
8. Draw line AC	

15.5.6 A Lemniscate "Toy" to Play With

The animation delineated in Table 15-6 is an adjustable construction that can be used to experiment with different configurations—some of which trace the Lemniscate of Bernoulli.

Table 15-6: A Lemniscate of Bernoulli Toy

1. Draw horizontal line AB through the middle of the screen	14. Let E_2 be the image when E is translated by length m_2 at 0°
2. Let C be a random point near the top of the screen	15. Draw line segment EE_2
3. Construct line L_1 parallel to line AB through point C	16. Draw circle EE_1 with center at E and passing through point E_1
4. Let D be a random point just underneath point C	17. Let H be a random point on the circumference of circle EE_1
5. Construct line L_2 parallel to line AB through point D	18. Draw line segments E_2H and EH
6. Let E be a random point on line AB	19. Let I be the midpoint of line segment E_2H
7. Let F be a random point on L_1	20. Construct P_1 ⊥ to line segment E_2H through point I
8. Let G be a random point on L_2	21. Let E_3 be the reflection of point E across perpendicular P_1
9. Hide lines L_1, L_2, and AB	22. Draw line segments E_2E_3 and E_3H
10. Draw line segments CF and DG	23. Let J be the midpoint of line segment E_3H
11. Let m_1 be a measure of the length of line segment CF	24. Let K be a random point on line segment E_3H
12. Let m_2 be a measure of the length of line segment DG	25. Trace point K and change its color
13. Let E_1 be the image when E is translated by length m_1 at 0°	26. Animate point H around circle EE_1

Three points are adjustable here. Points E, F, and K can all be dragged to affect the curve that the animation traces. Case 1: If point E and/or point F are dragged so that distance CF is less than distance DG (i.e., CF < DG), we get the trace of the Lemniscate of Bernoulli whenever point K and point J coincide. As point K is dragged toward either end of its line segment (i.e., away from the midpoint, J), we get Lemniscate-like curves where one loop is smaller than the other. When point K starts getting close to the endpoint of line segment E_3H, the small loop disappears and we get one large loop with a slight protrusion on one side. Finally, when point K coincides with the endpoint of the line segment, we get a circle. Case 2: If point E and/or point F are dragged so that distance CF is equal to distance DG (i.e., CF = DG), point K traces a circle no matter where point K is dragged on line segment E_3H. Case 3: If point E and/or point F are dragged so that distance CF is greater than distance DG (i.e., CF > DG), point K traces closed curves that are not quite completely circular but flattened slightly on one side.
It is interesting to experiment with this "toy."

15.5.7 The Lemniscate of Bernoulli as a Compass-Only Construction

See Chapter 6 for a discussion of what is meant by the GSP-version of a compass-only construction. Steps 6 and 7 below in Table 15-7 are the non-compass steps, but necessary because of GSP's inability to correctly manipulate point A' if point A' is constructed as the intersection of circle CA and circle DA.

Table 15-7: The Lemniscate of Bernoulli as a Compass-Only Construction

1. Draw circle AB with center at A and passing through point B	8. Draw circle A'C with center at A' and passing through point C
2. Let C be a random point on the circumference of circle AB	9. Let E and F be the two intersections of circles A'C and CA
3. Let D be a random point external to circle AB	10. Draw circle EF with center at E and passing through point F
4. Draw circle CA with center at C and passing through point A	11. Let G be the unlabeled intersection of circle EF and circle CA
5. Draw circle DA with center at D and passing through point A	12. Trace point G and change its color
6. Draw line segment CD	13. Animate point C around circle AB
7. Let A' be the reflection of A across line segment CD	

15.5.8 The Lemniscate of Bernoulli as the Inverse of a Hyperbola

As the title to this subsection suggests, the inverse of a rectangular hyperbola with respect to its center is the Lemniscate of Bernoulli. Check it out with the construction delineated in Table 15-8.

Table 15-8: The Lemniscate of Bernoulli as the Inverse of a Hyperbola

1. Draw circle AB with center at A and passing through point B	11. Draw line EG
2. Let C be a random point on the circumference of circle AB	12. Let m_1 be a measure of the distance from point E to point G
3. Draw horizontal line segment AD > line segment AB	13. Draw circle EH with center at E and passing through point H
4. Let E be the midpoint of line segment AD	14. Let m_2 be a measure of the distance from point E to point H
5. Draw line segment CD	15. Calculate $m_3 = (m_2)^2 / m_1$
6. Let F be the midpoint of line segment CD	16. Let E' be the image when E is translated by m_3 at 0°
7. Draw line AC	17. Draw circle EE' with center at E and passing through point E'
8. Construct $P_1 \perp$ to line segment CD through point F	18. Let I and J be the two intersections of circle EE' with line EG
9. Let G be the intersection of perpendicular P_1 and line AC	19. Trace either point I or point J and change its color
10. Construct the locus of point G as point C traverses circle AB	20. Animate point C around circle AB

15.5.9 The Osculating Circle of the Lemniscate of Bernoulli

Table 15-9 contains a construction for the osculating circle of the Lemniscate of Bernoulli as well as a few other surprises.

Table 15-9: The Osculating Circle of the Lemniscate of Bernoulli

1. Draw Circle AB with center at A and passing through point B	18. Draw line AD
2. Let C be a random point on the circumference of circle AB	19. Construct $P_4 \perp$ to P_1 through point H
3. Let D be a random point external to circle AB	20. Let point J be the intersection of line AD and P_4
4. Draw line segment CD	21. Construct $P_5 \perp$ to P_4 through point J
5. Let E be the midpoint of line segment CD	22. Let point K be the intersection of line AC and P_5
6. Construct $P_1 \perp$ to line segment CD through point E.	23. Construct $P_6 \perp$ to line AC through point K
7. Draw line segment AD	24. Let point L be the intersection of perpendiculars P_4 and P_6
8. Let F be the midpoint of line segment AD	25. Draw line segment FL
9. Construct $P_2 \perp$ to P_1 through point F	26. Let M be the midpoint of line segment FL
10. Let point G be the intersection of perpendiculars P_1 and P_2	27. Let point N be the intersection of line GI with P_4
11. Construct the locus of G as point C traver4ses circle AB	28. Construct $P_7 \perp$ to line GI through point N
12. Draw line AC	29. Let point O be the intersection of perpendiculars P_2 and P_7
13. Let point H be the intersection of line AC and P_1	30. Draw line MO
14. Draw line segment FH	31. Let point P be the intersection of line MO and line GI
15. Let I be the midpoint of line segment FH	32. Draw circle PG with center at P and passing through point G
16. Draw line GI	33. Make circle PG thick and change its color
17. Construct $P_3 \perp$ to line GI through point G	34. Animate point C around circle AB

Surprises? Oh yes, construct the locus of point H as point C travels on circle AB and you will find that the locus is that of a rectangular hyperbola. Further, if you construct circle LH, i.e., the circle centered at point L and passing through point H, you will see when you rerun the animation that circle LH is the osculating circle of the hyperbola. What an extremely remarkable construction!

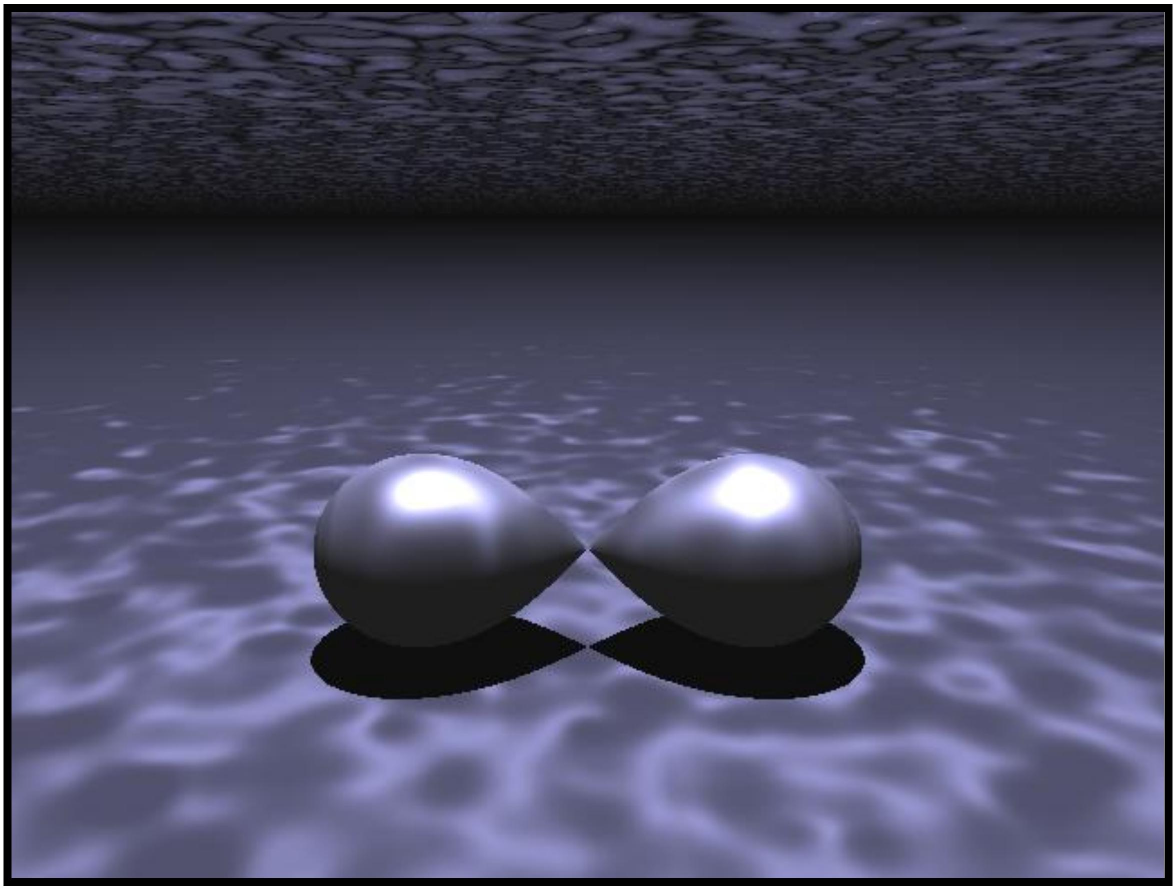


Figure 15-4: The Lemniscate of Bernoulli as a Solid of Revolution

Made to look like it's resting on the sea floor, the Lemniscate of Bernoulli has been rotated about the x-axis to form the solid in the picture above. The solid has been given a finish that reflects the light coming from above and also the watery surroundings.

Chapter 16 – The Folium of Descartes

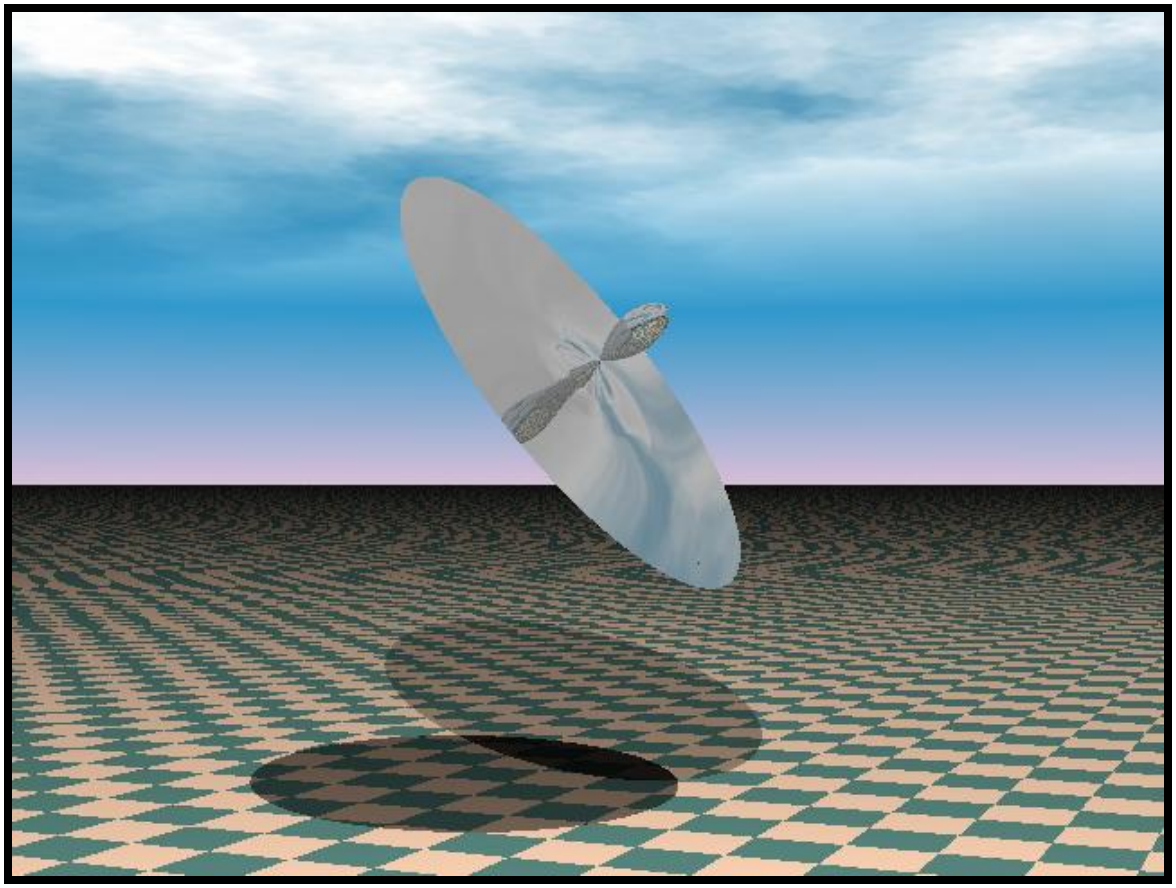


Figure 16-1: The Folium of Descartes as a Solid of Revolution

To form the object in the above picture, the Folium of Descartes was truncated along its asymptote and then rotated about the line y = x. The object was then placed above the checkered plane and given a semi-reflective surface. One can see that it reflects the checkered plane in its loop and the loop itself is then reflected in the disk formed by its wings. Light sources have been placed so as to give intersecting shadows on the plane below.

16.1 Introduction

The curve that is known today as the Folium of Descartes was first discussed by Descartes in 1638. (Folium, of course, means leaf.) Although Descartes found the correct shape of the curve in the first quadrant, he believed that the leaf shape was repeated in all four quadrants, thereby rendering a curve that looked like the four petals of a flower. (See Figure 16-2 for a graph of this curve.) It is said that Descartes devised the curve to challenge Fermat's extremum-finding techniques. Whether Fermat was successful or not is unknown; the story may simply be untrue. However, the problem to determine the tangent to the curve was definitely proposed to Roberval who also wrongly believed that the curve had the form of a jasmine flower. His name (*fleur de jasmine*) was later changed. As will be seen in the dynamic geometry section of this chapter, the problem of finding the tangent to the curve is quite a formidable one and takes some doing.

16.2 Equations and Graph of the Folium of Descartes

The Folium of Descartes is the curve described by the Cartesian equation

$$x^3 + y^3 = 3axy \qquad \textbf{Equation 16-1}$$

Clearly, the polar form follows directly by substituting $x = r\cos\theta$ and $y = r\sin\theta$ resulting in

$$r = \frac{3a\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta} \qquad \textbf{Equation 16-2}$$

Further, a parametric representation can be derived by letting $y = xt$ in Equation 16-1. Making this substitution and solving for x and, in turn y, we have

$$(x, y) = \frac{3at}{1+t^3}(1, t) \quad -\infty < t < +\infty \qquad \textbf{Equation 16-3}$$

In the parametric form, the curve has three arcs. For $-1 < t < 0$, the curve is located in the second quadrant, with $t = 0$ corresponding to the origin. For $t < -1$, the curve occupies the fourth quadrant, and approaches the origin as $t \to -\infty$. The loop in the first quadrant corresponds to $0 \le t < \infty$, going counterclockwise with increasing t.

The equation of the tangent to the Folium of Descartes at the point $t = q$ is

$$(1-2q^3)y = q(2-q^3)x - 3aq^2 \qquad \textbf{Equation 16-4}$$

Figure 16-2 represents a graph of the curve.

16.3 Analytical and Physical Properties of the Folium of Descartes

Based on the Folium of Descartes' parametric representation found in Equation 16-3, i.e., $x = 3at / (1 + t^3)$ and $y = 3at^2 / (1 + t^3)$, the following subsections contain an analysis of the Folium of Descartes.

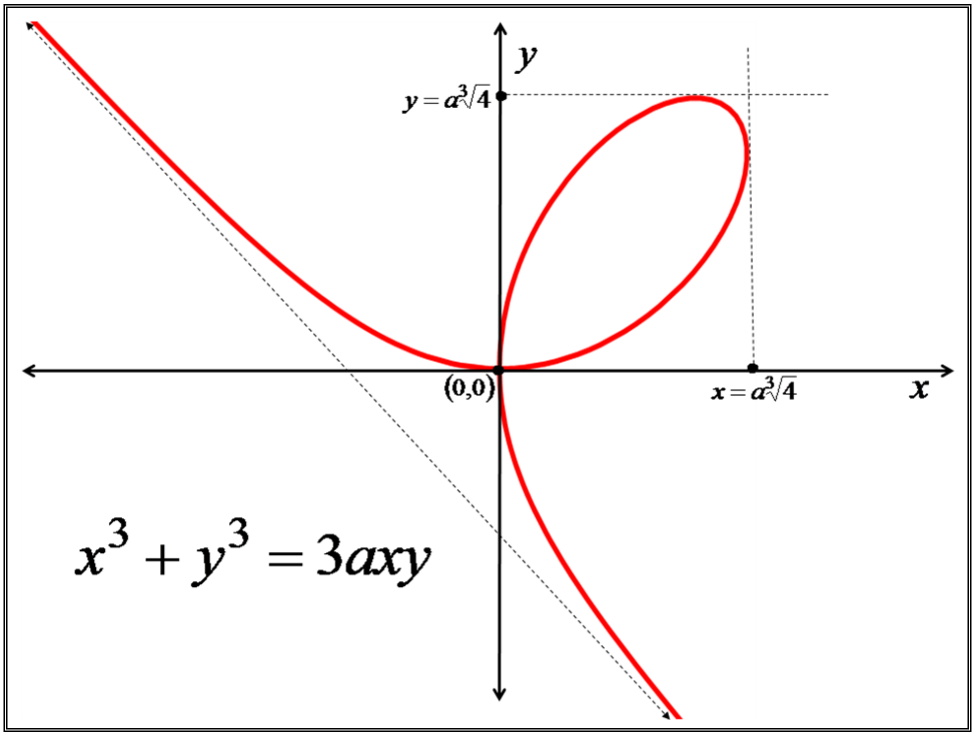


Figure 16-2: Graph of the Folium of Descartes

16.3.1 Derivatives of the Folium of Descartes

- $\dot{x} = \dfrac{3a\left(1-2t^3\right)}{\left(1+t^3\right)^2}$.
- $\ddot{x} = \dfrac{18at^2\left(t^3-2\right)}{\left(1+t^3\right)^3}$.
- $\dot{y} = \dfrac{3at\left(2-t^3\right)}{\left(1+t^3\right)^2}$.
- $\ddot{y} = \dfrac{6a\left(1-7t^3+t^6\right)}{\left(1+t^3\right)^3}$.
- $y' = \dfrac{t\left(2-t^3\right)}{1-2t^3}$.
- $y'' = \dfrac{2\left(1+t^3\right)^4}{3a\left(1-2t^3\right)^3}$.

16.3.2 Metric Properties of the Folium of Descartes

To calculate the area of the loop, the easiest way to proceed is to convert the parametric form of the Folium of Descartes to a polar form and use the area formula

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta .$$

Upon doing this, we get

$$r^2 = x^2 + y^2 = \frac{9a^2t^2}{\left(1+t^3\right)^2} + \frac{9a^2t^4}{\left(1+t^3\right)^2} = \frac{9a^2t^2\left(1+t^2\right)}{\left(1+t^3\right)^2}.$$

Since

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1} t,$$

we have

$$d\theta = \frac{dt}{1+t^2}.$$

Therefore,

$$A = \frac{1}{2}\int_0^{\infty} \frac{9a^2t^2\left(1+t^2\right)}{\left(1+t^3\right)^2} \cdot \frac{dt}{1+t^2} = \frac{9a^2}{2}\int_0^{\infty} \frac{t^2 dt}{\left(1+t^3\right)^2}.$$

To evaluate this integral, let $u = 1 + t^3$. Then $du = 3t^2 dt$ and the limits become 1 to ∞. Hence,

$$A = \frac{3a^2}{2}\int_1^{\infty} \frac{du}{u^2} = \frac{3a^2}{2}.$$

If p is the distance from the origin to the tangent of the Folium of Descartes, then

$$p = \frac{-3at^2}{\sqrt{t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1}}.$$

If r is the radial distance, that is, the distance from the origin to the curve, then

$$r = \frac{3at}{1+t^3}\sqrt{1+t^2} .$$

16.3.3 Curvature of the Folium of Descartes

If ρ represents the radius of curvature for the Folium of Descartes, then

$$\rho = \frac{3a\left(1+4t^2-4t^3-4t^5+4t^6+t^8\right)^{3/2}}{2\left(1+t^3\right)^4}.$$

If (α, β) denotes the coordinates of the center of curvature for the Folium of Descartes, then

$$\alpha = \frac{-3at^3\left(8 - 15t - 12t^3 + 6t^4 + 6t^6 - 6t^7 - t^9\right)}{2\left(1+t^3\right)^4} \quad \text{and}$$

$$\beta = \frac{3a\left(1 + 6t^2 - 6t^3 - 6t^5 + 12t^6 + 15t^8 - 8t^9\right)}{2\left(1+t^3\right)^4}$$

16.3.4 Angles for the Folium of Descartes

If ψ is the tangential-radial angle for the Folium of Descartes, then

$$\tan\psi = \frac{t\left(1+t^3\right)}{1+2t^2-2t^3-t^5}.$$

If ϕ denotes the tangential angle of the Folium of Descartes, then

$$\tan\phi = \frac{t\left(2-t^3\right)}{1-2t^3}.$$

If θ represents the radial angle for the Folium of Descartes, then

$$\tan\theta = t.$$

16.4 Geometric Properties of the Folium of Descartes

- Intercept: (0, 0).
- Extrema: $\left(a\sqrt[3]{2}, a\sqrt[3]{4}\right)$ is y-maximum; $\left(a\sqrt[3]{4}, a\sqrt[3]{2}\right)$ is x-maximum.
- Extent: $-\infty < x < +\infty$; $-\infty < y < +\infty$; $-\infty < t < +\infty$.
- Discontinuity: $t = -1$.
- Symmetry: The Folium of Descartes is symmetric about the line $y = x$.
- Asymptote: $x + y + a = 0$.
- Loop: $0 \le t < \infty$.

16.5 Dynamic Geometry of the Folium of Descartes

The next few sections delineate dynamic geometry constructions for the Folium of Descartes and its tangent. Interestingly enough, every one of the following constructions specifies the use of the inversion of a point. Recall that the inversion of point C with respect to circle AB means that if point C' is the inverted point then $AB^2 = AC \cdot AC'$ and C' will lie on line AC. The inversions that follow are therefore generally accomplished by calculating the quantity $m = (AB^2/AC) - AC$ and then translating point C at the appropriate angle by the calculated distance. However, this calculation of m and the subsequent translation does not seem like a "straight-edge and compass construction." Nonetheless, the inversion of a point with respect to a circle can be done with a simple straight-edge and compass and we show how to make such a construction in the Appendix. Therefore, when encountering inversion in any of the constructions in this text, consider the calculation method alluded to above to be merely a shortcut for the "straight-edge and compass" methodology.

16.5.1 The Folium of Descartes Using Polar Coordinates

For the construction found in Table 16-1, make sure that the grid form and the coordinate form under the graph menu of GSP are set for polar representation and that the angle units in the preference window under the display menu are set for radian measure.

Table 16-1: The Folium of Descartes Using Polar Coordinates

1. Create x-y axes with origin at point A and unit point B = (1, 0)	11. Calculate $m_3 = m_1 + m_2$
2. Draw circle AB with center at A and passing through point B	12. Let G be the plot of m_3 and t as r, θ, i.e., G = $(r,\theta) = (m_3, t)$
3. Draw circle CD with center at C and passing through point D	13. Draw ray AG
4. Let E be a random point on the circumference of circle CD	14. Let m_4 be the measure of the distance from point A to point B
5. Draw line segments CD and CE	15. Let m_5 be the measure of the distance from point A to point G
6. Let t be the measure of ∠DCE in radians	16. Calculate $m_6 = (m_4^2 / m_5) - m_5$
7. Let F be a random point in the plane	17. Let G' be the image when G is translated by m_6 at ∠BAG
8. Label the r-coordinate of point F as a	18. Trace point G' and change its color
9. Calculate $m_1 = \cos^2 t / (3a\sin t)$	19. Animate point E around circle CD
10. Calculate $m_2 = \sin^2 t / (3a\cos t)$	

If, in the Cartesian equation for the Folium of Descartes, one makes the substitution $x = r\cos t$ and $y = r\sin t$, it is easy to see that

$$\frac{3a}{r} = \cot t \cos t + \sin t \tan t .$$

Hence, the curve can be constructed by taking the inverse with respect to the unit circle of the point with polar coordinates

$$\left(\frac{\cot t \cos t + \sin t \tan t}{3a}, t \right).$$

Therefore, steps 5 – 9 make the necessary calculations for the coordinates and then plot the point. Steps 11 – 15 construct the plotted point’s inverse. Drag point F to change the value of a.

16.5.2 A Real Geometric Construction for the Folium of Descartes

The previous construction is what one might call a quasi-geometric construction in that it relies heavily on the plotting capability of GSP to achieve its animation. The only geometry involved is the inversion of point G. However, Table 16-2 presents a construction that is 100 percent geometric.

Table 16-2: A Real Geometric Construction for the Folium of Descartes

1. Draw horizontal line AB	13. Construct $P_3 \perp$ to line L_2 through point G
2. Draw circle AB with center at A and passing through point B	14. Let H be the intersection of perpendicular P_2 and line AC
3. Construct $P_1 \perp$ to Line AB through point A.	15. Let I be the intersection of perpendicular P_3 and line AC
4. Let C be a random point on the circumference of circle AB	16. Draw line segment HI
5. Draw line AC	17. Let J be the midpoint of line segment HI
6. Let point D be the point diametrically opposite point B	18. Let J' be the inversion of point J with respect to circle AB*
7. Let L_1 be the line parallel to line AC through point D	a. Let m_1 be a measure of the distance of point A to point B
8. Let E be either intersection of P_1 with circle AB	b. Let m_2 be a measure of the distance of point A to point J
9. Let L_2 be the line parallel to line AC through point E	c. Calculate $m_3 = (m_1^2/m_2) - m_2$
10. Let point F be the intersection of line L_1 and P_1	d. Translate point J by m_3 at $\angle$BAJ
11. Construct $P_2 \perp$ to line L_1 through point F	19. Trace point J' and change its color
12. Let point G be the intersection of line L_2 and line AB	20. Animate point C around circle AB

*Steps a, b, c, and d are the sub-steps necessary to accomplish step 18.

We will now continue this construction to add the Folium's tangent. This becomes quite a complicated construction, mostly due to the fact that the screen gets quite crowded with all the lines that are required. However, if you persevere, it is well worth the effort. The resulting animation is spectacular!

Table 16-2 (Continued): A Real Geometric Construction for the Folium of Descartes

21. Let K be the intersection of perpendicular P_2 and line AB	32. Construct $P_6 \perp$ to P_3 through point N
22. Let K' be the image when point K is reflected across line L_1	33. Let point O be the intersection of perpendiculars P_5 and P_6
23. Let L be the intersection of perpendiculars P_1 and P_3	34. Draw line AO
24. Let L' be the image when point L is reflected across line L_2	35. Construct $P_7 \perp$ to line AO through point J
25. Let L" be the image when point L' is reflected across line AC	36. Draw line segment JJ'
26. Construct $P_4 \perp$ to P_3 through point L"	37. Let P be the midpoint of line segment JJ'
27. Let point M be the intersection of perpendiculars P_2 and P_4	38. Construct $P_8 \perp$ to line AC through point P
28. Construct $P_5 \perp$ to line AC through point J	39. Let Q and R be random points on perpendicular P_7
29. Let K" be the image when point K' is reflected across line AC	40. Let Q' and R' be the reflections of Q and R across P_8
30. Draw line segment MK"	41. Draw line Q'R'
31. Let N be the midpoint of line segment MK"	42. Make line Q'R' thick and change its color

Of course, line Q'R' is the tangent. For the best looking animation and to really be able to see the tangent touch the curve as the animation executes, hide all of the construction elements except the tracing point and the tangent line. It's gorgeous!

16.5.3 An Alternate Folium of Descartes Construction

If you thought that the previous construction was complicated and created a cluttered screen, get a load of the one in Table 16-3. However, it's very interesting—give it a try. This construction is also continued in Table 16-3 (Continued) just as we did in the previous section, to add the Folium's tangent. In this case, it's also an alternative construction for the tangent line. Again, the screen gets quite crowded with all the lines that are required. As before, however, if you persevere, it is well worth the effort. When you get done and are successful, for a non-cluttered animation, hide all construction elements except the tangent line (P_{11}) and the tracing point (J'). There is something quite breathtaking about this particular curve.

Table 16-3: An Alternate Construction for the Folium of Descartes

1. Draw horizontal line AB	16. Let point H be the intersection of line L_1 and perpendicular P_4
2. Draw circle AB with center at A and passing through point B	17. Let H' be the image when H is rotated about point F by 90°
3. Construct $P_1 \perp$ to line AB through point A	18. Let H" be the inverse of point H' with respect to circle AB*
4. Let C be a random point on the circumference of circle AB	a. Let m_1 be a measure of the distance from point A to B
5. Draw line AC	b. Let m_2 be a measure of the distance from point A to H'
6. Construct $P_2 \perp$ to line AC through point B	c. Calculate $m_3 = (m_1^2/m_2) - m_2$
7. Let D be the intersection of perpendicular P_2 and line AC	d. Let H" be the translation of H' by m_3 at ∠BAH'
8. Construct $P_3 \perp$ to line AB through point D	19. Construct $P_6 \perp$ to line AB through point H"
9. Let E be the intersection of perpendicular P_3 and line AB	20. Let I be the intersection of line AB and perpendicular P_6
10. Construct P_4 to line AC through point E	21. Let line L_2 be the parallel to perpendicular P_4 through point A
11. Let F be the intersection of perpendicular P_4 and line AC	22. Let line L_3 be the parallel to line AC through point I
12. Construct $P_5 \perp$ to P_2 through point E	23. Let point J be the intersection of lines L_2 and L_3
13. Let point G be the intersection of perpendiculars P_2 and P_5	24. Let J' be the image when J is rotated about point A by 90°
14. Draw line FG	25. Trace point J' and change its color
15. Let line L_1 be the parallel to line FG through point B	26. Animate point C around circle AB

*Steps a, b, c, and d are the sub-steps necessary to accomplish step 18.

Now for the continuation:

Table 16-3 (Continued): An Alternate Construction for the Folium of Descartes

27. Let K be either intersection of circle AB and P_1	37. Let M be the intersection of perpendiculars P_1 and P_9
28. Bisect ∠BAK	38. Draw circle AM with center at A and passing through M
29. Let J" be the reflection of point J' across the angle bisector	39. Let K' be the inverse of point K with respect to circle AM**
30. Construct $P_7 \perp$ to line AB through point J'	i. Let m_7 be a measure of the distance from A to point M
31. Let L be the intersection of line AB and perpendicular P_7	j. Let m_8 be a measure of the distance from point A to point K
32. Draw circle AL with center at A and passing through point L	k. Calculate $m_9 = (m_7^2/m_8) - m_8$
33. Let B' be the inverse of point B with respect to circle AL*	l. Let K' be the image when K is translated by m_9 at ∠BAK
e. Let m_4 be a measure of the distance from point A to point L	40. Let K" be the image when K' is dilated about A by 3
f. Let m_5 be a measure of the distance from point A to point B	41. Construct $P_{10} \perp$ to P_1 through point K"
g. Calculate $m_6 = (m_4^2/m_5) - m_5$	42. Let point N be the intersection of perpendiculars P_8 and P_{10}
h. Let B' be the image when B is translated by m_6	43. Draw line segment NJ"
34. Let B" be the image when B' is dilated about A by 3	44. Construct $P_{11} \perp$ to line segment NJ" through point J'
35. Construct $P_8 \perp$ to line AB through point B"	45. Make perpendicular P_{11} thick and change its color
36. Construct $P_9 \perp$ to P_8 through point J'	

* Steps e, f, g, and h are the sub-steps necessary to accomplish step 33.
**Steps i, j, k, and l are the sub-steps necessary to accomplish step 39.

16.5.4 A Variant of the Folium of Descartes Construction

Table 16-4 contains a construction that is simply a variant of the previous construction, but interesting in its own right.

Table 16-4: Variant of the Folium of Descartes Construction

1. Draw horizontal line AB	12. Draw line segment AF
2. Construct $P_1 \perp$ to line AB through point A	13. Let G be the midpoint of line segment AF
3. Draw circle AB with center at A and passing through point B	14. Let G_1 be the image when G is translated by vector D' → G
4. Let C be a random point on the circumference of circle AB	15. Let G_2 be the inversion of G_1 with respect to circle AB*
5. Draw line AC	a. Let m_1 be a measure of the distance from A to B
6. Construct $P_2 \perp$ to line AC through point B	b. Let m_2 be a measure of the distance from A to G_1
7. Let point D be the intersection of perpendiculars P_1 and P_2	c. Calculate $m_3 = m_1^2 / m_2 - m_2$
8. Let E be the intersection of line AC and perpendicular P_2	d. Translate point G_1 by distance m_3 at ∠BAG_1
9. Construct $P_3 \perp$ to line AB through point B	16. Let G_3 be the image when G_2 is dilated about point A by 3
10. Let D' be the image when D is rotated about point E by 90°	17. Trace point G_3 and change its color
11. Let F be the intersection of line AC and perpendicular P_3	18. Animate point C around circle AB

*Steps a, b, c, and d are the sub-steps necessary to accomplish step 15.

We will not continue with the tangent for this construction as it is essentially a duplicate of the tangent construction done in the previous section.

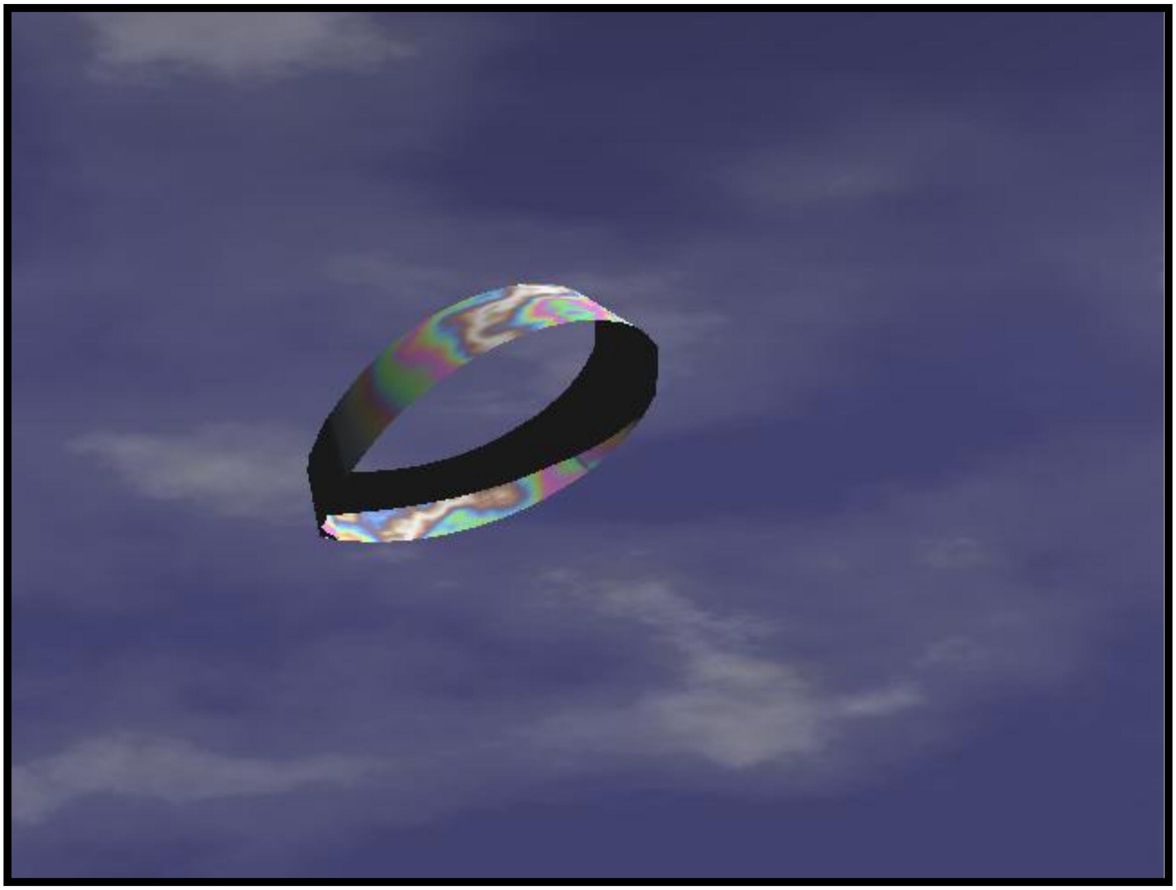


Figure 16-3: The Loop of the Folium of Descartes in Three Dimensions

The loop of the Folium of Descartes has been extruded to create the three-dimensional object seen floating in the sky above. It has been given an iridescent finish which makes it look as though it were made from opal. A light source has been placed so as to shadow the inside portion of the loop.

Chapter 17 – The Lemniscate of Gerono

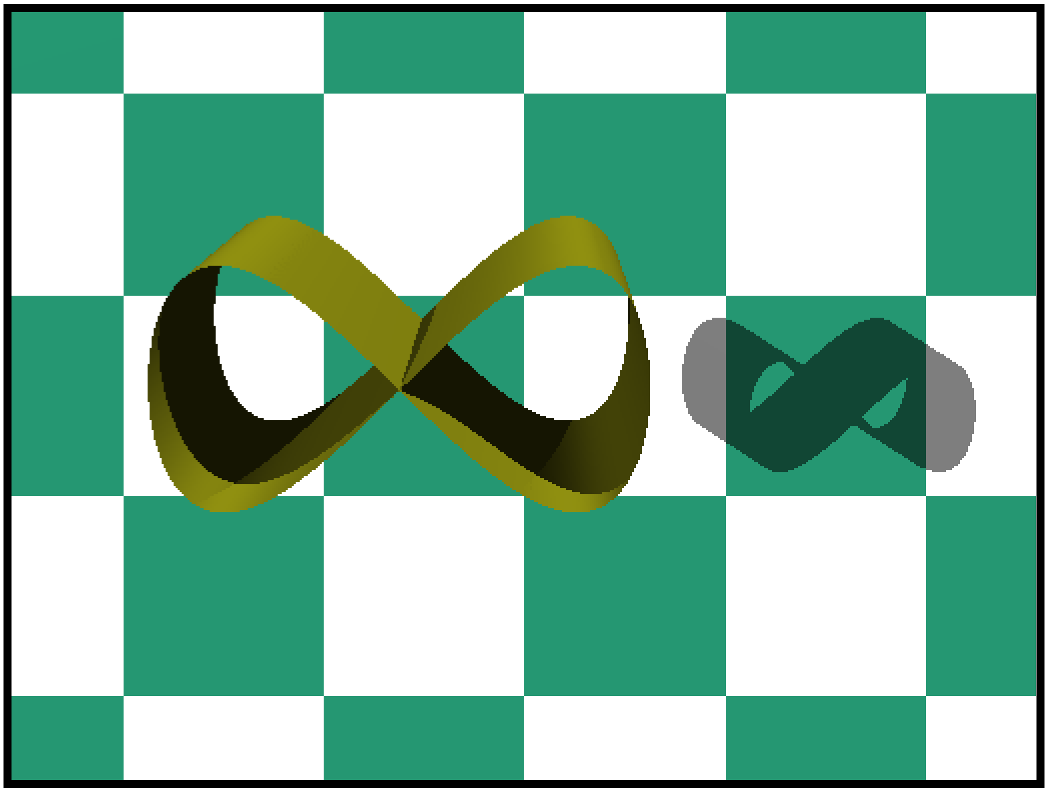


Figure 17-1: The Lemniscate of Gerono Rendered in Three Dimensions

The cross-section of the object above is the curve known as the Lemniscate of Gerono. To obtain this rendering, the curve was extruded into the third dimension, given a golden colored finish, and placed above the green and white checkered plane. Unlike previous renderings, this time the viewpoint is looking down on the object so there is no horizon line. Light sources have been situated so as to illuminate the object from above and slightly to the side, thereby casting the shadow down upon the plane and to the side of the object. This also causes the inside loops to be partially shadowed.

17.1 Introduction

The curve for this chapter is the Lemniscate of Gerono; this curve is sometimes referred to as the Eight Curve or the Bowtie Curve (due to its shape). Camille Christophe Gerono was a French mathematics teacher who was born in Paris in 1799, lived there all his life, and died in Paris in 1891. He published many papers on geometry and the Diophantine analysis. The curve was first studied by Gregoire of St. Vincent in 1647 and further studied by Cramer in 1750; in 1895, Aubry officially named the curve in Gerono's honor.

17.2 Equations and Graph of the Lemniscate of Gerono

The Lemniscate of Gerono is defined by the Cartesian equation

$$x^4 = a^2\left(x^2 - y^2\right) \quad \textbf{Equation 17-1}$$

By making the usual transformation to polar coordinates, that is, $x = r\cos\theta$ and $y = r\sin\theta$, we obtain the polar equation as

$$r^2 = a^2 \sec^4\theta \cdot \cos 2\theta \quad \textbf{Equation 17-2}$$

Now if we let $y = x \sin t$, Equation 17-1 will yield a parametric representation which is

$$(x, y) = a\cos t(1, \sin t), \quad -\pi \le t \le \pi \quad \textbf{Equation 17-3}$$

An alternate parametric representation can be obtained by letting $u = \tan(t/2)$. After much manipulation, this substitution into Equation 17-3 gives

$$(x, y) = \frac{a\left(1-u^2\right)}{1+u^2}\left(1, \frac{2u}{1+u^2}\right) \quad -\infty \le u \le +\infty \quad \textbf{Equation 17-4}$$

Figure 17-2 portrays the graph of the Lemniscate of Gerono.

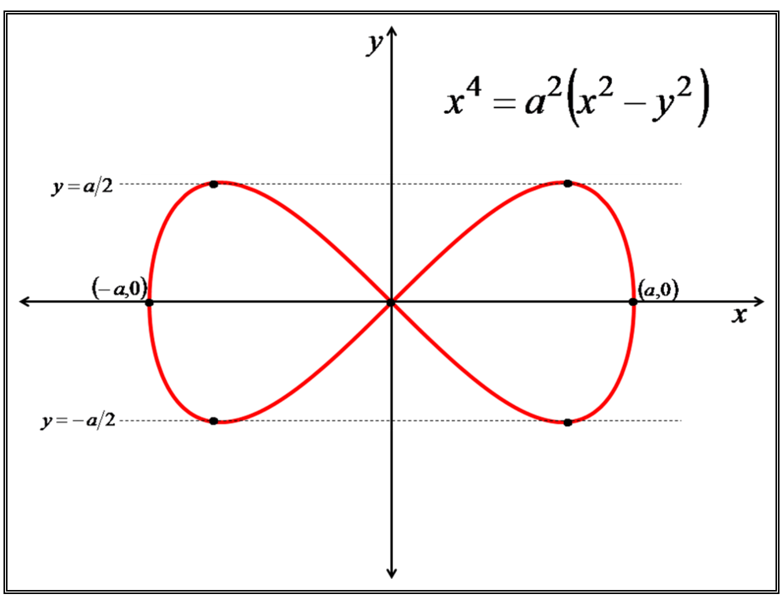


Figure 17-2: Graph of the Lemniscate of Gerono

The equation of the tangent line to the Lemniscate of Gerono at the point $t = q$ is

$$\sin q \cdot y = \left(2\sin^2 q - 1\right)\cdot x + a\cos^3 q. \quad \textbf{Equation 17-5}$$

17.3 Analytical and Physical Properties of the Lemniscate of Gerono

Based on the Lemniscate of Gerono's parametric representation found in Equation 17-3, that is, $x = a \cos t$ and $y = a \sin t \cos t$, the following subsections contain an analysis of the Lemniscate of Gerono.

17.3.1 Derivatives of the Lemniscate of Gerono

- $\dot{x} = -a\sin t$.
- $\ddot{x} = -a\cos t$.
- $\dot{y} = a\left(1 - 2\sin^2 t\right)$.
- $\ddot{y} = -4a\sin t\cos t$.
- $y' = \dfrac{2\sin^2 t - 1}{\sin t}$.
- $y'' = -\dfrac{\cos t\left(1 + 2\sin^2 t\right)}{\sin^3 t}$.

17.3.2 Metric Properties of the Lemniscate of Gerono

One method to calculate the area enclosed by the Lemniscate of Gerono is to first note that the two loops of the curve are symmetric and therefore the area enclosed by one loop is simply half of the entire area. Further, each loop is symmetric about the x-axis, so the area enclosed by the curve in the first quadrant is one-quarter of the entire area. Therefore, solving the Cartesian equation for y, one obtains for the area in the first quadrant

$$A = \frac{1}{a}\int_0^a x\sqrt{a^2 - x^2}\,dx.$$

Now, a substitution of $x = a \sin\theta$ yields the following

$$A = a^2\int_0^{\pi/2} \sin\theta\cos^2\theta \cdot d\theta.$$

Of course, the value of this simple integral is merely $a^2/3$ and therefore the total area enclosed by the curve is $4a^2/3$. Alternately, one could substitute $u = a^2 - x^2$ in the first integral; however, the ultimate result is, of course, the same.

If the Lemniscate of Gerono is rotated about the *x*-axis, the volume of the resulting solid of revolution can also be calculated. For the portion of the solid's volume in the first and fourth quadrant, we have

$$V = \pi \int_0^a y^2 dx = \frac{\pi}{a^2} \int_0^a \left(a^2 x^2 - x^4\right) dx$$

This expression integrates directly without any intermediate substitution necessary. Its value is $2\pi a^3/15$, thereby making the entire volume of the solid $4\pi a^3/15$.

If r represents the distance from the origin to the Lemniscate of Gerono, then

$$r = a \cos t \sqrt{1 + \sin^2 t}\,.$$

If p denotes the distance from the origin to the tangent line of the Lemniscate of Gerono, then

$$p = \frac{-a\cos^3 t}{\sqrt{1 - 3\sin^2 t + 4\sin^4 t}}\,.$$

17.3.3 Curvature of the Lemniscate of Gerono

If ρ represents the radius of curvature for the Lemniscate of Gerono, then

$$\rho = \frac{a\left(1 - 3\sin^2 t + 4\sin^4 t\right)^{3/2}}{\cos t\left(1 + 2\sin^2 t\right)}\,.$$

If (α, β) denotes the coordinates of the center of curvature for the Lemniscate of Gerono, then

$$\alpha = \frac{2a\sin^2 t\left(3 - 6\sin^2 t + 4\sin^4 t\right)}{\cos t\left(1 + 2\sin^2 t\right)} \quad \text{and} \quad \beta = \frac{2a\sin^3 t\left(2 - 3\sin^2 t\right)}{\cos t\left(1 + 2\sin^2 t\right)}.$$

17.3.4 Angles for the Lemniscate of Gerono

If ψ is the tangential-radial angle for the Lemniscate of Gerono, then

$$\tan\psi = -\frac{1}{2} \cdot \cot^2 t \cdot \csc t\,.$$

If ϕ is the tangential angle for the Lemniscate of Gerono, then

$$\tan\phi = \frac{2\sin^2 t - 1}{\sin t}\,.$$

If θ is the radial angle for the Lemniscate of Gerono, then

$$\tan\theta = \sin t\,.$$

17.4 Geometric Properties of the Lemniscate of Gerono

- Intercepts: $(a, 0)$; $(0, 0)$, $(–a, 0)$.
- Extrema: $(a, 0)$ is x-maximum; $(–a, 0)$ is x-minimum;
  $\left(\frac{a\sqrt{2}}{2}, \frac{a}{2}\right)$ and $\left(\frac{-a\sqrt{2}}{2}, \frac{a}{2}\right)$ are y-maxima;
  $\left(\frac{a\sqrt{2}}{2}, -\frac{a}{2}\right)$ and $\left(\frac{-a\sqrt{2}}{2}, -\frac{a}{2}\right)$ are y-minima.
- Point of Inflection: $(0, 0)$.
- Extent: $-\pi \le t \le \pi$; $-a \le x \le a$; $-a/2 \le y \le a/2$.
- The Lemniscate of Gerono is symmetric about the x-axis, the y-axis, and the origin.
- There are two loops: (1) $-\pi \le t \le -\pi/2$ and $\pi/2 \le t \le \pi$; (2) $-\pi/2 \le t \le \pi/2$

17.5 Dynamic Geometry of the Lemniscate of Gerono

The following subsections delineate two different constructions for the Lemniscate of Gerono as well as constructions for the curve's tangent line and its osculating circle.

17.5.1 The Lemniscate of Gerono Made Easy

The Lemniscate of Gerono can be constructed by taking the locus of the foot of the perpendicular through the point $(\cos\theta, \sin\theta)$ dropped upon the line halfway between the point $(\cos 2\theta, \sin 2\theta)$ and the x-axis. The construction of Table 17-1 illustrates this.

Table 17-1: The Lemniscate of Gerono Made Easy

1. Draw horizontal line AB	8. Let D be the intersection of perpendicular P_2 and line AB
2. Draw circle AB with center at A and passing through point B	9. Draw line segment B'D
3. Let C be a random point on the circumference of circle AB	10. Let E be the midpoint of line segment B'D
4. Draw line AC	11. Construct $P_3 \perp$ to P_2 through point E
5. Construct $P_1 \perp$ to line AB through point C	12. Let point F be the intersection of perpendiculars P_1 and P_3
6. Let B' be the image when point B is reflected across line AC	13. Trace point F and change its color
7. Construct $P_2 \perp$ to line AB through point B'	14. Animate point C around circle AB

17.5.2 The Tangent Line to the Lemniscate of Gerono

We will use the same construction for the Lemniscate here as we did in the previous construction; however, we will add the steps necessary to construct the tangent line. Table 17-2 contains the construction. Of course, once that you have constructed the tangent line, it's "duck soup" to construct the Lemniscate's pedal curves. To do so, let point H be a random point anywhere on the screen and drop a perpendicular from point H to the Lemniscate's tangent (i.e., P_5 in the construction above). Point I, the intersection of that perpendicular and the tangent, will trace the desired pedal curves. Drag point H

Table 17-2: The Tangent to the Lemniscate of Gerono

1. Draw horizontal line AB	11. Construct $P_3 \perp$ to P_2 through point E
2. Draw circle AB with center at A and passing through point B	12. Let point F be the intersection of perpendiculars P_1 and P_3
3. Let C be a random point on the circumference of circle AB	13. Trace point F and change its color
4. Draw line AC	14. Construct $P_4 \perp$ to P_2 through point C
5. Let B' be the image when point B is reflected across line AC	15. Let point G be the intersection of perpendiculars P_2 and P_4
6. Construct $P_1 \perp$ to line AB through point C	16. Draw line AG
7. Construct $P_2 \perp$ to line AB through point B'	17. Construct $P_5 \perp$ to line AG through point F
8. Let D be the intersection of perpendicular P_2 and line AB	18. Make perpendicular P_5 thick and change its color
9. Draw line segment B'D	19. Animate point C around circle AB
10. Let E be the midpoint of line segment B'D	

around to various positions on the screen to trace different members of the family of pedals. Experiment with different positions of point H; it is particularly interesting to place point H on a perpendicular to line AB through point A—one gets a very symmetric pedal.

17.5.3 The Osculating Circle of the Lemniscate of Gerono

Well, the constructions in this chapter have been pretty easy up to now. Get a load of this one! See Table 17-3.

Table 17-3: The Osculating Circle of the Lemniscate of Gerono

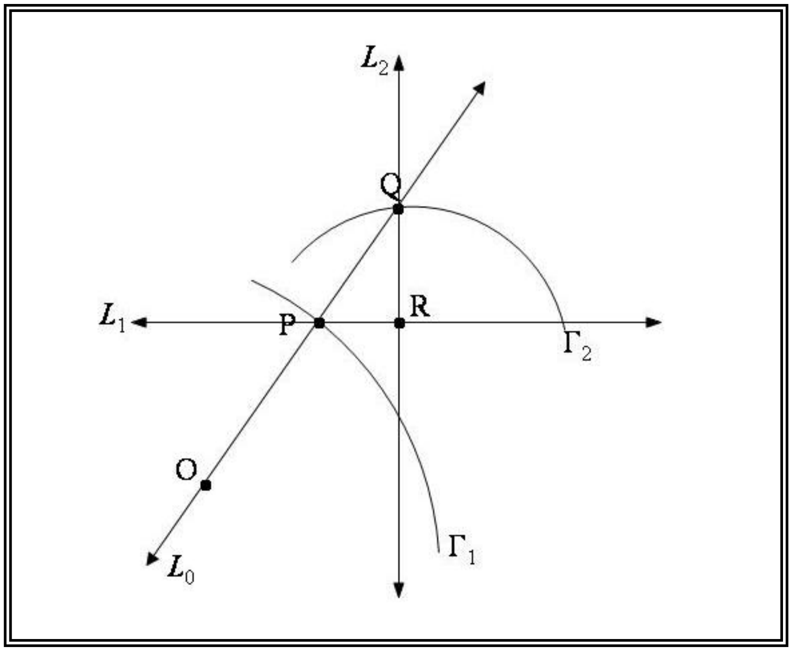
1. Draw horizontal line AB	22. Let A_2 be the image when A is translated by vector I → A
2. Draw circle AB with center at A and passing through point B	23. Draw line A_1A_2
3. Let C be a random point on the circumference of circle AB	24. Construct $P_6 \perp$ to line A_1A_2 through point B'
4. Draw line AC	25. Let J be a random point on perpendicular P_6
5. Construct $P_1 \perp$ to line AB through point C	26. Let A_3 be the image when A is translated by vector J → A
6. Let B' be the image when point B is reflected across line AC	27. Let K be a second random point on perpendicular P_6
7. Construct $P_2 \perp$ to line AB through point B'	28. Let A_4 be the image when A is translated by vector K → A
8. Let D be the intersection of perpendicular P_2 and line AB	29. Draw line A_3A_4
9. Draw line segment B'D	30. Let A_5 be the image when A is reflected across line A_3A_4
10. Let E be the midpoint of line segment B'D	31. Let B_1 be the image when B is reflected across line A_3A_4
11. Construct $P_3 \perp$ to P_2 through point E	32. Draw line A_5B_1
12. Let point F be the intersection of perpendiculars P_1 and P_3	33. Let point L be the intersection of line A_5B_1 and line A_1A_2
13. Trace point F and change its color	34. Construct $P_7 \perp$ to line AG through point L
14. Construct $P_4 \perp$ to P_2 through point C	35. Let M be the intersection of perpendicular P_7 and line AG
15. Let point G be the intersection of perpendiculars P_2 and P_4	36. Draw line MG'
16. Let G' be the image when G is rotated about point A by 90°	37. Construct $P_8 \perp$ to line MG' through point G'
17. Draw line AG	38. Let N be the intersection of line AG and perpendicular P_8
18. Construct $P_5 \perp$ to line AG through point G	39. Let F' be the image when F is translated by vector N → A
19. Let H be a random point on perpendicular P_1	40. Draw circle F'F with center at F' and passing through point F
20. Let A_1 be the image when A is translated by vector H → A	41. Change the color of circle F'F and make it thick
21. Let I be a second random point on perpendicular P_1	42. Animate point C around circle AB

This is a spectacular looking animation! If you're interested in seeing the evolute of the Lemniscate of Gerono, trace point F' and rerun the animation. The evolute is very weird and very symmetric; it makes for a nice graphic.

17.5.4 An Alternate Construction for the Lemniscate of Gerono

At first, this construction may seem very similar to that of section 17.5.1; however, it is actually quite different. There is a type of derived curve that was not addressed in Chapter 1. Refer to Figure 17-3 for the following brief discussion. The derived curve is called a hyperbolism and is defined in the following way. Given two curves, Γ_1 and Γ_2, a point O, and a line L_0 through point O intersecting Γ_1 and Γ_2 in points P and Q, respectively, draw line L_1 through point P parallel to the *x*-axis and draw line L_2 through point Q perpendicular to the *x*-axis. If point R is the intersection point of lines L_1 and L_2, then the locus of point R for all of the possible L_0 lines is the curve defined to be the hyperbolism of Γ_1 and Γ_2 with respect to point O. Point O is, of course, called the pole point.

Figure 17-3: The Hyperbolism



Well, given this brief introduction to the hyperbolism, it turns out that the Lemniscate of Gerono is a hyperbolism of two tangent circles such that the smaller circle has half the radius of the larger circle, is inside the larger circle, and the pole point is the center of the larger circle. In other words, if the pole point is the origin, and if the larger circle has equation $x^2 + y^2 = a^2$ then the smaller circle must have equation $x^2 + y^2 = ax$. Table 17-4 contains the construction based on this idea.

Table 17-4: An Alternate Construction for the Lemniscate of Gerono

1. Create *x*-*y* axes with origin at A and unit point B = (1, 0)	9. Construct $P_2 \perp$ to P_1 through point E
2. Draw circle AB with center at A and passing through point B	10. Let point F be the intersection of perpendiculars P_1 and P_2
3. Let C be a random point on circle AB in the 1st quadrant	11. Trace point F and change its color (say yellow)
4. Let B' be the image when B is dilated about point A by ½	12. Drag point C to the second quadrant so that point D appears
5. Draw circle B'B with center at B' and passing through point B	13. Construct $P_3 \perp$ to P_1 through point D
6. Draw line AC	14. Let point G be the intersection of perpendiculars P_1 and P_3
7. Let D and E be the intersections of line AC and circle B'B*	15. Trace point G and change its color (say yellow)
8. Construct $P_1 \perp$ to the *x*-axis through point C	16. Animate point C around circle AB

*GSP does not handle this situation correctly; however, the construction is structured to account for this mishandling. Point A will get labeled a second time (i.e., double labeled) as point D.

The horizontal diameter of circle AB will also get traced as this animation is run; of course, this diameter is not part of the Lemniscate. If GSP managed this situation correctly, the diameter would not get traced and point E would traverse the lower half of circle B'B when point C is dragged to the second quadrant instead of point D; point D would not be necessary, and therefore neither would perpendicular P_3 nor point G. In other words, point F would trace the entire Lemniscate. If you are confused after reading this last paragraph, select both line AC and circle B'B and use the "point of intersection" command under GSP’s "construction" menu when executing step 7. This guarantees that point E will appear after executing step 7 and point A will get double labeled as point D.

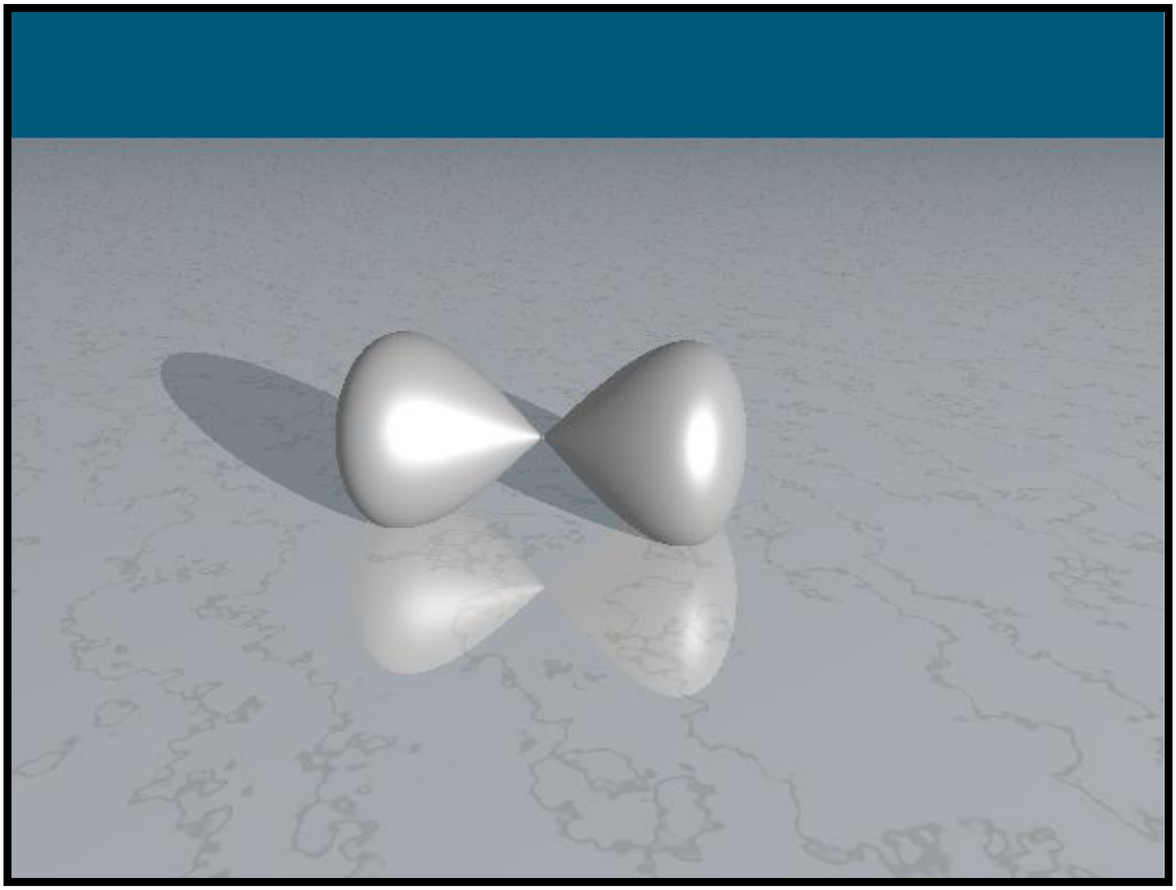


Figure 17-4: The Lemniscate of Gerono as a Solid of Revolution

The plane curve known as the Lemniscate of Gerono was rotated about the x-axis to create the object in the figure above. It was then situated so as to appear that it is resting upon the grayish, marble-like floor. The floor itself has been given a partially reflective finish and the object can be seen reflected in the floor. A light source has been located so as to shine on the object and cast its shadow behind on the floor.

Chapter 18 – The Cross and Bullet Nose Curves

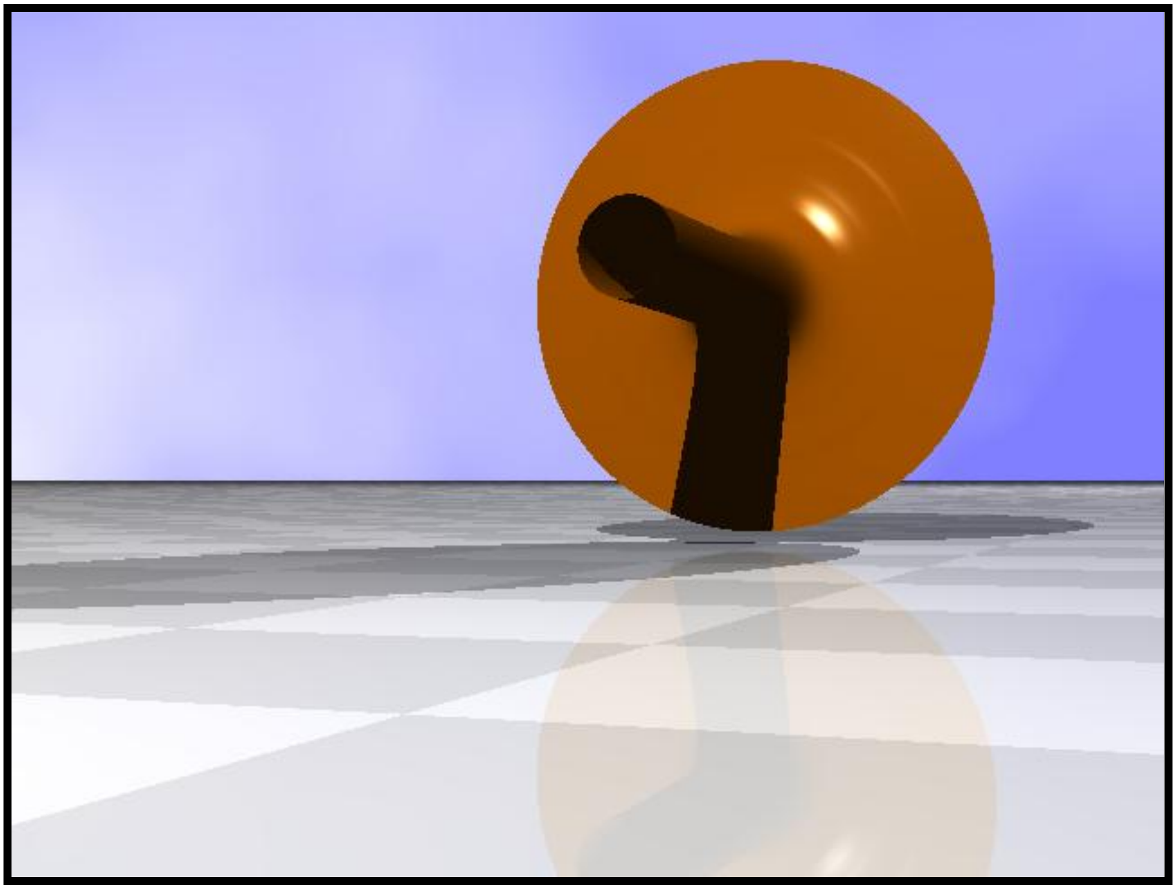


Figure 18-1: The Solid of Revolution Formed from the Cross Curve

One branch of the Cross Curve obtained from the equation $1/x^2 + 4/y^2 = 1$ was revolved about the y-axis to obtain the object pictured above. In order to do this, the curve was truncated along both its x and y asymptotes, which is why one sees the circular disk-shaped "base" and the long cylindrical protuberance rising from that base. The resulting object was then rotated to be oriented as seen and placed above the gray and white checkered plane. The plane has been made slightly reflective so as to reflect the image of the object. Light sources have been located so as to cast the shadows on the plane just under and behind the object. Note also how the cylindrical protuberance casts a shadow on the base itself.

18.1 Introduction

Instead of one curve, this chapter is devoted to two curves, namely the Cross Curve and the Bullet Nose Curve. There is an interesting connection between the two and that is why both are included in one chapter. We will discuss that connection later in the chapter. Alternate names for the Cross Curve are the Stauroid, the Equilateral Cruciform Curve, and the Policeman on Point-Duty Curve. This last alternate name undoubtedly stems from the fact that, visually, the curve looks like a city street intersection where there might very well be a policeman at the center directing traffic. However, names like that, humorous as they may be, tend to lessen the mystique and romance of the curves; Cross Curve is much preferred! The Bullet Nose Curve derives its name from its shape which (with a little imagination) looks like two bullets nose-to-nose.

18.2 Equations and Graph of the Cross and Bullet Nose Curves

If one takes the equation of the ellipse[6] given in Equation 14-7 of Chapter 14 and changes the sign of every exponent in that equation, one arrives at the Cartesian equation for the Cross Curve, that is,

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=1 \quad \textbf{Equation 18-1}$$

The usual substitutions of $x = r\cos\theta$ and $y = r\sin\theta$ into Equation 18-1 yields the polar form of the Cross Curve, which is

$$r^2 = a^2\sec^2\theta + b^2\csc^2\theta \quad \textbf{Equation 18-2}$$

Finally, making the substitution of $y = (b/a)\cdot x\cdot\cot t$ into Equation 18-1 and solving for x, one obtains $x = a\sec t$. Then, similarly solving for y one obtains $y = b\csc t$. Therefore, a parametric representation for the Cross Curve is simply

$$(x,y)=(a\sec t, b\csc t) \quad -\pi < t < \pi \quad \textbf{Equation 18-3}$$

The equation of the tangent to the Cross Curve at the point $t = q$ is

$$a\cdot y + b\cot^3 q\cdot x = ab\csc^3 q \quad \textbf{Equation 18-4}$$

Figure 18-2 portrays a graph of the Cross Curve.

In a similar vein, if one takes the equation of the hyperbola given in Equation 14-11 of Chapter 14 and changes the sign of every exponent in that equation, one arrives at the Cartesian equation for the Bullet Nose Curve, that is,

$$\frac{a^2}{x^2}-\frac{b^2}{y^2}=1 \quad \textbf{Equation 18-5}$$

[6] This connection with the ellipse is not completely coincidental, as the first dynamic geometry construction of this chapter will clearly demonstrate.

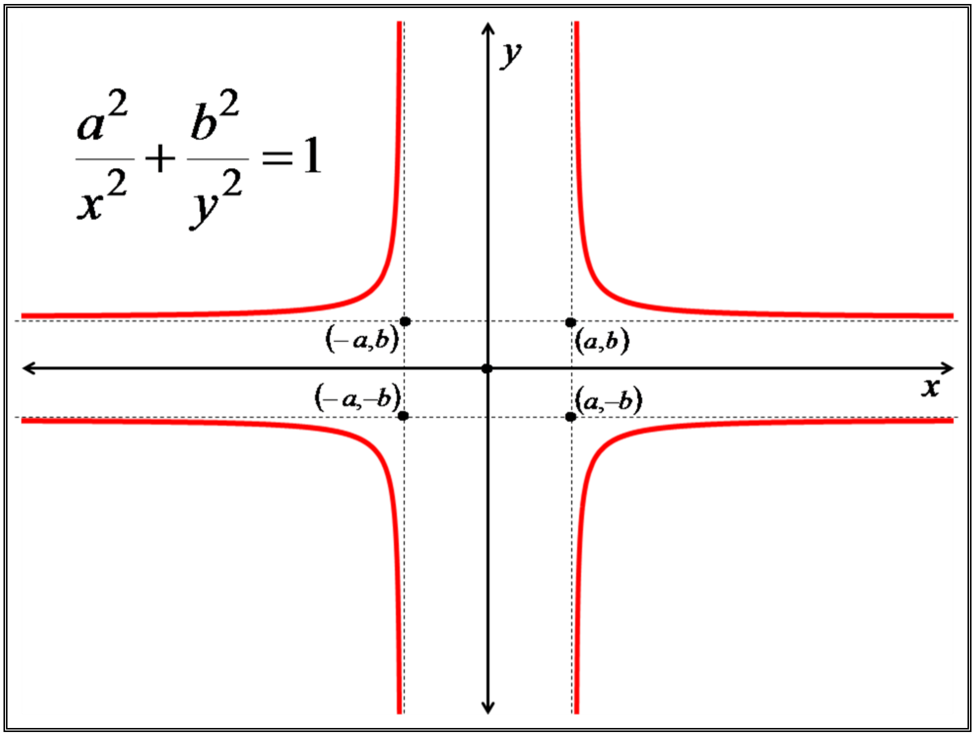


Figure 18-2: Graph of the Cross Curve

The usual substitutions of $x = r\cos\theta$ and $y = r\sin\theta$ into Equation 18-5 yields the polar form of the Bullet Nose Curve, which is

$$r^2 = a^2 \sec^2\theta - b^2 \csc^2\theta \qquad \textbf{Equation 18-6}$$

Finally, making the substitution of $y = (b/a)\cdot x\cdot\csc t$ into Equation 18-5 and solving for x, one obtains $x = a\cos t$. Then, similarly solving for y one obtains $y = b\cot t$. Therefore, a parametric representation for the Bullet Nose Curve is simply

$$(x, y) = (a\cos t, b\cot t) \quad -\pi < t < \pi \qquad \textbf{Equation 18-7}$$

The equation of the tangent to the Bullet Nose Curve at the point $t = q$ is

$$a\cdot y = b\csc^3 q\cdot x - ab\cot^3 q \qquad \textbf{Equation 18-8}$$

Figure 18-3 depicts a graph of the Bullet Nose Curve.

18.3 Analytical and Physical Properties of Both Curves

Based on the Cross Curve's parametric representation found in Equation 18-3 and based on the Bullet Nose Curve's parametric representation found in Equation 18-7, the following is an analysis of both curves.

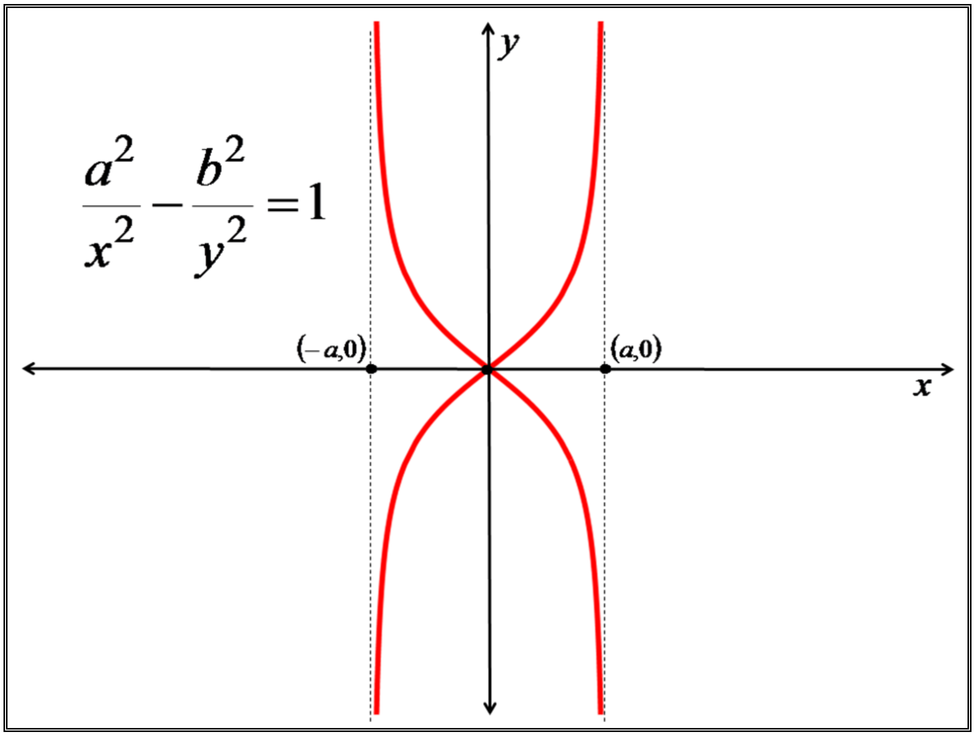


Figure 18-3: Graph of the Bullet Nose Curve

18.3.1 Derivatives of the Cross and Bullet Nose Curves

CROSS CURVE	BULLET NOSE CURVE
$\dot{x}=a\sec t\tan t$.	$\dot{x}=-a\sin t$.
$\ddot{x}=a\sec^3 t\left(1+\sin^2 t\right)$.	$\ddot{x}=-a\cos t$.
$\dot{y}=-b\csc t\cot t$.	$\dot{y}=-b\csc^2 t$.
$\ddot{y}=b\csc^3 t\left(1+\cos^2 t\right)$.	$\ddot{y}=2b\cot t\csc^2 t$.
$y'=-\frac{b}{a}\cot^3 t$.	$y'=\frac{b}{a}\csc^3 t$.
$y''=\frac{3b\cos^4 t}{a^2\sin^5 t}$.	$y''=\frac{3b\cos t}{a^2\sin^5 t}$.

18.3.2 Metric Properties of the Cross and Bullet Nose Curves

If r denotes the distance between the origin and the curve, then

CROSS CURVE	BULLET NOSE CURVE
$r=\sqrt{a^2\sec^2 t+b^2\csc^2 t}$.	$r=\cot t\sqrt{a^2\sin^2 t+b^2}$.

If p denotes the distance from the origin to the tangent of the curve, then

CROSS CURVE | BULLET NOSE CURVE

$$p = \frac{ab}{\sqrt{a^2 \sin^6 t + b^2 \cos^6 t}}. \qquad p = \frac{ab\cos^3 t}{\sqrt{a^2 \sin^6 t + b^2}}.$$

18.3.3 Curvature of the Cross and Bullet Nose Curves

If ρ denotes the radius of curvature of the curve, then

CROSS CURVE | BULLET NOSE CURVE

$$\rho = \frac{\left(a^2 \sin^6 t + b^2 \cos^6 t\right)^{3/2}}{3ab \sin^4 t \cos^4 t}. \qquad \rho = -\frac{\left(a^2 \sin^6 t + b^2\right)^{3/2}}{3ab \sin^4 t \cos t}$$

If (α, β) denotes the coordinates of the center of curvature for the curve, then for the Cross Curve

$$\alpha = \frac{a^2 \sin^6 t + 3a^2 \sin^4 t + b^2 \cos^6 t}{3a \sin^4 t \cos t} \quad \text{and} \quad \beta = \frac{a^2 \sin^6 t + 3b^2 \cos^4 t + b^2 \cos^6 t}{3b \sin t \cos^4 t}.$$

For the Bullet Nose Curve

$$\alpha = -\frac{a^2 \sin^6 t - 3a^2 \sin^4 t \cos^2 t + b^2}{3a \sin^4 t \cos t} \quad \text{and} \quad \beta = \frac{a^2 \sin^6 t + b^2\left(3\cos^2 t + 1\right)}{3b \sin t \cos t}.$$

18.3.4 Angles for the Cross and Bullet Nose Curves

If θ represents the radial angle of the Curve, then

CROSS CURVE | BULLET NOSE CURVE

$$\tan\theta = \frac{b}{a}\cot t. \qquad \tan\theta = \frac{b}{a}\csc t.$$

If ψ denotes the tangential-radial angle of the curve, then

CROSS CURVE | BULLET NOSE CURVE

$$\tan\psi = -\frac{ab}{a^2 \sin^4 t - b^2 \cos^4 t}. \qquad \tan\psi = \frac{ab \sin t \cos^2 t}{a^2 \sin^4 t + b^2}$$

If ϕ denotes the tangential angle of the curve, then

CROSS CURVE	BULLET NOSE CURVE
$\tan\phi = -\frac{b}{a}\cot^3 t$.	$\tan\phi = \frac{b}{a}\csc^3 t$

18.4 Geometric Properties of the Cross and Bullet Nose Curves

	CROSS CURVE	BULLET NOSE CURVE
Intercepts:	None.	(0, 0)
Extent:	$-\pi < t < \pi$, $t \neq 0$, $t \neq \pm\,\pi/2$ $x > a$ and $x < -a$ $y > b$ and $y < -b$	$-\pi < t < \pi$ $-a < x < a$ $-\infty < y < +\infty$
Inflection:	None	(0, 0)
Discontinuity:	$t = 0$, $t = \pm\,\pi/2$ $x = \pm\, a$, $x = \pm\,\infty$ $y = b$, $y = \pm\,\infty$.	$t = 0$
Symmetry:	$x = 0$, $y = 0$, (0, 0)	$x = 0$, $y = 0$, (0, 0)
Asymptotes:	$x = \pm\, a$ $y = \pm\, b$.	$x = \pm\, a$

18.5 Dynamic Geometry of the Cross and Bullet Nose Curves

The next seven subsections delineate constructions for the Cross Curve, the construction of the Cross Curve's tangent, the construction of the Cross Curve's osculating circle, construction of the Bullet Nose Curve, construction of the tangent to the Bullet Nose Curve, and the osculating circle of the Bullet Nose Curve.

18.5.1 The Cross Curve from the Tangent to an Ellipse

If through the points of intersection of a tangent to an ellipse with the two axes of the ellipse, lines perpendicular to each of the axes are drawn, the locus of the intersection point of those perpendiculars is the Cross Curve. In simpler words, this says construct an ellipse and then construct its tangent. Extend the semi-major and semi-minor axes of the ellipse until they intersect the tangent. Drop perpendiculars to the two axes through those two intersection points. Where the two perpendiculars intersect is a point on the Cross Curve, as the construction of Table 18-1 illustrates.

Steps 1 to 9 are the construction of the ellipse and its tangent, P_1. Line AD is the major axis of the ellipse while perpendicular P_2 is the minor axis. Points H and I are the two points where the ellipse's tangent intersects the axes and perpendiculars P_3 and P_4 are simply the two perpendiculars whose intersection point generates the Cross Curve.

Neat! In step 8, if line segment AC and P_1 do not intersect, simply drag point D until they do.

Table 18-1: The Cross Curve from the Tangent to an Ellipse

1. Draw circle AB with center at A and passing through point B	11. Draw line segment AD
2. Let C be a random point on the circumference of circle AB	12. Let G be the midpoint of line segment AD
3. Draw line segment AC	13. Construct $P_2 \perp$ to line AD through point G
4. Let D be a random point anywhere in the plane	14. Let H be the intersection of perpendicular P_1 and line AD
5. Draw line segment CD	15. Let point I be the intersection of perpendiculars P_1 and P_2
6. Let E be the midpoint of line segment CD	16. Construct $P_3 \perp$ to line AD through point H
7. Construct $P_1 \perp$ to line segment CD through point E	17. Construct $P_4 \perp$ to P_2 through point I
8. Let F be the intersection of line segment AC and P_1	18. Let point J be the intersection of perpendiculars P_3 and P_4
9. Construct the locus of point F as point C traverses circle AB	19. Trace point J and change its color
10. Draw line AD	20. Animate point C around circle AB

18.5.2 An Alternate Construction of the Cross Curve

Given a vertical line *L*, the Cross Curve is formed as the locus of points P such that the distance between P and the *x*-axis is the same as the distance from the origin, O, to where line OP intersects line *L*. In other words, AP = OB in Figure 18-4. The steps of Table 18-2 illustrate this construction.

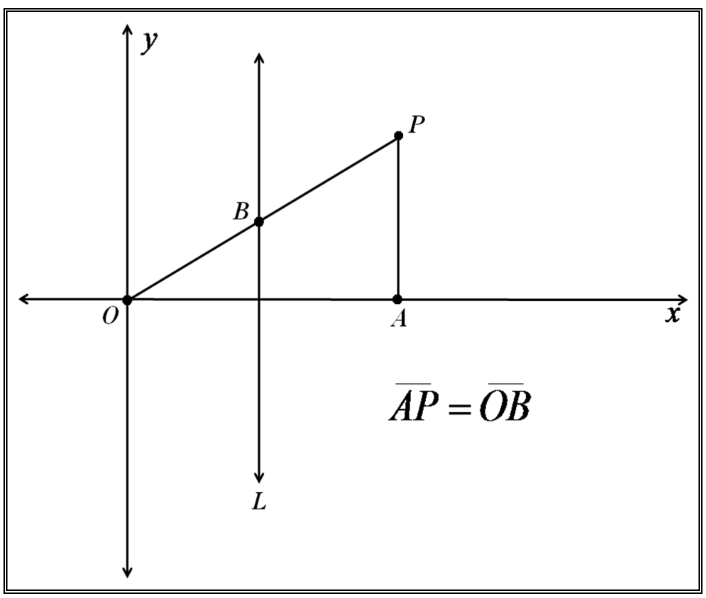


Figure 18-4: An Alternate Construction for the Cross Curve

Table 18-2: An Alternate Construction for the Cross Curve

1. Create *x*-*y* axes with origin A and unit point B = (1, 0)	10. Construct circle C_2 centered at F of radius = line segment AE
2. Draw circle AB with center at A and passing through point B	11. Construct $P_2 \perp$ to the *x*-axis through point F
3. Let C be a random point on the circumference of circle AB	12. Let G be one of the intersections of circle C_2 with P_2
4. Draw line AC	13. Construct $P_3 \perp$ to the *y*-axis through point G
5. Let D be any random point on the *x*-axis	14. Let H be the intersection of perpendicular P_3 and line AC
6. Construct $P_1 \perp$ to the *x*-axis through point D	15. Trace point H and change its color
7. Let E be the intersection of perpendicular P_1 and line AC	16. Let H' be the image when H is reflected across the *x*-axis
8. Let F be a second random point on the *x*-axis	17. Trace point H'
9. Draw line segment AE	18. Animate point C around circle AB

Point H only gives us the curve in two quadrants, hence we must create point H' in order to get all four branches.

18.5.3 The Tangent to the Cross Curve

Here, the tangent to the Cross Curve is constructed. The construction of the curve itself is done slightly differently than that of the construction in section 18.5.1; however, it's basically the same construction. Refer to Table 18-3.

Table 18-3: The Tangent to the Cross Curve

1. Create *x*-*y* axes with origin A and unit point B = (1, 0)	13. Let point H be the intersection of perpendiculars P_3 and P_4
2. Draw circle AB with center at A and passing through point B	14. Trace point H and change its color
3. Draw circle AC centered at A passing through C and AC < AB	15. Let I be the intersection of perpendicular P_3 and line AD
4. Let D be a random point on the circumference of circle AB	16. Let J be the intersection of perpendicular P_4 and line AD
5. Draw line AD	17. Construct $P_5 \perp$ to P_3 through point I
6. Construct $P_1 \perp$ to line AD through point D	18. Construct $P_6 \perp$ to P_4 through point J
7. Let point E be one of the intersections of circle AC and line AD	19. Let point K be the intersection of perpendiculars P_5 and P_6
8. Construct $P_2 \perp$ to line AD through point E	20. Draw line AK
9. Let point F be the intersection of the *x*-axis and perpendicular P_1	21. Construct $P_7 \perp$ to line AK through point H
10. Construct $P_3 \perp$ to the *x*-axis through point F	22. Change the color of P_7 and make it thick
11. Let G be the intersection of the *y*-axis and perpendicular P_2	23. Animate point D around circle AB
12. Construct $P_4 \perp$ to the *y*-axis through point G	

Here we have simply used a different construction for the ellipse than we did in section 18.5.1. If you drop a perpendicular from point E to the *x*-axis and another perpendicular to the *y*-axis from point D, the intersection of those two perpendiculars will trace the ellipse. Although the ellipse's tangent is not constructed, perpendiculars P_1 and P_2 are both parallel to that tangent so the effect is the same for determining a point on the Cross Curve.

18.5.4 The Osculating Circle of the Cross Curve

This is a very complex construction, but well worth the effort. As we have learned, to construct the osculating circle we first must locate the center of curvature. So many other interesting things can be created once one has constructed the center of curvature: the radius of curvature, the osculating circle, the evolute, etc. See Table 18-4.

Table 18-4: The Osculating Circle of the Cross Curve

1. Create the *x*-*y* axes with origin A and unit point B = (1, 0)	22. Let point L be the intersection of perpendiculars P_7 and P_8
2. Draw circle AB with center at A and passing through point B	23. Draw line AL
3. Draw circle AC centered at A passing through C and AC < AB	24. Construct $P_9 \perp$ to line AL through point A
4. Let D be a random point on the circumference of circle AB	25. Let M be the intersection of the *y*-axis with P_6
5. Draw line AD	26. Draw circle AL with center at A and passing through L
6. Construct $P_1 \perp$ to line AD through point D	27. Let N be the intersection of circle AL and perpendicular P_9
7. Let point E be the intersection of the *x*-axis and perpendicular P_1	28. Let G' be the image when G is translated by vector E → G
8. Construct $P_2 \perp$ to the *x*-axis through point E	29. Construct $P_{10} \perp$ to the *x*-axis through point G'
9. Let point F be the intersection of perpendicular P_2 and line AD	30. Let I' be the image when I is translated by vector A → I
10. Construct $P_3 \perp$ to line AD through point F	31. Construct $P_{11} \perp$ to the *y*-axis through point I'
11. Let G be the intersection of the *x*-axis with perpendicular P_3	32. Let O be the intersection of perpendiculars P_{10} and P_{11}
12. Let point H be one of the intersections of circle AC and line AD	33. Draw line OA
13. Construct $P_4 \perp$ to line AD through point H	34. Construct $P_{12} \perp$ to line AL through point O
14. Let point I be the intersection of the *y*-axis with perpendicular P_4	35. Let P be the intersection of perpendicular P_{12} and line AL
15. Construct $P_5 \perp$ to the *y*-axis through point I	36. Draw line PN
16. Let point J be the intersection of perpendicular P_5 and line AD	37. Construct $P_{13} \perp$ to line PN through point N
17. Construct $P_6 \perp$ to line AD through point J	38. Let Q be the intersection of perpendicular P_{13} and line AL
18. Let point K be the intersection of perpendiculars P_1 and P_5	39. Let K' be the image when K is translated by vector Q → A
19. Trace point K and change its color	40. Draw circle K'K centered at K' and passing through K
20. Construct perpendicular P_7 to the *y*-axis through point F	41. Make circle K'K thick and change its color
21. Construct perpendicular P_8 to the *x*-axis through point J	42. Animate point D around circle AB

Of course, circle K'K is the osculating circle while point K' is the center of curvature. Tracing point K' will draw the evolute of the Cross Curve.

18.5.5 The Bullet Nose Curve

One might say that the ellipse is to the Cross Curve as the hyperbola is to the Bullet Nose Curve because, if you construct a hyperbola and then construct its tangent, and construct perpendiculars where the tangent intersects the coordinate axes, the locus of the intersection point of those two perpendiculars is the Bullet Nose Curve. Sound familiar? It should. Substitute ellipse for hyperbola and substitute Cross Curve for Bullet Nose Curve and the statement is the one addressed in section 18.5.1. If seeing is believing, try this construction (Table 18-5).

Table 18-5: The Bullet Nose Curve

1. Draw horizontal line segment AB	11. Construct $P_3 \perp$ to line segment AE through point F
2. Let C be a random point on line segment AB	12. Let G be the intersection of line BE and perpendicular P_3
3. Draw circle BC with center at B and passing through point C	13. Trace point G and change its color
4. Let D be the midpoint of line segment AB	14. Let point H be the intersection of P_3 and line segment AB
5. Let E be a random point on the circumference of circle BC	15. Let point I be the intersection of perpendiculars P_1 and P_3
6. Construct $P_1 \perp$ to line segment AB through point D	16. Construct $P_4 \perp$ to line segment AB through point H
7. Draw line BE	17. Construct $P_5 \perp$ to P_1 through point I
8. Draw line segment AE	18. Let point J be the intersection of perpendiculars P_4 and P_5
9. Construct $P_2 \perp$ to line BE through point E	19. Trace point J and change its color
10. Let F be the midpoint of line segment AE	20. Animate point E around circle BC

Steps 1 – 13 execute the construction of the hyperbola and its tangent, P_3. Steps 14 – 17 locate the points where the hyperbola's tangent intersects the axes and then drop perpendiculars to the axes through those points, namely, P_4 and P_5. Finally, point J, the intersection of the two perpendiculars, is supposed to be a point on the Bullet Nose Curve. Run the animation and, indeed, the Bullet Nose Curve is produced by the trace of point J.

18.5.6 The Tangent to the Bullet Nose Curve

As an exercise in perseverance, Table 18-6 shows one way to construct the tangent to the Bullet Nose curve.

Table 18-6: The Tangent to the Bullet Nose Curve

1. Create *x*-*y* axes with origin A and unit point B = (1, 0)	20. Let C' be the image when C is reflected across line AB'
2. Draw circle AB with center at A and passing through point B	21. Construct $P_7 \perp$ to the *x*-axis through point C'
3. Let C be a random point on the circumference of circle AB	22. Let point J be the intersection of perpendiculars P_6 and P_7
4. Draw line AC	23. Draw line AJ
5. Draw circle AD centered at A passing through D and AD < AB	24. Construct L_1 parallel to line AJ through point H
6. Construct $P_1 \perp$ to line AC through point C	25. Let K be the intersection of parallel line L_1 and the *y*-axis
7. Let point E be the intersection of perpendicular P_1 and the *x*-axis	26. Construct $P_8 \perp$ to the *y*-axis through point K
8. Construct $P_2 \perp$ to the *x*-axis through point E	27. Let point L be the intersection of perpendiculars P_5 and P_8
9. Let F be the intersection of circle AD with the positive *x*-axis	28. Construct the locus of L as point C traverses circle AB
10. Construct $P_3 \perp$ to the *x*-axis through point F	29. Construct $P_9 \perp$ to the *y*-axis through point C
11. Let point G be the intersection of line AC and perpendicular P_3	30. Let M be the intersection of P_9 and the *y*-axis
12. Construct $P_4 \perp$ to the *y*-axis through point G	31. Construct $P_{10} \perp$ to line AC through point M
13. Let point H be the intersection of perpendiculars P_2 and P_4	32. Let N be the intersection of perpendicular P_{10} and line AC
14. Construct the locus of point H as point C traverses circle AB	33. Construct $P_{11} \perp$ to the *y*-axis through point N
15. Construct $P_5 \perp$ to the *x*-axis through point C	34. Let point O be the intersection of P_3 and P_{11}
16. Let I be the intersections of the positive *y*-axis with circle AD	35. Draw line AO
17. Construct $P_6 \perp$ to the *y*-axis through point I	36. Construct $P_{12} \perp$ to line AO through point L
18. Let B' be the image when B is rotated about point A by 45°	37. Make P_{12} thick and color it
19. Draw line AB'	38. Animate point C around circle AB

18.5.7 The Osculating Circle of the Bullet Nose Curve

Table 18-7 is another exercise in perseverance, but well worth the effort. We will not only construct the osculating circle and the curve, but also the curve's tangent. Give it a try.

Table 18-7: The Osculating Circle of the Bullet Nose Curve

1. Draw circle AB centered at A and passing through point B	22. Construct $P_9 \perp$ to line AH through point A
2. Let C be a random point on the circumference of circle AB	23. Draw circle AH centered at A and passing through point H
3. Draw line AB	24. Let I be either intersection of circle AH with P_9
4. Construct $P_1 \perp$ to line AB through point C	25. Let J be a random point on perpendicular P_1
5. Let D be a random point on line AB	26. Let A_1 be the image when A is translated by vector J → A
6. Construct $P_2 \perp$ to line AB through point D	27. Let K be a second random point on perpendicular P_1
7. Draw line AC	28. Let A_2 be the image when A is translated by vector K → A
8. Construct $P_3 \perp$ to line AC through point A	29. Draw line A_1A_2
9. Let point E be the intersection of perpendiculars P_2 and P_3	30. Let point L be the intersection of perpendiculars P_3 and P_7
10. Construct $P_4 \perp$ to P_1 through point E	31. Let L' be the image when L is translated by vector G → L
11. Let point F be the intersection of perpendiculars P_1 and P_4	32. Construct $P_{10} \perp$ to P_1 through point L'
12. Construct the locus of point F as point C traverses circle AB	33. Let point M be the intersection of line A_1A_2 and P_{10}
13. Make the locus thick and change its color	34. Construct $P_{11} \perp$ to Line AH through point M
14. Construct $P_5 \perp$ to P_1 through point C	35. Let point N be the intersection of line AH and P_{11}
15. Construct $P_6 \perp$ to P_3 through point E	36. Draw line segment IN
16. Let point G be the intersection of line AB and P_6	37. Construct $P_{12} \perp$ to line segment IN through point I
17. Construct $P_7 \perp$ to line AB through point G	38. Let point O be the intersection of P_{12} and line AH
18. Let point H be the intersection of perpendiculars P_5 and P_7	39. Let F_1 be the image when F is translated by vector O → A
19. Draw line AH	40. Draw circle F_1F centered at F_1 and passing through point F
20. Construct $P_8 \perp$ to line AH through point F	41. Make circle F_1F thick and change its color
21. Make P_8 thick and change its color	42. Animate point C around circle AB

18.5.8 Both Curves in One Construction

The construction of Table 18-8 should seem familiar, but there is a nice surprise when finished.

Table 18-8: Both Curves in One Construction

1. Draw circle AB with center at A and passing through point B	10. Draw line segment AC
2. Let C be a random point anywhere outside of circle AB	11. Let G be the midpoint of line segment AC
3. Draw line AC	12. Construct $P_3 \perp$ to line AC through point G
4. Let D be a random point on the circumference of circle AB	13. Let point H be the intersection of perpendiculars P_1 and P_3
5. Draw line segment CD	14. Construct $P_4 \perp$ to P_2 through point H
6. Let E be the midpoint of line segment CD	15. Let point I be the intersection of perpendiculars P_2 and P_4
7. Construct $P_1 \perp$ to line segment CD through point E	16. Trace point I and change its color
8. Let F be the intersection of line AC and perpendicular P_1	17. Animate point D around circle AB
9. Construct $P_2 \perp$ to line AC through point F	

Now, after having executed the animation above, drag point C so that it is inside of circle AB and rerun the animation. Lo and behold—the Cross Curve!

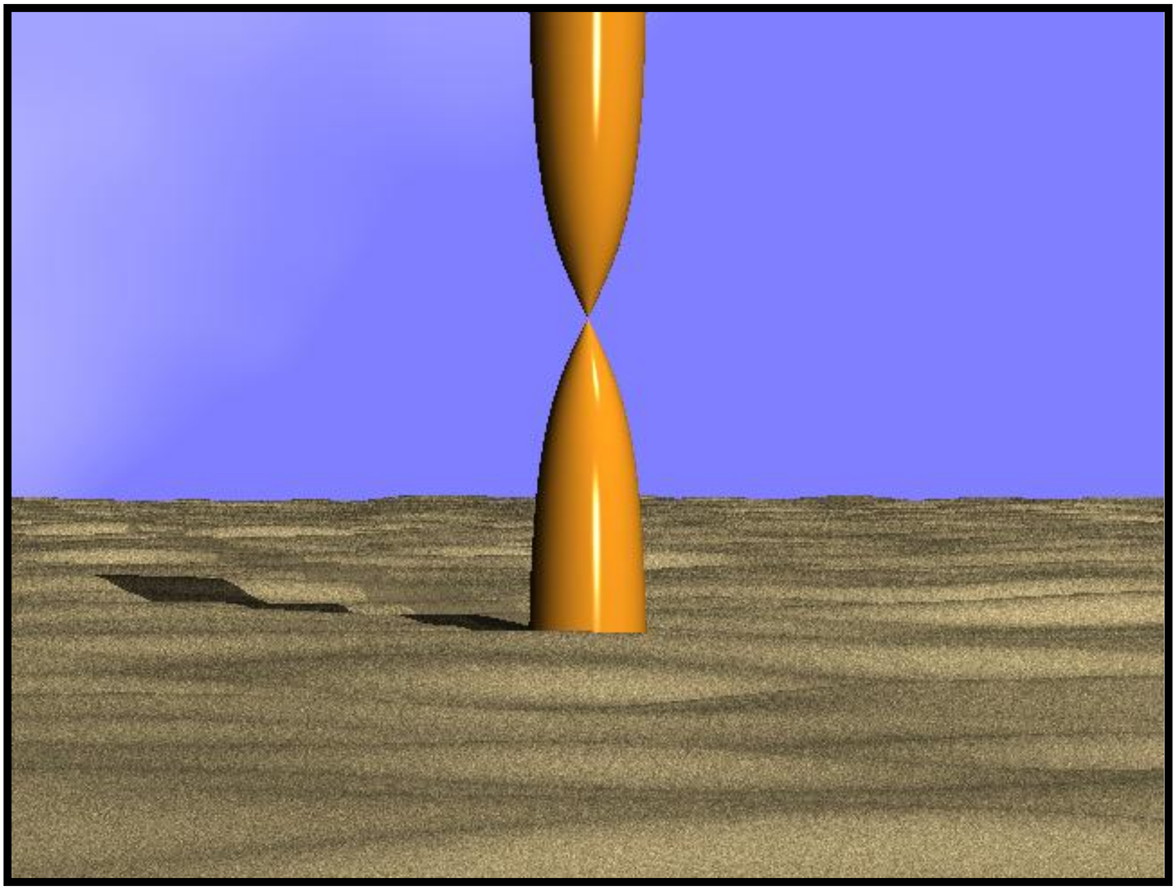


Figure 18-5: The Solid of Revolution from the Bullet Nose Curve

The Bullet Nose Curve was revolved about the y-axis to achieve the object pictured above. The object was then placed so as to appear to be embedded in the desert-like landscape which extends to the horizon. The object has been given a copper colored finish and light sources have been placed so as to cast the object's shadow onto the landscape. (Seeing this rendering, one can understand why the curve was named the Bullet Nose Curve.)

Chapter 19 – The Piriform Curve

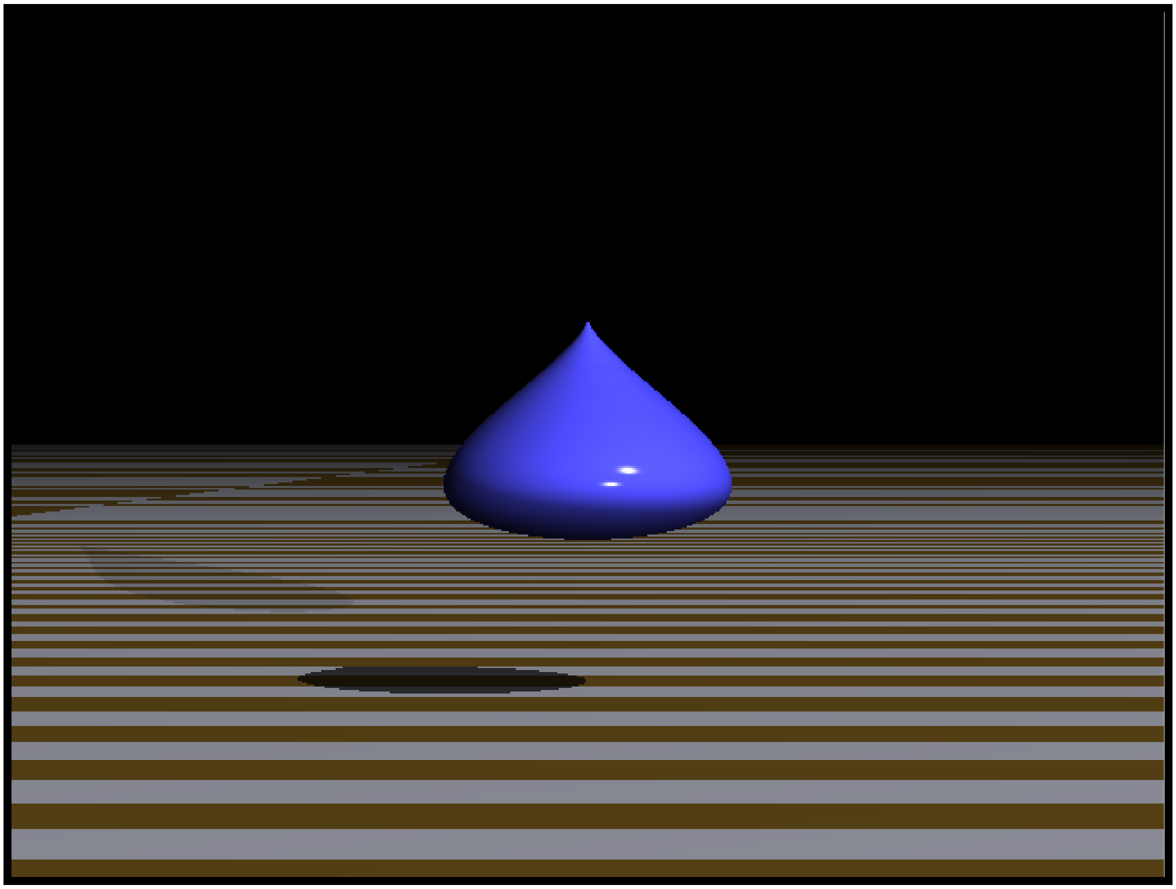


Figure 19-1: The Piriform as a Solid of Revolution

Figure 19-1 is the solid of revolution created when the plane Piriform Curve is rotated about the x-axis. The solid was then given a lustrous bright-blue finish and placed over the horizontally striped plane. Light sources have been placed so as to cast the solid's shadows onto the plane, one in the foreground and one in the background.

19.1 Introduction

The Piriform is a quartic curve, often called the Pear-Shaped Quartic. Actually, the curve looks to be shaped more like a Hershey Kiss than a pear; however, the Hershey Kiss shaped quartic sounds rather dull; Piriform is preferable. In fact, *pirum* is Latin for pear and it is doubtful whether there is a Latin word for Hershey Kiss.

The first to study the curve was the French mathematician G. de Longchamps in 1886, so this curve is of a much later vintage than most of the curves that we have encountered so far. The curve is defined as follows (see Figure 19-2): Given point A on

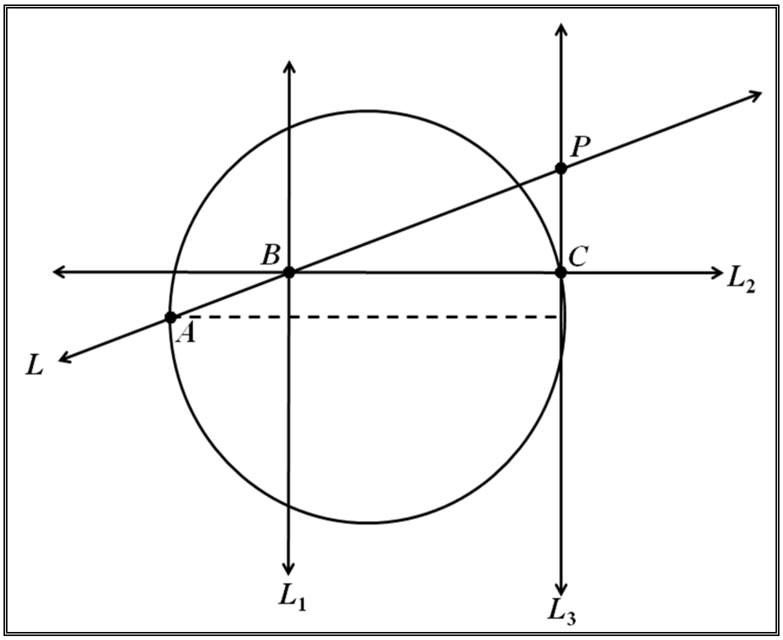


Figure 19-2: Definition of the Piriform Curve

the circumference of a circle and a line L_1 that is perpendicular to the diameter through point A, draw an arbitrary line L through point A that crosses L_1 in point B. Draw a line L_2 perpendicular to L_1 through point B that intersects the circle in point C. Draw a line L_3 perpendicular to L_2 through point C that intersects line L in point P. The locus of P for all possible lines L is the Piriform.

19.2 Equations and Graph of the Piriform Curve

Again, referring to Figure 19-2, if we assume that point A is the origin, the diameter through A is the x-axis, and the radius of the circle is a, then the equation of the circle is

$$(x-a)^2 + y^2 = a^2.$$

Further, if we assume that line L has equation $y = mx$, where m is the slope (the y-intercept being the origin—zero), and that line L_1 has equation $x = a^2/b$, then point B, which is on both lines L and L_1, has coordinates $(a^2/b,\ a^2m/b)$. Similarly, point C which is both on line L_2 and the circle, has coordinates

$$\left(a+\frac{a}{b}\sqrt{b^2-a^2m^2},\frac{a^2m}{b}\right).$$

And finally, point P which is both on L and L_3 has coordinates

$$\left(a+\frac{a}{b}\sqrt{b^2-a^2m^2},am+\frac{am}{b}\sqrt{b^2-a^2m^2}\right).$$

If we now define the parameter t by the relationship $am = b\cos t$ for $-\pi/2 \le t \le \pi/2$, we have as the coordinates of point P the following relationships: $x = a\,(1 + \sin t)$ and $y = b\cos t\,(1 + \sin t)$, giving us a parametric representation of the Piriform, that is,

$$(x,y)=(1+\sin t)(a,b\cos t) \qquad -\pi/2\le t\le 3\pi/2 \qquad \textbf{Equation 19-1}$$

Of course, to arrive at a Cartesian equation, we simply eliminate t from Equation 19-1. To do this, we write $\sin t = x/a - 1$ and $\cos t = ay/bx$, then square both quantities and add to give

$$a^4y^2=b^2x^3(2a-x) \qquad \textbf{Equation 19-2}$$

and the polar equation becomes

$$b^2r^2\cos^4\theta=2ab^2r\cos^3\theta-a^4\sin^2\theta \qquad \textbf{Equation 19-3}$$

Figure 19-3 depicts the graph of the Piriform.

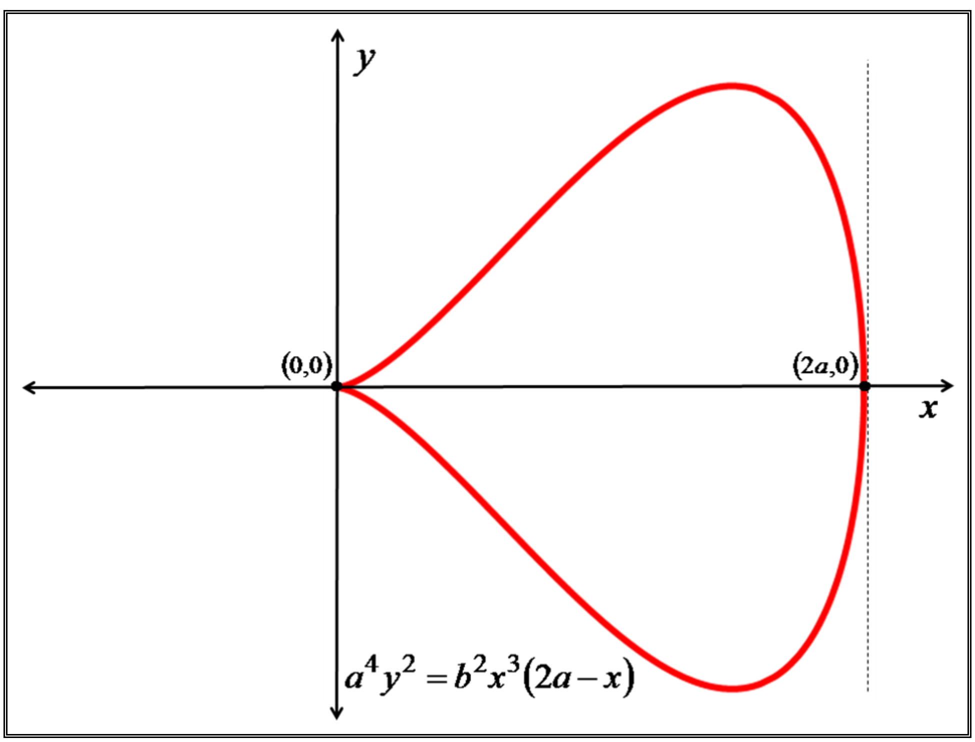


Figure 19-3: Piriform Graph

And last but not least, the equation of the Piriform's tangent at the point $t = q$ is

$$y = \frac{b(1+\sin q)(1-2\sin q)}{a\cos q}\cdot x + \frac{b\sin q(1+\sin q)}{\cos q} \qquad \textbf{Equation 19-4}$$

19.3 Analytical and Physical Properties of the Piriform Curve

Based on the Piriform's parametric representation found in Equation 19-1, i.e., $x = a\,(1 + \sin t)$ and $y = b\cos t\,(1 + \sin t)$, the following is an analysis of the Piriform curve.

19.3.1 Derivatives of the Piriform Curve

- $\dot{x} = a\cos t$.
- $\ddot{x} = -a\sin t$.
- $\dot{y} = b(1+\sin t)(1-2\sin t)$.
- $\ddot{y} = -b\cos t(1+4\sin t)$.
- $y' = \dfrac{b(1+\sin t)(1-2\sin t)}{a\cos t}$.
- $y'' = \dfrac{b\left(2\sin^3 t - 3\sin t - 1\right)}{a^2\cos^3 t}$.

19.3.2 Metric Properties of the Piriform Curve

Since the Piriform is symmetric about the x-axis, the total area enclosed by the curve can be calculated by obtaining the area of the portion of the curve above the x-axis and then doubling the result. The portion of the curve above the x-axis is generated by values of t such that $-\pi/2 \le t \le \pi/2$. Hence,

$$A = 2\int_{-\pi/2}^{\pi/2} y(t)\dot{x}(t)dt = 2ab\int_{-\pi/2}^{\pi/2}(1+\sin t)\cos^2 t\,dt = \pi ab.$$

Similarly, the volume of the solid of revolution obtained when the Piriform is rotated about the x-axis can be calculated by

$$V = \pi\int_{-\pi/2}^{\pi/2}[y(t)]^2\,\dot{x}(t)dt = \frac{8\pi ab^2}{5}.$$

If p denotes the distance from the origin to the Piriform's tangent, then

$$p = \frac{ab\sin t(1+\sin t)^2}{\sqrt{a^2\cos^2 t + b^2(1+\sin t)^2(1-2\sin t)^2}}.$$

If r denotes the radial distance, then

$$r=(1+\sin t)\sqrt{a^2+b^2\cos^2 t}\,.$$

19.3.3 Curvature of the Piriform Curve

If ρ stands for the radius of curvature of the Piriform, then

$$\frac{\left[a^2\cos^2 t+b^2(1+\sin t)^2(1-2\sin t)^2\right]^{3/2}}{ab(1+\sin t)(2\sin^2 t-2\sin t-1)}\,.$$

If (α, β) denotes the coordinates of the center of curvature of the Piriform, then

$$\alpha=\frac{(1+\sin t)\left[a^2(\sin t-2)-b^2(1+\sin t)(1-2\sin t)^3\right]}{a(2\sin^2 t-2\sin t-1)} \quad \text{and}$$

$$\beta=\frac{a^2\cos^3 t+6b^2\sin t\cos t(1+\sin t)^2(\sin t-1)}{b(1+\sin t)(2\sin^2 t-2\sin t-1)}\,.$$

19.3.4 Angles for the Piriform Curve

If θ represents the radial angle, then

$$\tan\theta=\frac{b}{a}\cos t\,.$$

If ψ denotes the tangential-radial angle, then

$$\tan\psi=-\frac{ab\sin t(1+\sin t)}{\cos t\left[a^2+b^2(1+\sin t)(1-2\sin t)\right]}\,.$$

If ϕ denotes the tangential angle, then

$$\tan\phi=\frac{b(1+\sin t)(1-2\sin t)}{a\cos t}\,.$$

19.4 Geometric Properties of the Piriform Curve

- Intercepts: $(-\pi/2, 0, 0)$; $(\pi/2, 2a, 0)$; $(3\pi/2, 0, 0)$.
- Extent: $-\pi/2 \le t \le 3\pi/2$; $0 \le x \le 2a$; $-\frac{3b\sqrt{3}}{4} \le y \le \frac{3b\sqrt{3}}{4}$.
- Symmetry: $y = 0$.
- Cusp: $(0, 0)$.

19.5 Dynamic Geometry of the Piriform Curve

Three dynamic geometry constructions follow; one to construct the curve, one to construct the curve's tangent, and the other to construct the curve's osculating circle.

19.5.1 The Piriform Curve Based on the Definition

In the introduction to this chapter, we defined the Piriform curve (see section 19.1 and Figure 19-2). The following construction of Table 19-1 follows directly from that definition.

Table 19-1: The Piriform Curve Directly from the Definition

1. Draw circle AB with center at A and passing through point B	8. Let G be the intersection of perpendicular P_2 and line CD
2. Let C be a random point external to circle AB	9. Draw line EG
3. Draw horizontal line CD	10. Construct $P_3 \perp$ to P_1 through point F
4. Construct $P_1 \perp$ to line CD through point A	11. Let H be the intersection of perpendicular P_3 and line EG
5. Let E be one intersection of perpendicular P_1 and circle AB	12. Trace point H and change its color
6. Let F be a random point on the circumference of circle AB	13. Animate point F around circle AB
7. Construct $P_2 \perp$ to line CD through point F	

Drag line CD up and/or down keeping it horizontal. When it intersects point E, the curve degenerates to a straight line.

19.5.2 The Tangent to the Piriform Curve

Table 19-2 contains a construction for the Piriform Curve and its tangent.

Table 19-2: The Tangent to the Piriform Curve

1. Create *x*-*y* axes with origin A and unit point B = (1, 0)	19. Let J be the midpoint of line segment HI
2. Draw circle AB with center at A, and passing through point B	20. Let J' be the image when point J is translated by vector A → J
3. Let C be a random point anywhere on the *x*-axis	21. Draw line segment AC
4. Construct $P_1 \perp$ to the *x*-axis through point C	22. Let point K be the midpoint of line segment AC
5. Let D be a random point on the circumference of circle AB	23. Let K' be the image when K is translated by vector F → K
6. Construct $P_2 \perp$ to P_1 through point D	24. Let L be the intersection of the positive *y*-axis and circle AB
7. Let point E be the intersection of perpendiculars P_1 and P_2	25. Draw line segment LJ'
8. Construct $P_3 \perp$ to the *x*-axis through point D	26. Draw line segment LK'
9. Let point F be diametrically opposite to point B	27. Construct line L_1 parallel to line segment LK' through point F
10. Draw line EF	28. Let point M be the intersection of parallel L_1 and the *y*-axis
11. Let G be the intersection of line EF and perpendicular P_3	29. Construct line L_2 parallel to line segment LJ' through point M
12. Trace point G and change its color	30. Let point N be the intersection of parallel L_2 and the *x*-axis
13. Draw line AD	31. Let O be the intersection of perpendicular P_2 and the *y*-axis
14. Let B' be the image when point B is reflected across line AD	32. Draw line NO
15. Construct $P_4 \perp$ to the *x*-axis through point B'	33. Construct $P_5 \perp$ to line NO through point G
16. Let H be the intersection of perpendicular P_3 and the *x*-axis	34. Make perpendicular P_5 thick and change its color
17. Let I be the intersection of perpendicular P_4 and the *x*-axis	35. Animate point D around circle AB
18. Draw line segment HI	

Drag point C to different positions of the plane to change the size of the Piriform.

19.5.3 The Osculating Circle to the Piriform Curve

Table 19-3 contains a construction for the osculating circle of the Piriform as well as the curve itself and the curve's tangent. The construction here for the curve is different from the previous two sections (as is the construction for the tangent different from the previous section). However, this construction is very complex (maybe the most complex in the book), albeit interesting, and the resulting animation makes it completely worthwhile. Of course, once we have constructed the Piriform's osculating circle, it implies that we have constructed the center of curvature (i.e., the center of the osculating circle) and, as we have learned, tracing the center of curvature traces the evolute. That is worth doing here because the evolute of the Piriform is very, very weird. Have a go at it!

Table 19-3: The Piriform's Osculating Circle

1. Create *x*-*y* axes with origin A and unit point B = (1, 0)	32. Let O' be the image when O is translated by vector N → O
2. Draw circle AB centered at A and passing through point B	33. Construct P_8 ⊥ to the *y*-axis through point O'
3. Let C be a random point on the circumference of circle AB	34. Let point P be the intersection of perpendiculars P_5 and P_8
4. Draw line AC	35. Draw line segment AP
5. Construct P_1 ⊥ to line AC through point A	36. Let line L_1 be parallel to line segment AP through point I
6. Let D be either intersection of circle AB with P_1	37. Make L_1 thick and change its color
7. Construct P_2 ⊥ to the *x*-axis through point D	38. Let B' be the image when point B is reflected across line AC
8. Let E be the point diametrically opposed to point B	39. Construct P_9 ⊥ to the *x*-axis through point B'
9. Let point F be the intersection of the *x*-axis and P_2	40. Let point Q be the intersection of the *x*-axis with P_9
10. Draw line segment EF	41. Let B" be the image when B' is translated by vector Q → B'
11. Let G be the midpoint of line segment EF	42. Construct P_{10} ⊥ to the *y*-axis through point B"
12. Let G' be the image when G is translated by vector A → G	43. Let point R be the intersection of P_{10} and the *y*-axis
13. Let A' be the image when A is translated by vector G' → A	44. Construct P_{11} ⊥ to line segment AP through point A
14. Draw circle AA' centered at A and passing through A'	45. Construct P_{12} ⊥ to the *y*-axis through point D
15. Let point H be either intersection of circle AA' with P_1	46. Let point S be the intersection of P_{12} and the *y*-axis
16. Construct P_3 ⊥ to the *y*-axis through point H	47. Draw line segment RS
17. Construct P_4 ⊥ to the *x*-axis through point A'	48. Let T be the midpoint of line segment RS
18, Let point I be the intersection of perpendiculars P_3 and P_4	49. Let T' be the image when T is translated by vector A → T
19. Construct the locus of point I as point C traverses circle AB	50. Let A" be the image when A is translated by vector T' → A
20. Make the locus thick and change its color	51. Construct P_{13} ⊥ to the *y*-axis through point A"
21. Construct P_5 ⊥ to the *x*-axis through point C	52. Let point U be the intersection of perpendiculars P_2 and P_{13}
22. Let point J be the intersection of P_5 and the *x*-axis	53. Construct P_{14} ⊥ to P_{11} through point U
23. Construct P_6 ⊥ to line AC through point J	54. Let point V be the intersection of perpendiculars P_{11} and P_{14}
24. Let point K be the intersection of line AC and P_6	55. Draw line segment PV
25. Draw circle AK centered at A and passing through point K	56. Construct P_{15} ⊥ to line segment PV through point P
26. Let point L be either intersection of the *y*-axis and circle AK	57. Let point W be the intersection of perpendiculars P_{11} and P_{15}
27. Let point M be either intersection of circle AA' with line AC	58. Let I' be the image when I is translated by vector W → A
28. Construct P_7 ⊥ to the *y*-axis through point M	59. Draw circle I'I centered at I' and passing through point I
29. Let point N be the intersection of the *y*-axis and P_7	60. Make circle I'I thick and change its color
30. Draw line segment AL	61. Animate point C around circle AB
31. Let O be the midpoint of line segment AL	

Of course, point I' is the center of curvature. As suggested earlier, trace it and rerun the animation to see the Piriform’s evolute.

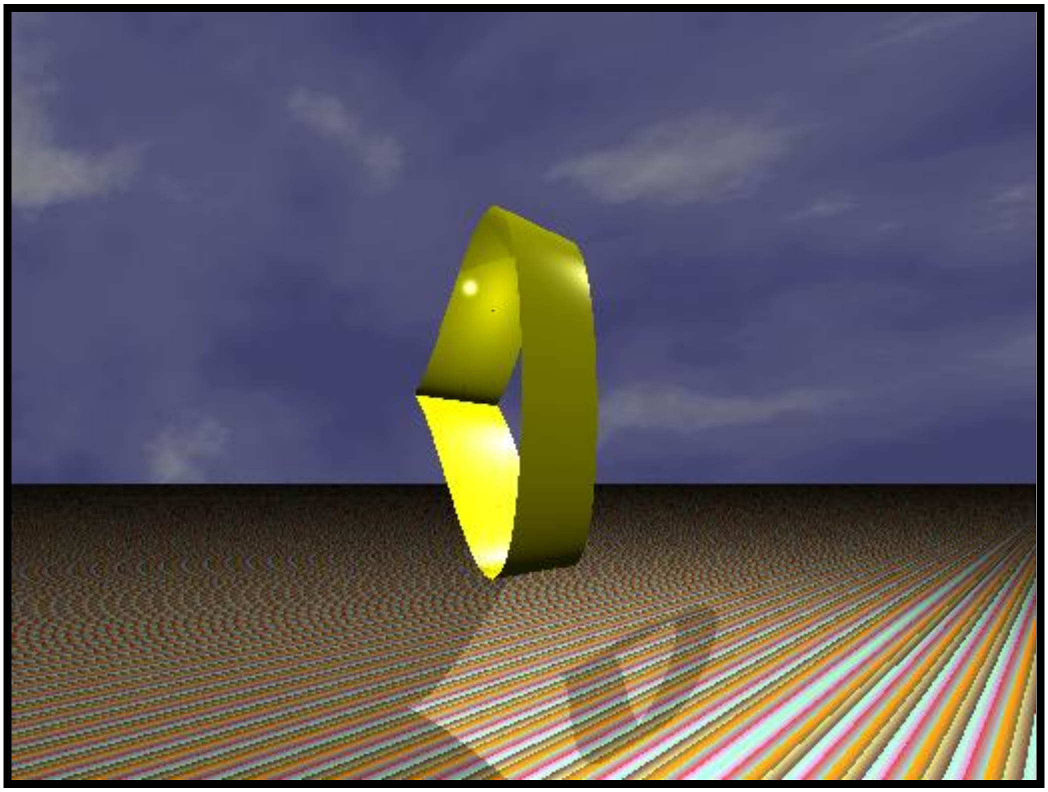


Figure 19-4: The Piriform in Three Dimensions

The Piriform has been extruded into the third dimension to render the object seen above. It was then placed above the vertically striped, multi-colored plane with light sources positioned so as to cast the shadows seen onto the plane. Note how the cusp is rendered twice in shadows.

Chapter 20 – The Kappa Curve

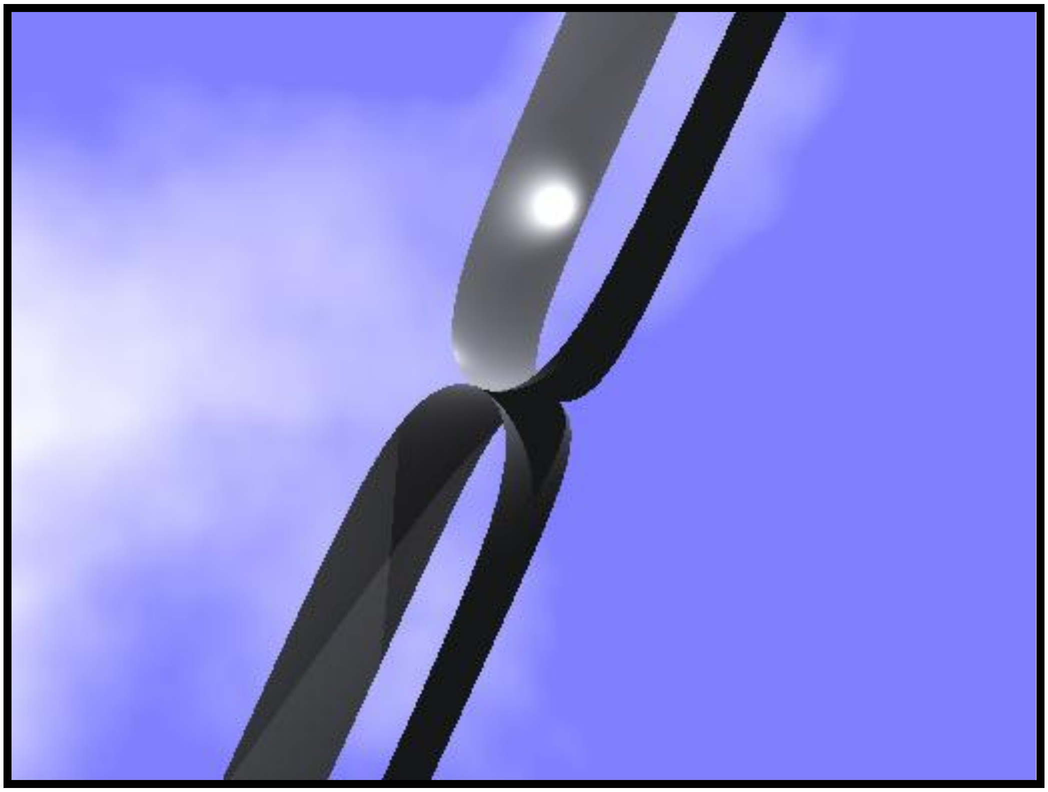


Figure 20-1: The Kappa Curve in Three Dimensions

The Kappa Curve was extruded into the third dimension to create the object seen above. It was then given a silvery-metallic finish and light sources were located so as to reflect off of the finish and cast the shadows seen on the lower portion of the object. A bright, summer sky was then used as the background.

20.1 Introduction

In this chapter we take up the curve called the Kappa Curve, so named because the curve somewhat resembles the Greek character kappa (κ). Another name for the curve is Gutschoven's Curve, named after G. van Gutschoven, the first person who studied the curve in 1662. Isaac Newton also studied the curve, as did Johann Bernoulli and de Sluze.

The Kappa Curve can be defined as follows (refer to Figure 20-2): Let point A be the origin and let the line L_1 be perpendicular to the y-axis, intersecting the y-axis in the point B. Let line L be an arbitrary line passing through the origin and intersecting line L_1 in the point C. If P is a point on line L such that AP = BC, then P is a point of the Kappa Curve.

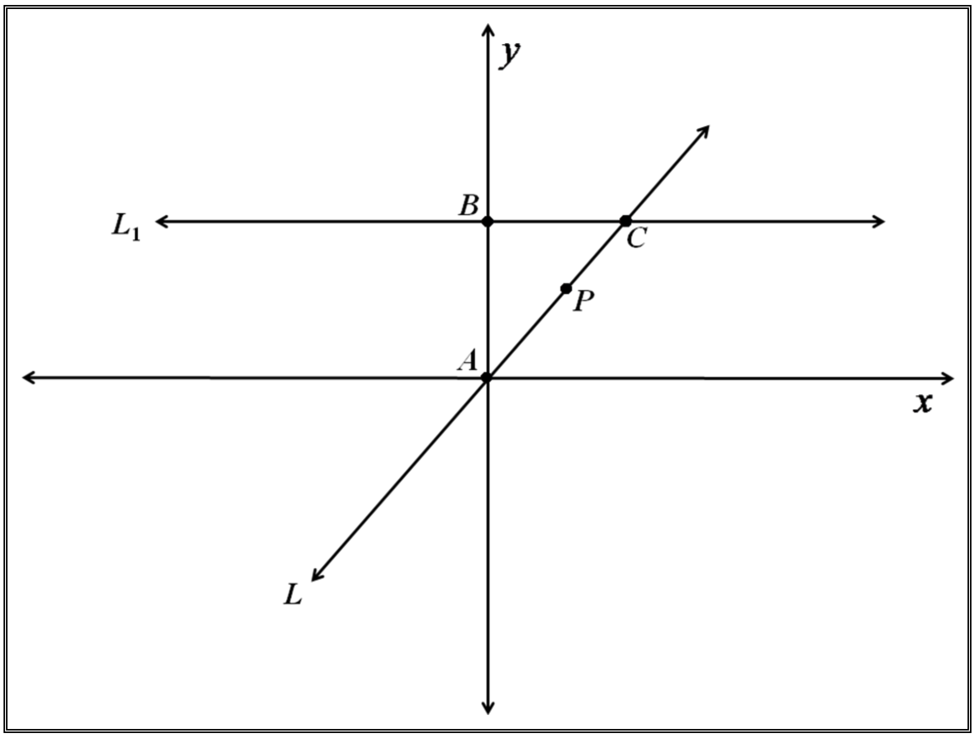


Figure 20-2: The Kappa Curve Definition

In other words, the locus of the point P for all possible lines L where AP = BC is the Kappa Curve.

20.2 Equations and Graph of the Kappa Curve

Assuming that the equation of line L_1 is $y = a$ and that of line L is $y = mx$, where m is the slope of line L, then the coordinates of point C are $(a/m, a)$ and distance BC is simply a/m. However, AP = $(x^2 + y^2)^{½}$ which equals, by definition, BC. Therefore, we have

$$\sqrt{x^2 + y^2} = \frac{a}{m}.$$

Squaring and rearranging gives the Cartesian equation for the Kappa Curve as

$$(x^2 + y^2)y^2 = a^2x^2 \quad \textbf{Equation 20-1}$$

Making the usual substitutions of $x = r\cos\theta$ and $y = r\sin\theta$, we easily get the polar equation of the Kappa Curve as

$$r = a\cot\theta \quad \textbf{Equation 20-2}$$

If in Equation 20-1 we make the substitution $y = x \tan t$, we find that $x = a \cos t \cot t$ and $y = a \cos t$; therefore, a parametric representation for the Kappa Curve is

$$(x, y) = a\cos t(\cot t, 1) \quad 0 < \mathrm{t} < 2\pi \quad \textbf{Equation 20-3}$$

Finally, the Kappa Curve's tangent at the point $t = q$ is

$$\cos q(1 + \sin^2 q)\cdot y = \sin^3 q \cdot x + a\cos^2 q \quad \textbf{Equation 20-4}$$

Figure 20-3 shows a graph of the Kappa Curve.

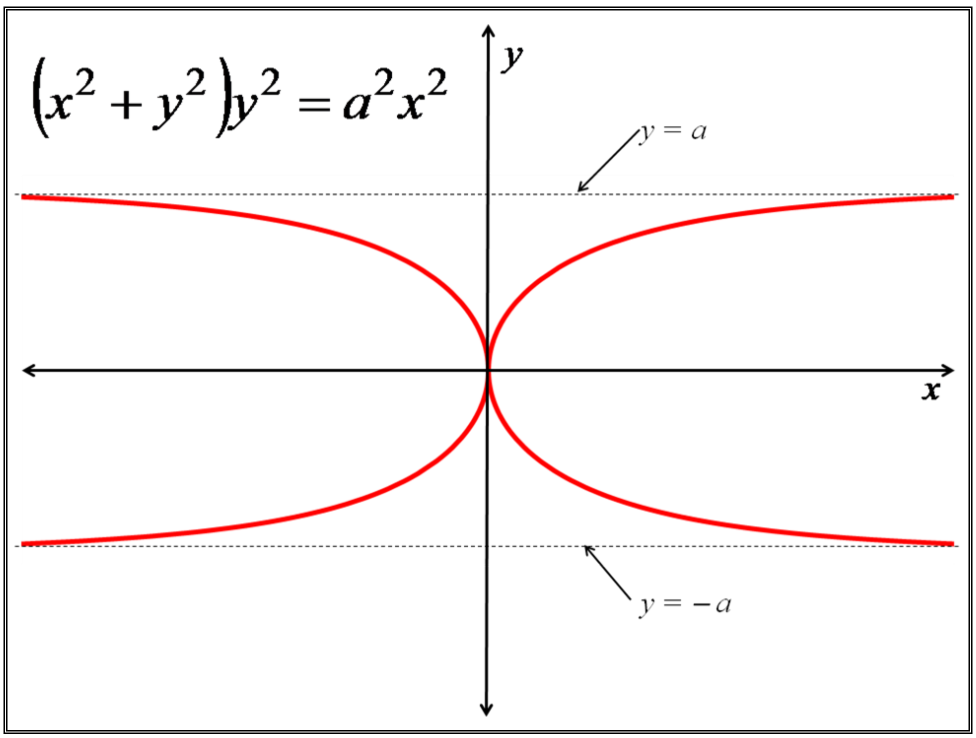


Figure 20-3: Graph of the Kappa Curve

20.3 Analytical and Physical Properties of the Kappa Curve

Based on the Kappa Curve's parametric representation found in Equation 20-3, i.e., $x = a{\cdot}\cos t{\cdot}\cot t$ and $y = a{\cdot}\cos t$, the following is an analysis of the Kappa Curve.

20.3.1 Derivatives of the Kappa Curve

- $\dot{x} = -a\cos t(1 + \csc^2 t)$.

- $\ddot{x} = a\left(2\csc^3 t - \csc t + \sin t\right)$.
- $\dot{y} = -a\sin t$.
- $\ddot{y} = -a\cos t$.
- $y' = \dfrac{\sin^3 t}{\cos t\left(1+\sin^2 t\right)}$.
- $y'' = \dfrac{\sin^4 t\left(\sin^2 t - 3\right)}{a\cos^3 t\left(1+\sin^2 t\right)^3}$.

20.3.2 Metric Properties of the Kappa Curve

If r denotes the distance from the origin to the Kappa Curve, then

$$r = a\cot t.$$

If p denotes the distance from the origin to the tangent to the Kappa Curve, then

$$p = -\frac{a\cos^2 t}{\sqrt{1+\sin^2 t - \sin^4 t}}.$$

20.3.3 Curvature of the Kappa Curve

If ρ represents the radius of curvature of the Kappa Curve, then

$$\rho = \frac{a\left(1+\sin^2 t - \sin^4 t\right)^{3/2}}{\sin^4 t\left(3-\sin^2 t\right)}.$$

If (α, β) denotes the coordinates of the center of curvature for the Kappa Curve, then

$$\alpha = \frac{a\left(3\sin^2 t - 4\right)}{\sin t\left(\sin^2 t - 3\right)} \quad \text{and} \quad \beta = \frac{a\cos t\left(1-\sin^2 t\right)\left(1+3\sin^2 t\right)}{\sin^4 t\left(\sin^2 t - 3\right)}.$$

20.3.4 Angles for the Kappa Curve

If θ denotes the radial angle, then

$$\theta = t.$$

If ψ denotes the tangential-radial angle, then

$$\tan\psi = -\sin t\cos t.$$

If ϕ denotes the tangential angle, then

$$\tan\phi = \frac{\sin^3 t}{\cos t\left(1+\sin^2 t\right)}.$$

20.4 Geometric Properties of the Kappa Curve

- Intercepts: (0, 0).
- Extent: $0 < t < 2\pi$; $-\infty < x < +\infty$; $-a < y < a$.
- Symmetry: $x = 0$; $y = 0$; (0, 0).
- Asymptotes: y = a; y = $-a$.
- Cusp: (0, 0).

20.5 Dynamic Geometry of the Kappa Curve

The next four subsections delineate constructions involving the Kappa Curve.

20.5.1 The Kappa Curve by Definition

The following construction of Table 20-1 is based directly on the definition of the Kappa Curve that was given in section 20.1.

Table 20-1: The Kappa Curve by Definition

1. Create *x-y* axes with origin A and unit point B = (1, 0)	7. Let point F be the intersection of line AE and parallel L_1
2. Let C be a random point on the *y*-axis	8. Draw line segment CF
3. Construct line L_1 parallel to the *x*-axis through point C	9. Construct circle C_2 centered at A and with radius equal to CF
4. Draw circle AD centered at A, and passing through point D	10. Let point G be either intersection of circle C_2 and line AE
5. Let E be a random point on the circumference of circle AD	11. Trace point G and change its color
6. Draw line AE	12. Animate point E around circle AD

20.5.2 An Alternate Construction of the Kappa Curve

At first glance, this construction (Table 20-2) may appear to be merely a minor variation of the previous construction—not so!

Table 20-2: An Alternate Construction of the Kappa Curve

1. Create *x-y* axes with origin at A and unit point B = (1, 0)	6. Let point D be either intersection of circle AB and P_1
2. Draw circle AB with center at A and passing through point B	7. Construct line L_1 parallel to the *x*-axis through point D
3. Let C be a random point on the circumference of circle AB	8. Let point E be the intersection of parallel L_1 and line AC
4. Draw line AC	9. Trace point E and change its color
5. Construct $P_1 \perp$ to line AC through point A	10. Animate point C around circle AB

Notice the reaction of the tracing point when the animation is executed for this construction. If the tracing point starts in the first quadrant, when it gets to the origin, it then traces the portion of the curve in the second quadrant, then the third quadrant, and finally, the fourth quadrant. In the previous construction, if the tracing point starts in the first quadrant, when it gets to the origin, it continues tracing in the fourth quadrant, then the third quadrant, and finally the second quadrant. A big difference between the two constructions!

20.5.3 The Tangent to the Kappa Curve

Table 20-3 contains a construction for the Kappa Curve and tangent.

Table 20-3: The Tangent to the Kappa Curve

1. Create *x-y* axes with origin at A and unit point B = (1, 0)	9. Construct the locus of point E as point C traverses circle AB
2. Draw circle AB with center at A and passing through point B	10. Construct $P_2 \perp$ to line AC through point E
3. Let C be a random point on the circumference of circle AB	11. Construct $P_3 \perp$ to the *x*-axis through point D
4. Draw line AC	12. Let point F be the intersection of perpendiculars P_2 and P_3
5. Construct $P_1 \perp$ to line AC through point A	13. Draw line AF
6. Let point D be either intersection of circle AB and P_1	14. Construct $P_4 \perp$ to line AF through point E
7. Construct line L_1 parallel to the *x*-axis through point D	15. Make perpendicular P_4 thick and change its color
8. Let point E be the intersection of parallel L_1 and line AC	16. Animate point C around circle AB

Of course, perpendicular P_4, the last perpendicular constructed, is the tangent line.

20.5.4 The Kappa Curve's Osculating Circle

Table 20-4 contains a construction for the osculating circle of the Kappa Curve.

Table 20-4: The Kappa Curve and Osculating Circle

1. Create *x-y* axes with origin at A and unit point B = (1, 0)	18. Let point H be the intersection of parallel L_2 and line AC
2. Draw circle AB with center at A and passing through point B	19. Let H' be the image when H is translated by vector E → H
3. Let C be a random point on the circumference of circle AB	20. Let E_1 be the image when E is translated by vector H' → E
4. Draw line AC	21. Let E_2 be the image when E is translated by vector F → E
5. Construct $P_1 \perp$ to line AC through point A	22. Let E_3 be the image when E_1 is translated by vector E_2 → F
6. Let point D be either intersection of circle AB and P_1	23. Draw line segment AE_3
7. Construct line L_1 parallel to the *x*-axis through point D	24. Construct line L_3 parallel to segment AE_3 through point G
8. Let point E be the intersection of parallel L_1 and line AC	25. Draw line AF
9. Construct the locus of point E as point C traverses circle AB	26. Construct $P_5 \perp$ to line AF through point E_3
10. Construct $P_2 \perp$ to line AC through point E	27. Let point I be the intersection of perpendicular P_5 and line AF
11. Construct $P_3 \perp$ to the *x*-axis through point D	28. Draw line segment GI
12. Let point F be the intersection of perpendiculars P_2 and P_3	29. Construct $P_6 \perp$ to line segment GI through point G
13. Draw line segment AF	30. Let J be the intersection of perpendicular P_6 and line AF
14. Construct $P_4 \perp$ to line segment AF through point A	31. Let E_4 be the image when E is translated by vector A → J
15. Draw circle AF with center at A and passing through point F	32. Draw circle E_4E with center at E_4 and passing through point E
16. Let point G be either intersection of P_4 with circle AF	33. Animate point C around circle AB
17. Construct line L_2 parallel to the x-axis through point F	

Circle E_4E is, of course, the osculating circle to the Kappa Curve. Trace point E_4 to see the evolute of the Kappa Curve, which looks very much like the Cissoid of Diocles and its mirror image—the two cusps pointing at one another.

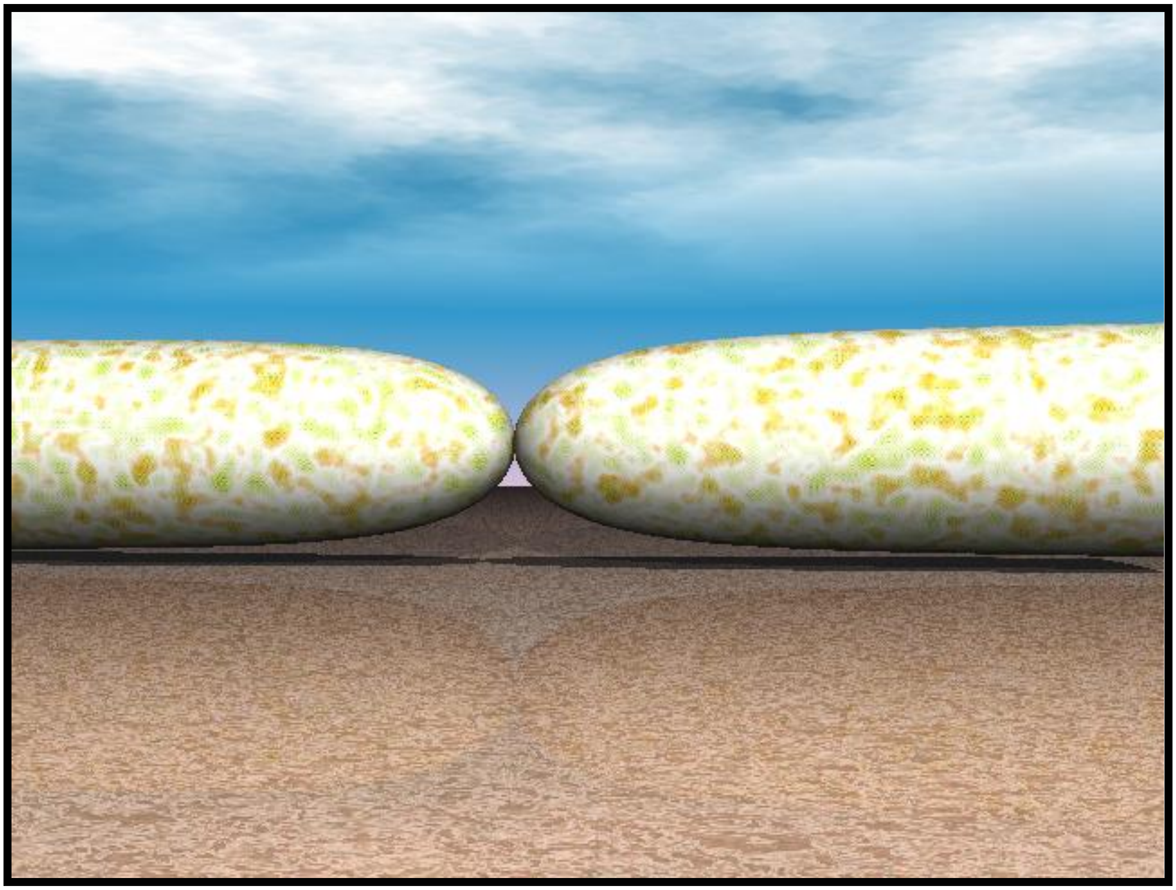


Figure 20-4: A Solid of Revolution of the Kappa Curve

Looking like two large, elongated eggs, the Kappa Curve was rotated about the x-axis to form the solid of revolution seen above. The object was then given a mottled yellow and white finish and placed just slightly above the brown plane which has just a hint of reflectivity. Light sources were located so as to cast the object's shadow onto the plane directly below the object.

Chapter 21 – Cayley's Sextic

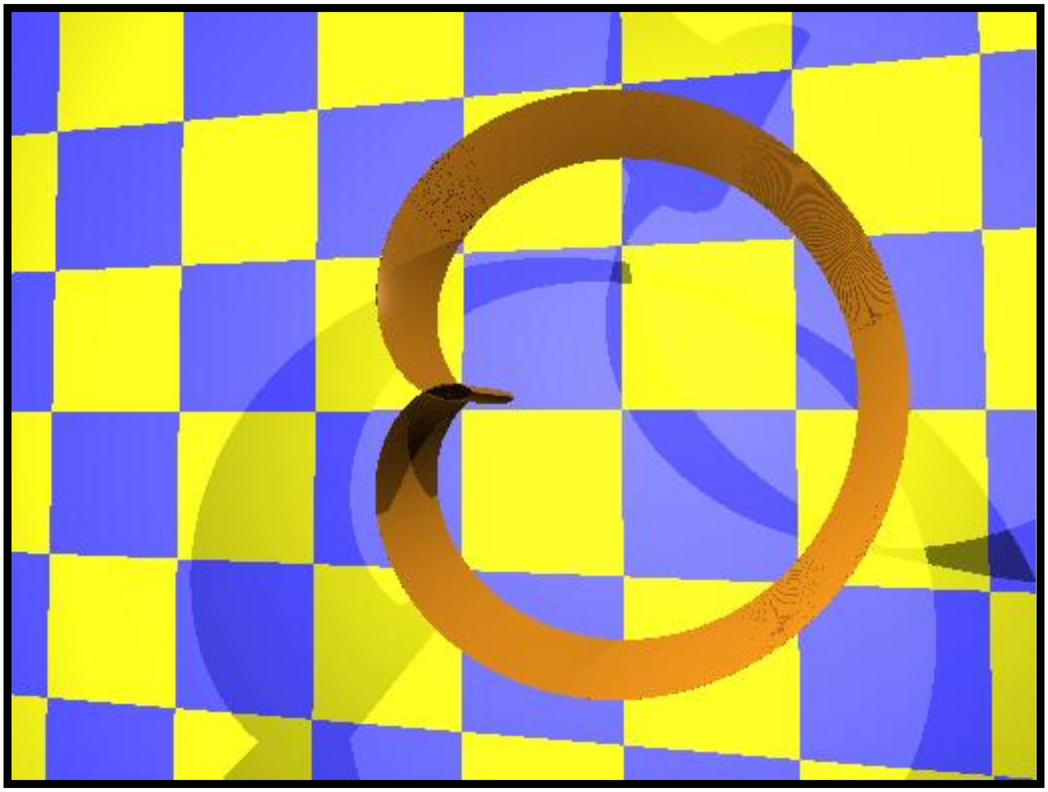


Figure 21-1: Cayley's Sextic in Three Dimensions

Cayley's Sextic was extruded into the third dimension, given a copper finish, and placed over the blue and yellow plane to create the object seen in the figure above. Light sources have been situated so as to cast shadows on the plane.

21.1 Introduction

This chapter is devoted to Cayley's Sextic. A Sextic curve is an algebraic curve of the sixth degree. Arthur Cayley was an English mathematician born in 1821 who lived to be 74 years old, dying in 1895. The curve named after him was first discovered by Colin Maclaurin, but it was Cayley who studied it in detail.

21.2 Equations and Graph of Cayley's Sextic

The Cartesian Equation for Cayley's Sextic is given by

$$4\left(x^2+y^2-ax\right)^3=27a^2\left(x^2+y^2\right)^2 \quad \textbf{Equation 21-1}$$

For the polar equation, we have

$$r=4a\cos^3\tfrac{\theta}{3} \quad \textbf{Equation 21-2}$$

And, a parametric representation is given by

$$\left(x,y\right)=4a\cos^3\tfrac{t}{3}\left(\cos t,\sin t\right) \quad 0<t<3\pi \quad \textbf{Equation 21-3}$$

The equation of the tangent at the point $t = q$ is

$$y+\cot\tfrac{4q}{3}\cdot x=4a\cos^3\tfrac{q}{3}\left(\sin q+\cos q\cot\tfrac{4q}{3}\right) \quad \textbf{Equation 21-4}$$

Figure 21-2 portrays a graph of Cayley's Sextic.

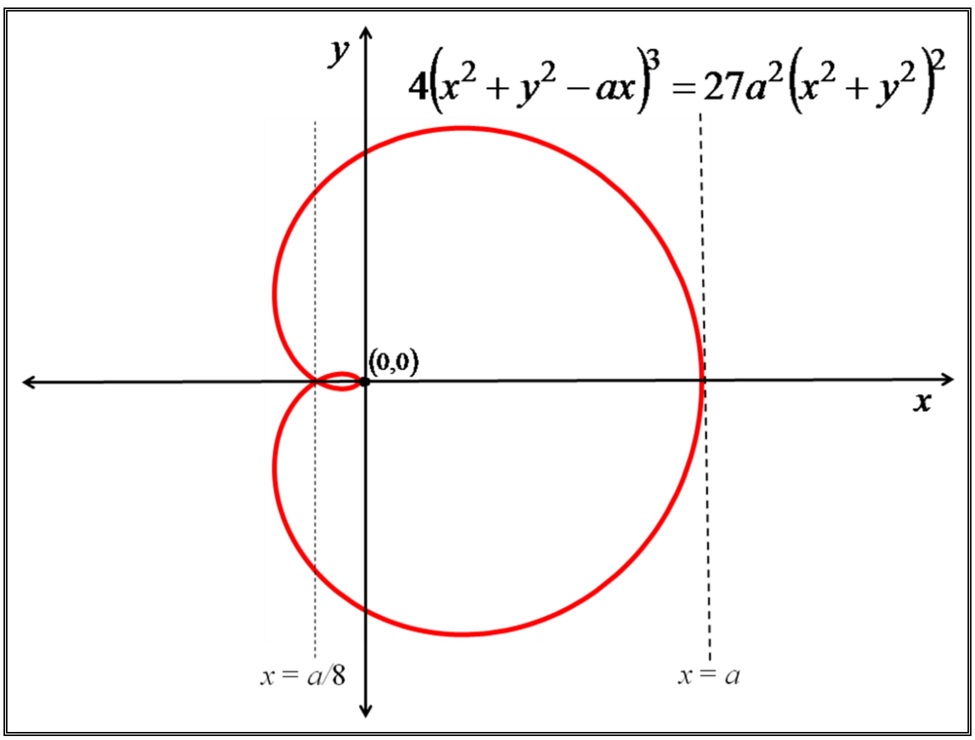


Figure 21-2: Graph of Cayley's Sextic

21.3 Analytical and Physical Properties of Cayley's Sextic

Based on the parametric representation found in Equation 21-3, i.e., $x = 4a\cos^3 t/3 \cdot \cos t$ and $y = 4a\cos^3 t/3 \cdot \sin t$, the following is an analysis of Cayley's Sextic.

21.3.1 Derivatives of Cayley's Sextic

- $\dot{x} = -4a\cos^2\frac{t}{3}\cdot\sin\frac{4t}{3}$
- $\ddot{x} = \dfrac{8a}{3}\cos\frac{t}{3}\left(\sin\frac{t}{3}\sin\frac{4t}{3} - 2\cos\frac{t}{3}\cos\frac{4t}{3}\right)$
- $\dot{y} = 4a\cos^2\frac{t}{3}\cdot\cos\frac{4t}{3}$
- $\ddot{y} = -\dfrac{8a}{3}\cos\frac{t}{3}\left(\sin\frac{t}{3}\cos\frac{4t}{3} + 2\cos\frac{t}{3}\sin\frac{4t}{3}\right)$
- $y' = -\cot\frac{4t}{3}$
- $y'' = -\dfrac{1}{3a\sin^3\frac{4t}{3}\cos^2\frac{t}{3}}$

21.3.2 Metric Properties of Cayley's Sextic

If r denotes the distance from the origin to the curve, then

$$r = 4a\cos^3\tfrac{t}{3}$$

If p denotes the distance from the origin to the tangent, then

$$p = -4a\cdot\cos^4\tfrac{t}{3}\,.$$

21.3.3 Curvature of Cayley's Sextic

If ρ represents the radius of curvature for Cayley's Sextic, then

$$\rho = 3a\cos^2\tfrac{t}{3}\,.$$

If (α, β) denotes the coordinates of the center of curvature for Cayley's Sextic, then

$$\alpha = a\cos^2\tfrac{t}{3}\left(4\cos\tfrac{t}{3}\cos t - 3\cos\tfrac{4t}{3}\right) \quad \text{and} \quad \beta = a\cos^2\tfrac{t}{3}\left(4\cos\tfrac{t}{3}\sin t - 3\sin\tfrac{4t}{3}\right).$$

21.3.4 Angles for Cayley's Sextic

If θ denotes the radial angle of Cayley's Sextic, then

$$\theta = t.$$

If ϕ denotes the tangential angle of Cayley's Sextic, then

$$\tan\phi = -\cot\tfrac{4t}{3}.$$

Finally, if ψ denotes the tangential-radial angle of Cayley's Sextic, then

$$\tan\psi = \frac{\cos t\cos\frac{4t}{3} + \sin t\sin\frac{4t}{3}}{\sin t\cos\frac{4t}{3} - \cos t\sin\frac{4t}{3}}.$$

21.4 Geometric Properties of Cayley's Sextic

- Intercepts: (0, 0); (a, 0); (–a/8, 0); $\left(0, \pm\frac{3\sqrt{3}}{8}a\right)$
- Extent: $|r| \le a$
- Symmetry: $y = 0$
- Node: (–a/8, 0).

21.5 Dynamic Geometry of Cayley's Sextic

The next four subsections contain dynamic geometry constructions dealing with Cayley's Sextic.

21.5.1 Cayley's Sextic as the Pedal Curve of a Cardioid

As can be seen from the construction contained in Table 21-1, Cayley's Sextic can be generated as the pedal curve of a Cardioid with respect to the Cardioid's cusp.

Table 21-1: Cayley's Sextic as the Pedal of a Cardioid

1. Draw circle AB with center at A and passing through point B	7. Construct $P_1 \perp$ to line segment CC' through point C'
2. Let C be a random point on the circumference of circle AB	8. Construct $P_2 \perp$ to P_1 through point B
3. Let A' be the image when A is rotated about point C by 180º	9. Let point D be the intersection of perpendiculars P_1 and P_2
4. Let C' be the image when C is rotated about A' by ∠BAC	10. Trace point D and change its color
5. Construct the locus of point C' as point C traverses circle AB	11. Animate point C around circle AB
6. Draw line segment CC'	

Of course, P_1 is the tangent to the Cardioid and therefore the pedal curve with respect to the Cardioid's cusp is simply the intersection of P_1 and the perpendicular to it through point B, the cusp.

21.5.2 An Alternate Construction for Cayley’s Sextic

Cayley’s Sextic is the roulette of a Cardioid with respect to an equal Cardioid and the cusp. Although GSP cannot produce a Cardioid rolling around another Cardioid, the construction of Table 21-2 is based on this concept.

Table 21-2: An Alternate Construction of Cayley's Sextic

1. Draw circle AB with center at A and passing through point B	6. Let C' be the image when C is translated by vector B' → C
2. Let C be a random point on the circumference of circle AB	7. Construct $P_1 \perp$ to line B'C through point C'
3. Draw line AC	8. Let B" be the image when B is reflected across P_1
4. Let B' be the image when point B is reflected across line AC	9. Trace point B" and change its color
5. Draw line B'C	10. Animate point C around circle AB

Note that the locus of point C' as point C traverses circle AB is that of a Cardioid.

21.5.3 A Tangent Construction for Cayley’s Sextic

Table 21-3 contains a construction for the tangent to Cayley’s Sextic.

Table 21-3: A Tangent to Cayley's Sextic

1. Draw circle AB with center at A and passing through point B	11. Let E be the intersection of perpendicular P_2 and line AB'
2. Let C be a random point on the circumference of circle AB	12. Construct $P_3 \perp$ to line AC' through point E
3. Draw line AC	13. Let F be the intersection of perpendicular P_3 and line AC'
4. Let B' be the image when point B is reflected across line AC	14. Trace point F and change its color
5. Draw line AB'	15. Let E' be the image when point E is reflected across line AC'
6. Let C' be the image when point C is reflected across line AB'	16. Draw line segment AE'
7. Draw line AC'	17. Construct $P_4 \perp$ to segment line AE' through point F
8. Construct $P_1 \perp$ to line AC' through point B'	18. Make P_4 thick and change its color
9. Let point D be the intersection of P_1 and line AC'	19. Animate point C around circle AB
10. Construct $P_2 \perp$ to line AB' through point D	

Note that steps 1 – 14 are a third alternate construction of Cayley’s Sextic.

21.5.4 The Osculating Circle to Cayley’s Sextic

Finally, the osculating circle for Cayley’s Sextic is found in Table 21-4.

Table 21-4: The Osculating Circle to Cayley's Sextic

1. Draw circle AB with center at A and passing through point B	20. Let G be either intersection of circle AE' and P_5
2. Let C be a random point on the circumference of circle AB	21. Construct $P_6 \perp$ to P_3 through point B'
3. Draw line AC	22. Let point H be the intersection of perpendiculars P_3 and P_6
4. Let B' be the image when point B is reflected across line AC	23. Let B" be the image when B' is dilated about point H by ⅔
5. Draw line AB'	24. Let F_1 be the image when F is dilated about A by – $^4/_3$
6. Let C' be the image when point C is reflected across line AB'	25. Let F_2 be the image when F_1 is translated by vector H → B"
7. Draw line AC'	26. Let E" be the image when E is translated by vector F → E
8. Construct $P_1 \perp$ to line AC through point B'	27. Construct $P_7 \perp$ to P_3 through point E"
9. Let D be the intersection of perpendicular P_1 and line AC	28. Construct $P_8 \perp$ to P_7 through point F_2
10. Construct $P_2 \perp$ to line AB' through point D	29. Let point I be the intersection of perpendiculars P_7 and P_8
11. Let E be the intersection of perpendicular P_2 and line AB'	30. Construct $P_9 \perp$ to line AE' through point I
12. Construct $P_3 \perp$ to line AC' through point E	31. Let J be the intersection of line AE' and perpendicular P_9
13. Let F be the intersection of line AC' and perpendicular P_3	32. Draw line segment GJ
14. Construct the locus of point F as point C traverses circle AB	33. Construct $P_{10} \perp$ to line segment GJ through point G
15. Let E' be the image when point E is reflected across line AC'	34. Let K be the intersection of perpendicular P_{10} and line AE'
16. Draw line AE'	35. Let F_3 be the image when F is translated by vector K → A
17. Construct $P_4 \perp$ to line AE' through point F	36. Draw circle F_3F with center at F_3 and passing through F
18. Draw circle AE' with center at A and passing through point E'	37. Animate point C around circle AB
19. Construct $P_5 \perp$ to line AE' through point A	

Circle F_3F is, of course, the osculating circle. As we have learned, point F_3 is the center of curvature and traces the evolute of, in this case, Cayley's Sextic. It is interesting to do so here. Trace point F_3 and rerun the animation. Lo and behold! The evolute is a Nephroid!

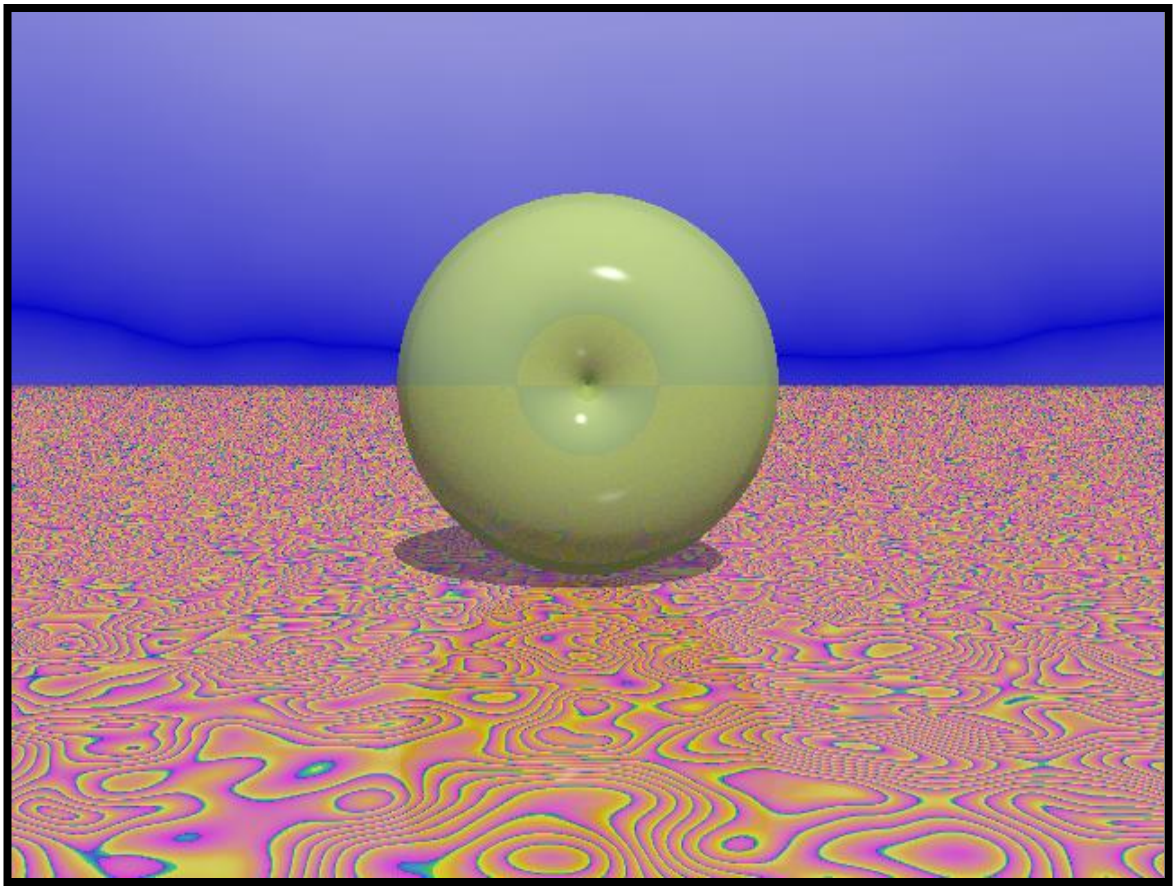


Figure 21-3: Cayley's Sextic as a Solid of Revolution

Cayley's Sextic was rotated about the x-axis to form the object seen above. It was then given a shiny, yellow-green finish and placed so as to just rest on the plane that has been given a turbulent undulating pattern. Note that the loop of Cayley's Sextic is not visible, as it would be inside of the object, thereby giving the object an appearance much like that of a Cardioid of revolution.

Chapter 22 – The String Tie Curve

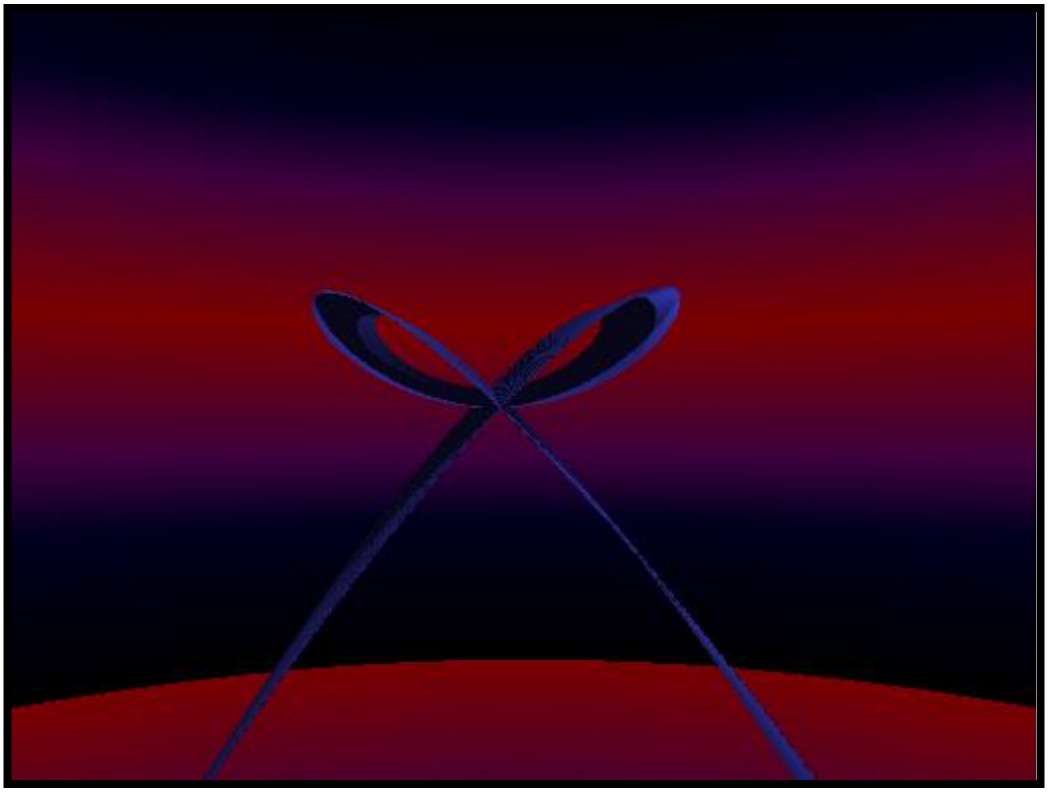


Figure 22-1: The String Tie Curve in Three Dimensions

The String Tie Curve has been extruded into the third dimension to render the object seen above. The object has then been given a bright blue finish and placed with a red-purplish background.

22.1 Introduction

A more apt name for this curve might be the Bowtie Curve; however, the Lemniscate of Gerono (Chapter 17) already has that nickname, so we are going to call this curve the String Tie Curve. It does rather look like one of those string ties that are often seen being worn (ugh!) with western style shirts (mostly in the western part of the U.S.A.). This curve is a much more modern curve then the majority that we have studied so far; it appears there is no official name for it.

22.2 Equations and Graph of the String Tie Curve

The Cartesian equation for the String Tie is given by

$$x^4 - ay\left(x^2 - y^2\right) = 0 \qquad \textbf{Equation 22-1}$$

For the polar equation, we have

$$r = a\sin\theta\left(\sec^2\theta - \sin^2\theta\sec^4\theta\right) \qquad \textbf{Equation 22-2}$$

If we let $y = xt$, and make this substitution into Equation 22-1, we get a parametric representation for the String Tie of

$$(x, y) = at\left(1 - t^2\right)(1, t) \qquad \textbf{Equation 22-3}$$

Finally, the equation of the tangent line to the String Tie at the point $t = q$ is

$$aq^2\left(1 - q^2\right)\left(1 - 3q^2\right)\cdot y = 2q\left(1 - 2q^2\right)\cdot x - aq^2\left(1 - q^2\right)^2 \qquad \textbf{Equation 22-4}$$

A graph of the String Tie is shown in Figure 22-2.

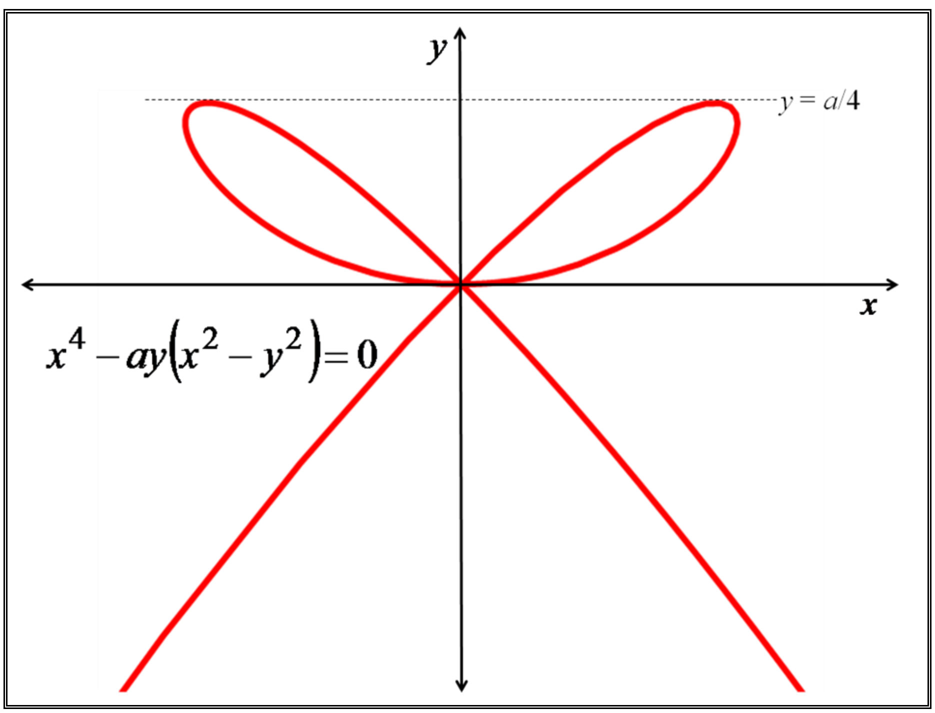


Figure 22-2: Graph of the String Tie Curve

22.3 Analytical and Physical Properties of the String Tie Curve

Based on the parametric representation of Equation 22-3, i.e., $x = at(1 - t^2)$ and $y = at^2 (1 - t^2)$, the next four subsections present an analysis of the String Tie Curve.

22.3.1 Derivatives of the String Tie Curve

- $\dot{x} = a\left(1-3t^2\right)$.
- $\ddot{x} = -6at$.
- $\dot{y} = 2at\left(1-2t^2\right)$.
- $\ddot{y} = 2a\left(1-6t^2\right)$.
- $y' = \dfrac{2t\left(1-2t^2\right)}{1-3t^2}$.
- $y'' = \dfrac{2\left(1-3t^2+6t^4\right)}{a\left(1-3t^2\right)^3}$.

22.3.2 Metric Properties of the String Tie Curve

If r represents the distance from the origin to the String Tie, then

$$r = at\left(1-t^2\right)\sqrt{1+t^2} \ .$$

If p denotes the distance from the origin to the String Tie's tangent, then

$$p = \frac{-at^2\left(1-t^2\right)^2}{\sqrt{1-2t^2-7t^4+16t^6}}.$$

22.3.3 Curvature of the String Tie Curve

If (α, β) denotes the coordinates of the center of curvature for the String Tie Curve, then

$$\alpha = \frac{4at^5\left(3-9t^2+8t^4\right)}{1-3t^2+6t^4} \quad \text{and} \quad \beta = \frac{a\left(1-3t^2-9t^4+55t^6-60t^8\right)}{2\left(1-3t^2+6t^4\right)}.$$

If ρ denotes the radius of curvature of the String Tie Curve, then

$$\rho = \frac{a\left(1-2t^2-7t^4+16t^6\right)^{3/2}}{2\left(1-3t^2+6t^4\right)}.$$

22.3.4 Angles for the String Tie Curve

If ψ denotes the tangential-radial angle of the String Tie, then

$$\tan\psi = \frac{t\left(1-t^2\right)}{1-t^2-4t^4}.$$

If θ denotes the radial angle of the String Tie, then

$$\tan\theta = t.$$

If ϕ denotes the tangential angle of the String Tie Curve, then

$$\tan\phi = \frac{2t\left(1-2t^2\right)}{1-3t^2}.$$

22.4 Geometric Properties of the String Tie Curve

- Intercepts: (0, 0).
- Symmetry: y-axis.
- Extent: $-\infty < y \le a/4$; $-\infty < x < \infty$.

22.5 Dynamic Geometry of the String Tie Curve

Three constructions for the String Tie Curve follow.

22.5.1 The Basic Construction of the String Tie Curve

This construction (see Table 22-1), as well as the two that follow, all construct the curve in the same way, which is the basic construction for the String Tie Curve.

Table 22-1: The String Tie's Basic Construction

1. Create x-y axes with origin A and unit point B = (1, 0)	9. Let F be the intersection of perpendicular P_2 and line AC
2. Draw circle AB with center at A and passing through point B	10. Construct $P_3 \perp$ to the y-axis through point F
3. Let C be a random point on the circumference of circle AB	11. Let point G be the intersection of perpendiculars P_1 and P_3
4. Draw line AC	12. Construct $P_4 \perp$ to the x-axis through point G
5. Let D be the intersection of the positive y-axis and circle AB	13. Let H be the intersection of line AC and perpendicular P_4
6. Construct $P_1 \perp$ to line AC through point D	14. Trace point H and change its color
7. Let E be the intersection of perpendicular P_1 and the x-axis	15. Animate point C around circle AB
8. Construct $P_2 \perp$ to the x-axis through point E	

22.5.2 The Tangent Construction to the String Tie Curve

If you have done (and retained) the previous construction, continue with that sketch and step 15 of this construction; if not, you will have to begin this one at step 1. In either case, this is a very nice construction and it's quite spectacular to view the final animation. Refer to Table 22-2.

Table 22-2: The Tangent to the String Tie Curve

1. Create *x-y* axes with origin A and unit point B = (1, 0)	16. Construct $P_6 \perp$ to the *y*-axis through point D
2. Draw circle AB with center at A and passing through point B	17. Let point I be the intersection of perpendiculars P_5 and P_6
3. Let C be a random point on the circumference of circle AB	18. Construct $P_7 \perp$ to the *y*-axis through point H
4. Draw line AC	19. Let point J be the intersection of perpendiculars P_1 and P_7
5. Let D be the intersection of the positive *y*-axis and circle AB	20. Let J' be the image when J is dilated about point G by 4
6. Construct $P_1 \perp$ to line AC through point D	21. Let G' be the image when G is translated by vector E → G
7. Let E be the intersection of perpendicular P_1 and the *x*-axis	22. Let I' be the image when I is translated by vector G' → J'
8. Construct $P_2 \perp$ to the *x*-axis through point E	23. Let I'' be the image when I' is reflected across line AC
9. Let F be the intersection of perpendicular P_2 and line AC	24. Construct $P_8 \perp$ to P_5 through point I''
10. Construct $P_3 \perp$ to the *y*-axis through point F	25. Construct $P_9 \perp$ to P_8 through point H
11. Let point G be the intersection of perpendiculars P_1 and P_3	26. Let point K be the intersection of perpendiculars P_8 and P_9
12. Construct $P_4 \perp$ to the *x*-axis through point G	27. Draw line AK
13. Let H be the intersection of line AC and perpendicular P_4	28. Construct $P_{10} \perp$ to line AK through point H
14. Construct the locus of point H as point C traverses circle AB	29. Make P_{10} thick and change its color
15. Construct $P_5 \perp$ to line AC through point A	30. Animate point C around circle AB

22.5.3 The Osculating Circle to the String Tie Curve

Utilize the first 28 steps of the previous construction and continue with step 29 of this construction (Table 22-3), or start all over again.

Table 22-3: The Osculating Circle of the String Tie Curve

1. Create *x-y* axes with origin A and unit point B = (1, 0)	29. Let L be the intersection of perpendicular P_1 and line AC
2. Draw circle AB with center at A and passing through point B	30. Let H_1 be the image when H is translated by vector L → H
3. Let C be a random point on the circumference of circle AB	31. Let H_2 be the image when H_1 is dilated about point F by 9
4. Draw line AC	32. Let H_3 be the image when H_2 is translated by vector F → H_2
5. Let D be the intersection of the positive *y*-axis and circle AB	33. Let H_4 be the image when H_3 is translated by vector F → H_1
6. Construct $P_1 \perp$ to line AC through point D	34. Let H_5 be the image when H_4 is translated by vector H → H_1
7. Let E be the intersection of perpendicular P_1 and the *x*-axis	35. Let F' be the image when F is translated by vector H_5 → F
8. Construct $P_2 \perp$ to the *x*-axis through point E	36. Draw line segment F'H
9. Let F be the intersection of perpendicular P_2 and line AC	37. Let M be the midpoint of line segment F'H
10. Construct $P_3 \perp$ to the *y*-axis through point F	38. Let M' be the image when M is translated by vector A → M
11. Let point G be the intersection of perpendiculars P_1 and P_3	39. Let A' be the image when A is translated by vector M' → A
12. Construct $P_4 \perp$ to the *x*-axis through point G	40. Let I''' be the image when I' is translated by vector A → I'
13. Let H be the intersection of line AC and perpendicular P_4	41. Construct $P_{11} \perp$ to P_5 through point I'''
14. Construct the locus of point H as point C traverses circle AB	42. Construct $P_{12} \perp$ to P_{11} through point A'
15. Construct $P_5 \perp$ to line AC through point A	43. Let point N be the intersection of perpendiculars P_{11} and P_{12}
16. Construct $P_6 \perp$ to the *y*-axis through point D	44. Draw circle AK with center at A and passing through point K
17. Let point I be the intersection of perpendiculars P_5 and P_6	45. Construct $P_{13} \perp$ to line AK through point A
18. Construct $P_7 \perp$ to the *y*-axis through point H	46. Let point O be one of the intersections of circle AK with P_{13}
19. Let point J be the intersection of perpendiculars P_1 and P_7	47. Construct $P_{14} \perp$ to line AK through point N
20. Let J' be the image when J is dilated about point G by 4	48. Let P be the intersection of perpendicular P_{14} and line AK
21. Let G' be the image when G is translated by vector E → G	49. Draw line segment OP
22. Let I' be the image when I is translated by vector G' → J'	50. Construct $P_{15} \perp$ to line segment OP through point O
23. Let I'' be the image when I' is reflected across line AC	51. Let Q be the intersection of perpendicular P_{15} and line AK
24. Construct $P_8 \perp$ to P_5 through point I''	52. Let H_6 be the image when H is translated by vector Q → A
25. Construct $P_9 \perp$ to P_8 through point H	53. Draw circle H_6H with center at H_6 and passing through H
26. Let point K be the intersection of perpendiculars P_8 and P_9	54. Make circle H_6H thick and change its color
27. Draw line AK	55. Animate point C around circle AB
28. Construct $P_{10} \perp$ to line AK through point H	

You may replace steps 31 – 33 with the following: Let point H_4 be the image when point H_1 is dilated about point F by a factor of 19. Early versions of GSP will not allow for a dilation this large. Steps 31 – 33 are a way around this limitation. A very complex and difficult construction but a truly amazing and beautiful result!

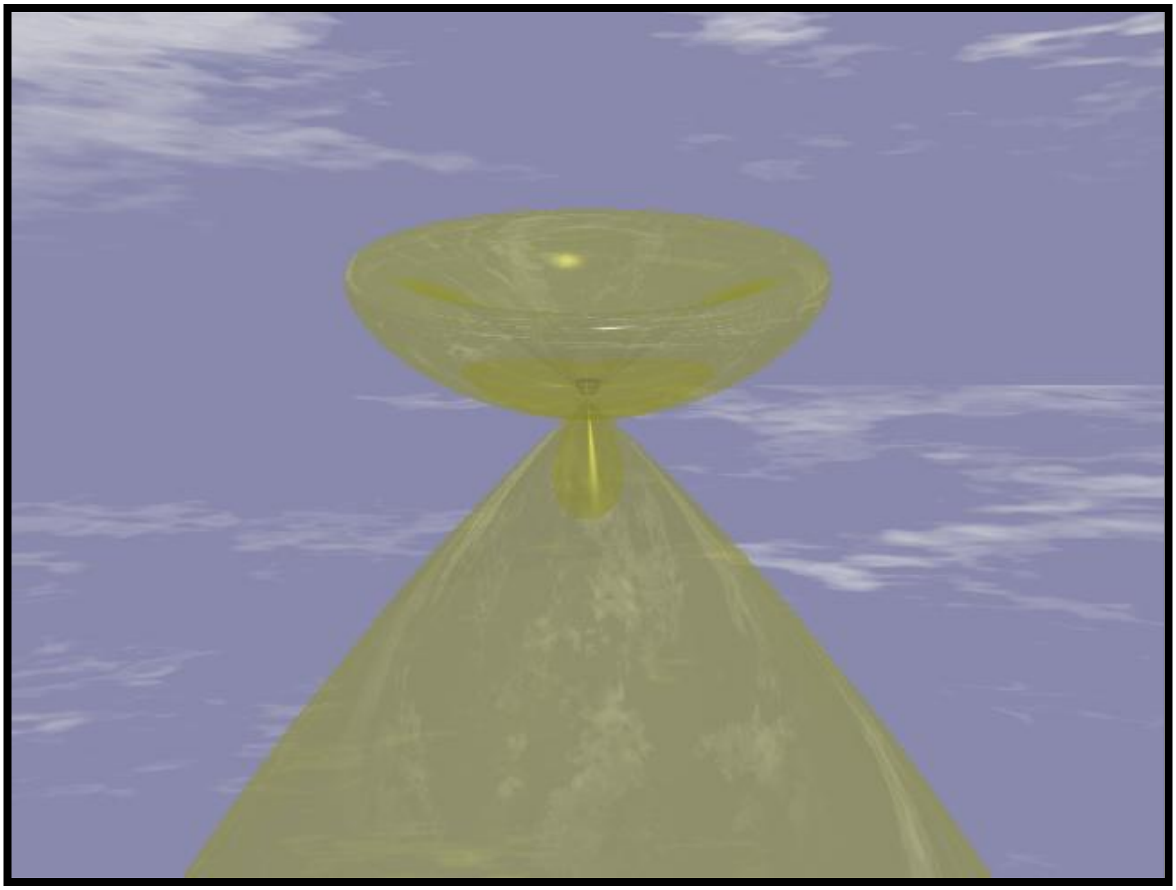


Figure 22-3: The String Tie as a Solid of Revolution

To obtain the object pictured above, the String Tie Curve was rotated about the y-axis. The upper, bowl-shaped portion is due to the revolution of the loop portion of the curve while the lower, rounded, conical portion is due to the revolution of the asymptotes. The object was then given a background of clouds and a yellowish-glass texture. The glass not only reflects the clouds, but reflects the upper bowl in the lower conical structure and vice-versa.

Chapter 23 – The Bowditch Curve

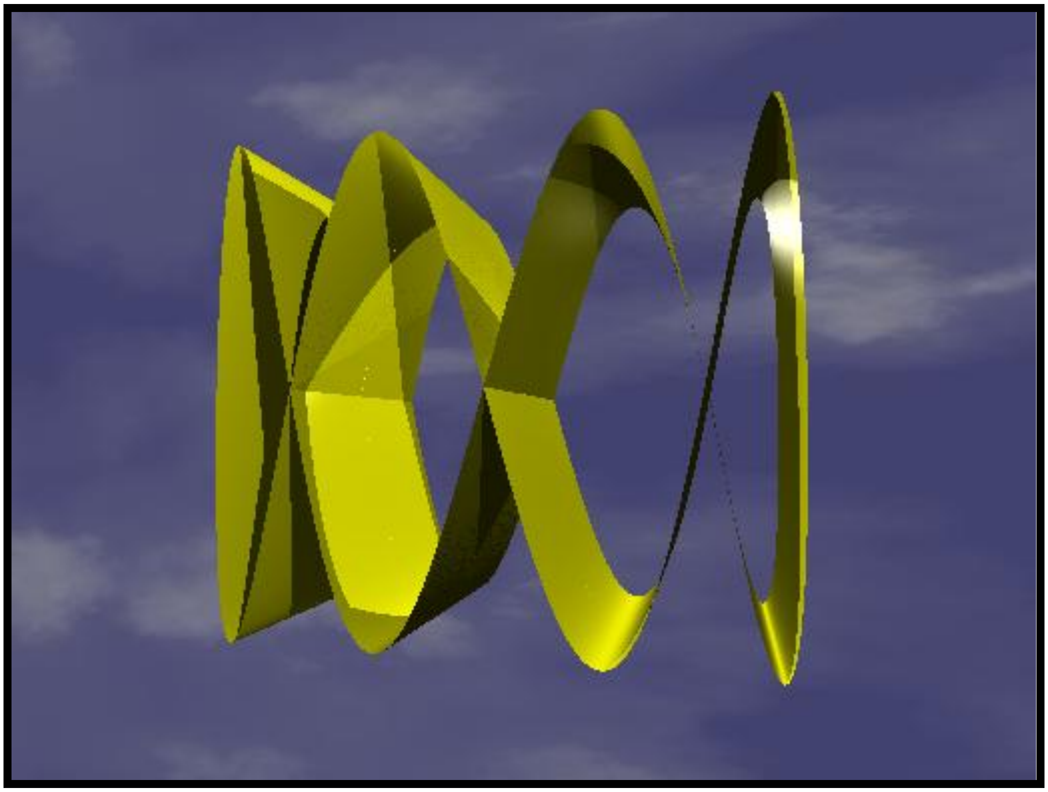


Figure 23-1: The Bowditch Curve in Three Dimensions

The Bowditch Curve specified by (a, b, d, k) = (9, 8, π, ¼) was extruded into the third dimension (normal to the plane of the paper), given a golden finish, and placed so as to appear to be floating in a blue sky with scattered, wispy clouds. Light sources have been placed so as to cast shadows.

23.1 Introduction

The Bowditch Curves are named for Nathaniel Bowditch, who was one of the first American mathematicians to receive international recognition. Nathaniel Bowditch was born in Salem, Massachusetts in 1773 and died in Boston in 1838. Bowditch studied the curves in 1815. Bowditch curves are also sometimes referred to as Lissajous (pronounced liz·a·jew) curves because they were studied (independently from Bowditch) by Jules-Antoine Lissajous in 1857. The curves have applications in physics, astronomy, and other sciences.

23.2 Equations and Graph of the Bowditch Curve

Bowditch Curves are the family of curves specified by the parametric equations

$$x = a\sin(k\,t + d) \quad \text{and} \quad y = b\sin t\,,$$

where *a*, *b*, *d*, and *k* are constants, so that the 4-tuple (*a*, *b*, *d*, *k*) completely specifies the curve. If the constant *k* is a rational number, then the curve is algebraic, and if *k* is not rational, the curve is transcendental. Further, if *k* is rational, the interval of definition of the curve is a function of *k*. That is, if $k = m/n$, where *m* and *n* are integers, then the interval of definition of the curve is $-n\pi \le t \le n\pi$. Hence, we can conclude that for rational values of the constant *k*, a parametric representation of the Bowditch Curves is

$$(x, y) = \left[a\sin\left(\frac{mt}{n} + d\right), b\sin t\right] \quad -n\pi \le t \le n\pi \qquad \textbf{Equation 23-1}$$

The equation of the tangent at the point $t = q$ is

$$y = \frac{bn\cos q}{am\cos\left(\frac{mq}{n} + d\right)} \cdot x + \frac{bm\sin q\cos\left(\frac{mq}{n} + d\right) - bn\cos q\sin\left(\frac{mq}{n} + d\right)}{m\cos\left(\frac{mq}{n} + d\right)}. \qquad \textbf{Equation 23-2}$$

Figures 23-2 to 23-6 depict a graph of the Bowditch Curves for a few selected values of the constants (*a*, *b*, *d*, *k*).

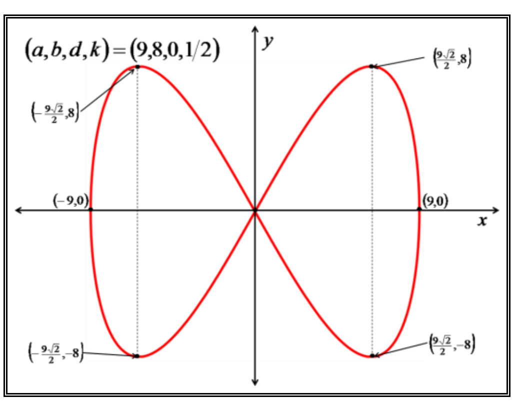


Figure 23-2: Graph of the Bowditch Curve for (*a*, *b*, *d*, *k*) = (9, 8, 0, ½)

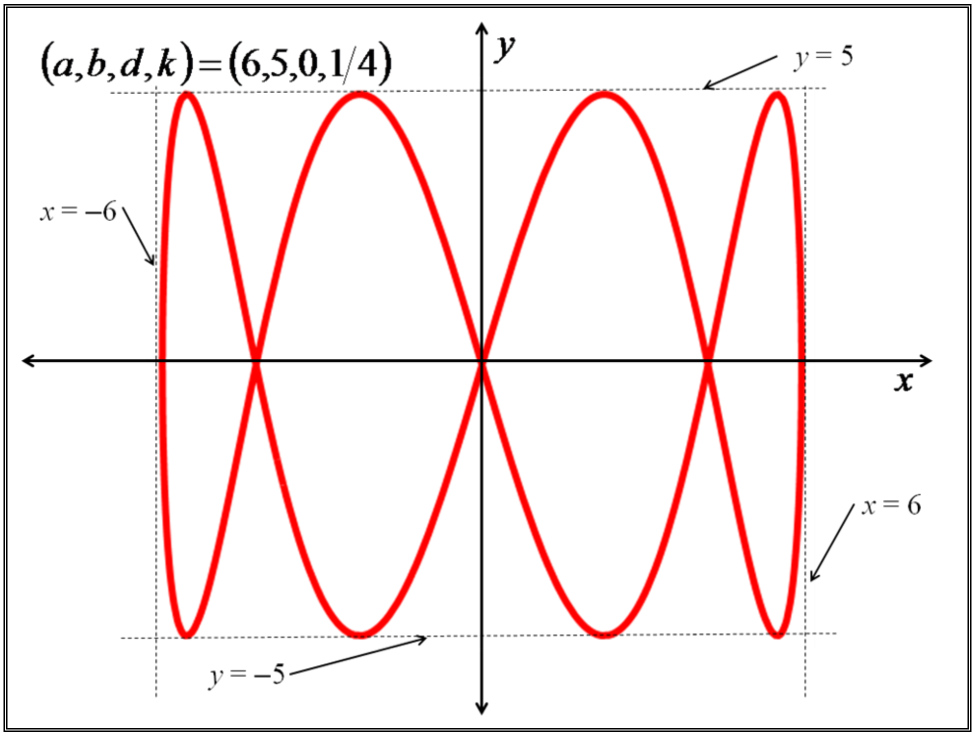


Figure 23-3: Graph of the Bowditch Curve for (*a, b, d, k*) = (6, 5, 0, ¼)

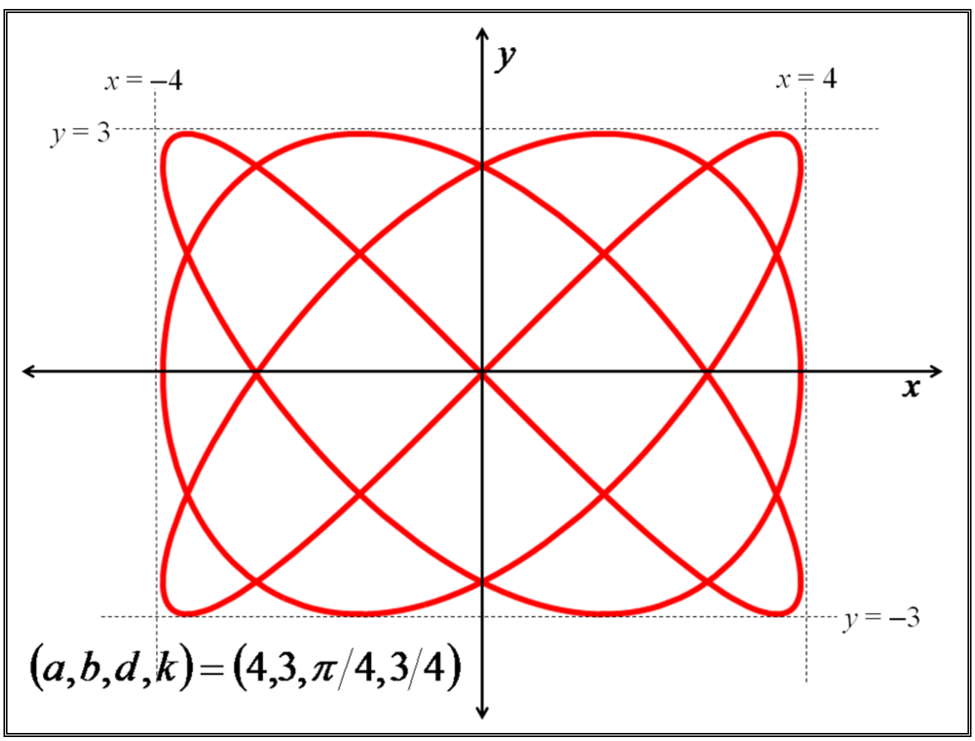


Figure 23-4: Graph of the Bowditch Curve for (*a, b, d, k*) = (4, 3, π/4, ¾)

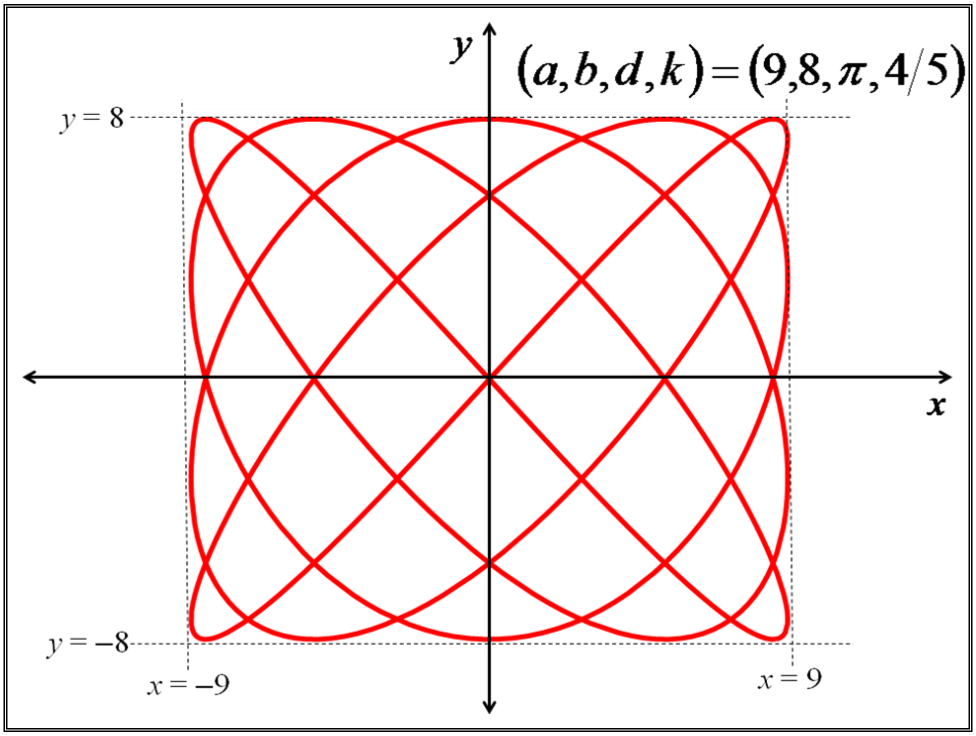


Figure 23-5: Graph of the Bowditch Curve for (*a, b, d, k*) = (9, 8, π, 4/5)

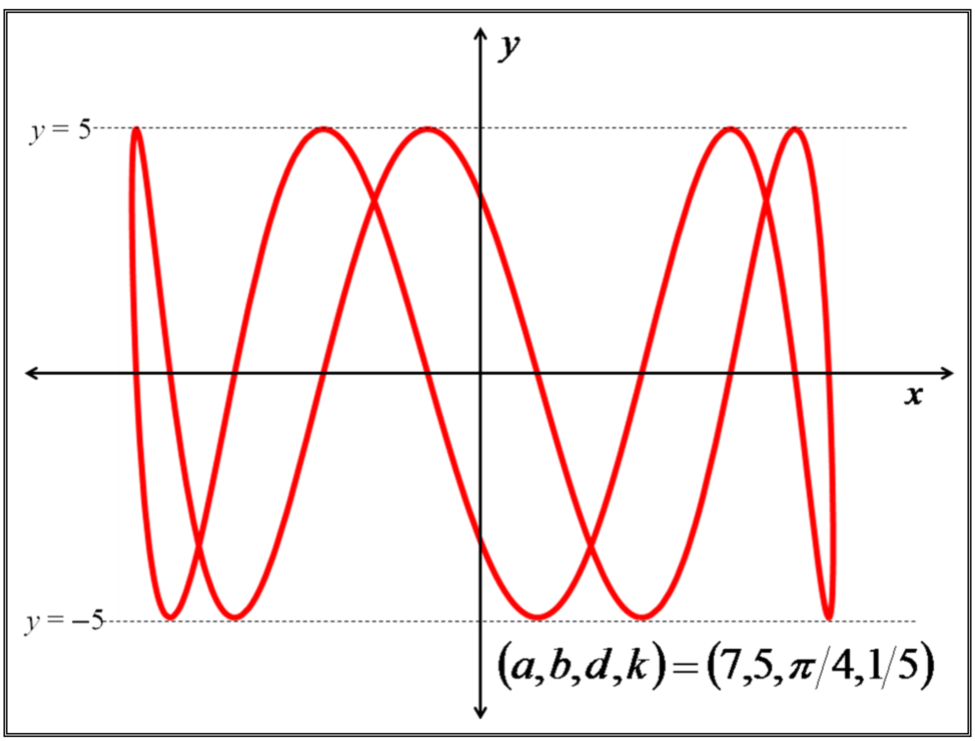


Figure 23-6: Graph of the Bowditch Curve for (*a, b, d, k*) = (7, 5, π/4, 1/5)

23.3 Analytical and Physical Properties of the Bowditch Curve

Based on the parametric representation of the Bowditch Curve found in Equation 23-1, the following paragraphs delineate further properties and characteristics of the Bowditch Curve.

23.3.1 Derivatives of the Bowditch Curve

- $\dot{x} = \frac{am}{n}\cos\left(\frac{mt}{n}+d\right).$
- $\ddot{x} = -\frac{am^2}{n^2}\sin\left(\frac{mt}{n}+d\right).$
- $\dot{y} = b\cos t.$
- $\ddot{y} = -b\sin t.$
- $y' = \dfrac{bn\cos t}{am\cos\left(\frac{mt}{n}+d\right)}.$
- $y'' = \dfrac{bn\left[m\cos t\sin\left(\frac{mt}{n}+d\right)-n\sin t\cos\left(\frac{mt}{n}+d\right)\right]}{a^2m^2\cos^3\left(\frac{mt}{n}+d\right)}$

23.3.2 Metric Properties of the Bowditch Curve

If r is the radial distance, then

$$r = \sqrt{a^2\sin^2\left(\frac{mt}{n}+d\right)+b^2\sin^2 t}.$$

If p is the distance from the origin to the tangent of the Bowditch curve, then

$$p = \frac{abm\sin t\cos\left(\frac{mt}{n}+d\right)-abn\sin\left(\frac{mt}{n}+d\right)\cos t}{\sqrt{a^2m^2\cos^2\left(\frac{mt}{n}+d\right)+b^2n^2\cos^2 t}}.$$

23.3.3 Curvature of the Bowditch Curve

If ρ denotes the radius of curvature, then

$$\rho = \frac{\left[a^2m^2\cos^2\left(\frac{mt}{n}+d\right)+b^2n^2\cos^2 t\right]^{3/2}}{abmn\left[m\cos t\sin\left(\frac{mt}{n}+d\right)-n\sin t\cos\left(\frac{mt}{n}+d\right)\right]}.$$

If (α, β) denote the coordinates of the center of curvature, then

$$\alpha = \frac{am\cos t\left[\sin^2\left(\frac{mt}{n}+d\right)-\cos^2\left(\frac{mt}{n}+d\right)\right]}{m\cos t\sin\left(\frac{mt}{n}+d\right)-n\sin t\cos\left(\frac{mt}{n}+d\right)}-\frac{a^2mn\sin t\sin\left(\frac{mt}{n}+d\right)\cos\left(\frac{mt}{n}+d\right)+b^2n^2\cos^3 t}{am\left[m\cos t\sin\left(\frac{mt}{n}+d\right)-n\sin t\cos\left(\frac{mt}{n}+d\right)\right]}$$

and

$$\beta = \frac{bn\cos 2t\cos\left(\frac{mt}{n}+d\right)+bm\sin t\cos t\sin\left(\frac{mt}{n}+d\right)}{m\cos t\sin\left(\frac{mt}{n}+d\right)-n\sin t\cos\left(\frac{mt}{n}+d\right)}+\frac{a^2m^2\cos^3\left(\frac{mt}{n}+d\right)}{bn\left[m\cos t\sin\left(\frac{mt}{n}+d\right)-n\sin t\cos\left(\frac{mt}{n}+d\right)\right]}$$

23.3.4 Angles for the Bowditch Curve

If ψ represents the tangential-radial angle, then

$$\tan\psi = \frac{abn\sin\left(\frac{mt}{n}+d\right)\cos t - abm\sin t\cos\left(\frac{mt}{n}+d\right)}{a^2m\sin\left(\frac{mt}{n}+d\right)\cos\left(\frac{mt}{n}+d\right)+b^2n\sin t\cos t}.$$

If θ denotes the radial angle, then for the Bowditch Curve

$$\tan\theta = \frac{b\sin t}{a\sin\left(\frac{mt}{n}+d\right)}.$$

If ϕ denotes the tangential angle, then for he Bowditch Curve

$$\tan\phi = \frac{bn\cos t}{am\cos\left(\dfrac{mt}{n}+d\right)}.$$

23.4 Geometric Properties of the Bowditch Curve

The curve is algebraic and unicursal if k is rational, and transcendental otherwise. The curve is entirely contained within a rectangle defined by $|x| \le a$, $|y| \le b$.

23.5 Dynamic Geometry of the Bowditch Curve

The next five subsections delineate constructions for the Bowditch Curve.

23.5.1 The Bowditch Curve

Table 23-1 contains a construction for the Bowditch Curve.

Table 23-1: The Bowditch Curve

1. Draw circle AB with center at A and passing through point B	8. Construct $P_1 \perp$ to line AB through point C'
2. Let C be a random point on the circumference of circle AB	9. Draw line AC'
3. Draw line AC	10. Let B" be the image when point B' is reflected across line AC'
4. Let B' be the image when point B is reflected across line AC	11. Construct $P_2 \perp$ to P_1 through point B"
5. Draw line AB'	12. Let point D be the intersection of perpendiculars P_1 and P_2
6. Let C' be the image when point C is reflected across line AB'	13. Trace point D and change its color
7. Draw line AB	14. Animate point C around circle AB

Note that for this construction, m/n = ¾.

23.5.2 The Tangent to the Bowditch Curve

Based on the previous construction of the Bowditch Curve, Table 23-2 contains a construction for the tangent to the curve.

Table 23-2: The Bowditch's Tangent

1. Draw circle AB with center at A and passing through point B	14. Make the locus thick and change its color
2. Let C be a random point on the circumference of circle AB	15. Construct $P_3 \perp$ to P_1 through point C'
3. Draw line AC	16. Construct P_4 to $\perp$ line AB through point B"
4. Let B' be the image when point B is reflected across line AC	17. Let line L_1 be the reflection of line AB across P_3
5. Draw line AB'	18. Let line L_2 be the reflection of P_3 across line L_1
6. Let C' be the image when point C is reflected across line AB'	19. Construct $P_5 \perp$ to line AB through point A
7. Draw line AB	20. Let line L_3 be the reflection of P_5 across P_4
8. Construct $P_1 \perp$ to line AB through point C'	21. Let line L_4 be the reflection of P_5 across L_3
9. Draw line AC'	22. Let point E be the intersection of lines L_2 and L_4
10. Let B" be the image when point B' is reflected across line AC'	23. Draw line AE
11. Construct $P_2 \perp$ to P_1 through point B"	24. Construct $P_6 \perp$ to line AE through point D
12. Let point D be the intersection of perpendiculars P_1 and P_2	25. Make P_6 thick and change its color
13. Construct the locus of point D as Point C traverses circle AB	26. Animate point C around circle AB

Perpendicular P_6 is, of course, the tangent. Again, note that for this construction, m/n = ¾.

23.5.3 The Bowditch Curve as a Compass-Only Construction

Table 23-3 contains the GSP version of a compass-only construction for the Bowditch Curve.

Table 23-3: The Bowditch Curve as a Compass-Only Construction

1. Draw circle AB with center at A and passing through point B	17. Draw line segment C_1C_2
2. Let C be a random point on the circumference of circle AB	18. Let B_3 be the reflection of point B_2 across line segment C_1C_2
3. Draw circle BC with center at B and passing through point C	19. Hide line segment C_1C_2
4. Draw circle CB with center at C and passing through point B	20. Draw circle B_3B_2 centered at B_3 and passing through point B_2
5. Draw line segment AC	21. Draw circle B_2B_3 centered at B_2 and passing through point B_3
6. Let B_1 be the reflection of point B across line segment AC	22. Let D and E be the intersections of circles B_2B_3 and B_3B_2
7. Draw circle B_1C with center at B_1 and passing through point C	23. Draw circle ED with center at E and passing through point D
8. Draw line segment AB_1	24. Let F be the unlabeled intersection of circles ED and B_2B_3
9. Let C_1 be the reflection of point C across line segment AB_1	25. Draw circle FB_3 with center at F and passing through point B_3
10. Draw circle C_1B_1 centered at C_1 and passing through point B_1	26. Let G and H be the two intersections of circles FB_3 and B_3B_2
11. Draw line segment AC_1	27. Draw circle GB_3 centered at G and passing through point B_3
12. Let B_2 be the reflection of point B_1 across line segment AC_1	28. Draw circle HB_3 centered at H and passing through point B_3
13. Draw circle BC_1 centered at B and passing through point C_1	29. Let I be the unlabeled intersection of circles HB_3 and GB_3
14. Draw line segment AB	30. Trace point I and change its color
15. Let C_2 be the reflection of point C_1 across line segment AB	31. Animate point C around circle AB
16. Draw circle C_2B_2 centered at C_2 and passing through point B_2	

This member of the Bowditch family of curves is the same member that is formed in the previous constructions, except this time the construction is the GSP version of a compass-only construction. The reason for hiding line segment C_1C_2 is because the location of point I in step 29 is ambiguous unless segment C_1C_2 is hidden. In other words, circles HB_3 and GB_3 and line segment C_1C_2 all intersect at the same point. GSP doesn't allow the placement of a point under this kind of circumstance, but hiding the line segment resolves the ambiguity. However, if this was a true compass-only construction, line segment C_1C_2 would not be present as point B_3 would have been created as the intersection of circles C_1B_1 and C_2B_2 instead of as a reflection of point B_3 across line segment C_1C_2.

23.5.4 An Alternate Construction for the Bowditch Curve and Tangent

At the risk of being redundant, here is an alternate construction for both the Bowditch Curve and its tangent (see Table 23-4).

Table 23-4: An Alternate Construction of the Bowditch Curve and Tangent

1. Draw horizontal line AB	15. Let C_8 be the image when C_6 is rotated about point A by 90°
2. Draw circle AB with center at A and passing through point B	16. Let E be the intersection of line AB and perpendicular P_2
3. Let C be a random point on the circumference of circle AB	17. Let C_9 be the image when C_7 is dilated about point A by 1¼
4. Let D also be a random point on circle AB	18. Let point F be the intersection of perpendiculars P_1 and P_3
5. Let C_1 be the image when C is rotated about A by ∠BAC	19. Construct the locus of F while point C traverses circle AB
6. Let C_2 be the image when C_1 is rotated about A by ∠BAC	20. Make the locus thick and change its color
7. Let C_3 be the image when C_2 is rotated about A by ∠BAC	21. Let C_{10} be the image when C_9 is rotated about A by ∠BAD
8. Let C_4 be the image when C_3 is rotated about A by ∠BAC	22. Construct $P_4 \perp$ to P_1 through point C_{10}
9. Construct $P_1 \perp$ to line AB through point C_3	23. Let point G be the intersection of perpendiculars P_2 and P_4
10. Let C_5 be the image when C_3 is rotated about point A by 90°	24. Draw line segment AG
11. Let C_6 be the image when C_4 is rotated about A by ∠BAD	25. Construct line L_1 parallel to line segment AG through point F
12. Construct $P_2 \perp$ to line AB through point C_5	26. Make line L_1 thick and change its color
13. Let C_7 be the image when C_4 is rotated about point A by 90°	27. Animate point C around circle AB
14. Construct $P_3 \perp$ to P_1 through point C_6	

Now, drag point D and rerun the animation. Neat, huh?

23.5.5 The Osculating Circle of the Bowditch Curve

Believe it or not, here is the osculating circle to this very complex curve (see Table 23-5).

Table 23-5: The Osculating Circle of the Bowditch Curve

1. Create *x*-*y* axes with origin A and unit point B = (1, 0)	26. Let point F' be the image when F is dilated about H by 4
2. Draw circle AB centered at A and passing through point B	27. Construct $P_8 \perp$ to P_7 through point F'
3. Let C be a random point on the circumference of circle AB	28. Let point I be the intersection of perpendiculars P_6 and P_8
4. Draw line segment AC	29. Draw line segment AI
5. Let B' be the reflection of point B across line segment AC	30. Let point J be the intersection of the *x*-axis with P_1
6. Draw line segment AB'	31. Let A' be the image when A is translated by vector J → A
7. Let C' be the reflection of point C across line segment AB'	32. Let A" be the image when A' is dilated about point A by 9
8. Draw line AC'	33. Construct $P_9 \perp$ to the *x*-axis through point A"
9. Let B" be the reflection of point B' across line AC'	34. Construct $P_{10} \perp$ to the *x*-axis through point B"
10. Draw line AB"	35. Let point K be the intersection of the *x*-axis with P_{10}
11. Construct $P_1 \perp$ to the *x*-axis through point C'	36. Let K_1 be the image when K is translated by vector B" → K
12. Construct $P_2 \perp$ to P_1 through point B"	37. Let K_2 be the image when K_1 is dilated about K by 4
13. Let point D be the intersection of perpendiculars P_1 and P_2	38. Let K_3 be the image when K_2 is dilated about K by 4
14. Construct the locus of point D as point C traverses circle AB	39. Construct $P_{11} \perp$ to P_{10} through point K3
15. Make the locus thick and change its color	40. Let point L be the intersection of perpendiculars P_9 and P_{11}
16. Construct $P_3 \perp$ to line AC' through point A	41. Construct $P_{12} \perp$ to line segment AI through point A
17. Construct $P_4 \perp$ to line AB" through point A	42. Construct $P_{13} \perp$ to P_{12} through point L
18. Let point E be the intersection of circle AB with P_3*	43. Let point M be the intersection of perpendiculars P_{12} and P_{13}
19. Let point F be the intersection of circle AB with P_4*	44. Draw line segment IM
20. Construct $P_5 \perp$ to the *x*-axis through point E	45. Construct $P_{14} \perp$ to line segment IM through point I
21. Let point G be the intersection of the *x*-axis and P_5	46. Let point N be the intersection of perpendiculars P_{12} and P_{14}
22. Let G' be the image when G is dilated about point A by 3	47. Let D' be the image when D is translated by vector N → A
23. Construct $P_6 \perp$ to the *x*-axis through point G'	48. Draw circle D'D centered at D' and passing through point D
24. Construct $P_7 \perp$ to the *x*-axis through point F	49. Make circle D'D thick and change its color
25. Let point H be the intersection of the *x*-axis with P_7	50. Animate point C around circle AB

* There are two intersections. In order to select the correct intersection, do the following. Position point C (by dragging it around circle AB) in the 1st quadrant so that point C' is in the 2nd quadrant and point B" is in the 3rd quadrant. Point E is then the intersection of P_3 with circle AB that is below the *x*-axis. Likewise, point F is the intersection of P_4 with circle AB that is below the *x*-axis.

Trace point D' and rerun the animation to get a look at the evolute to the Bowditch Curve! It's quite complex, but, as you might suspect, very symmetrical. Steps 37 and 38 may be combined into one step if you can dilate by a factor of 16.

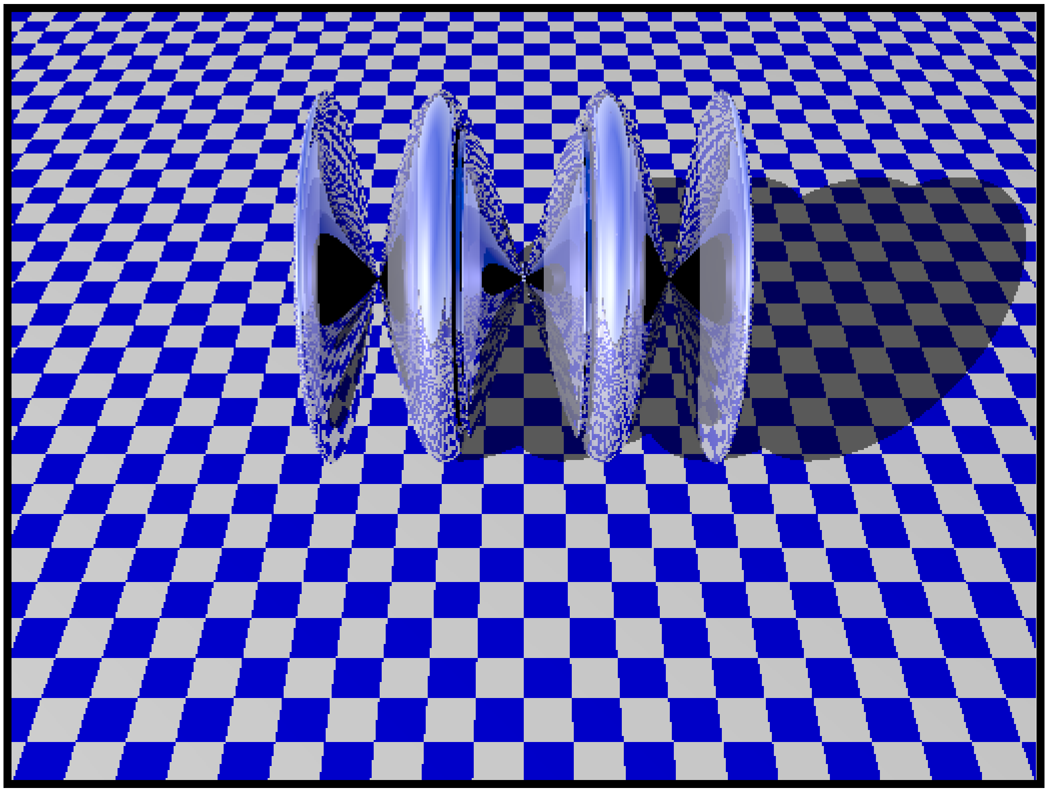


Figure 23-7: A Bowditch Curve Solid of Revolution

The Bowditch Curve with parameters (a, b, d, k) = (9, 8, 0, ¼) *has been rotated around the x-axis to form the solid of revolution seen in the picture above. The solid of revolution was then placed above the blue and white checkered plane and given a silvery reflective finish. Light sources have been placed so as to cast the shadows seen in the picture. The blue and white checkered plane can be seen reflected in the object's finish as well as adjacent lobes of the object itself.*

Appendix

A1. Inversion as a Geometric Construction

Throughout this text, various constructions call for the inversion of a point or possibly the inversion of an entire curve. In particular, the constructions in Chapter 16, "The Folium of Descartes," all call for inversion. Recall that the inversion of point C with respect to circle AB means that if point C' is the inverted point then $AB^2 = AC \cdot AC'$ and C' will lie on line AC. (AB represents the distance from point A to point B, AC represents the distance from point A to point C, and AC' denotes the distance from point A to point C'.) The method generally used in the text to perform any inversion called for is an algebraic methodology involving the above calculation and then the subsequent translation of the point to be inverted by that calculated distance along the line connecting points A and C. Here, we wish to show that inversion can be done by purely geometric means (i.e., with straight-edge and compass) and that the algebraic method used throughout the text is simply a shortcut for such a geometric method.

So, given circle AB (i.e., a circle with center at point A and passing through point B) and point C, a random point in the plane, this section shows that the inversion of point C with respect to circle AB is a straight-edge and compass construction.

1 Draw circle CA with center at point C and passing through point A.
2 Let D and E be the two intersections of circle AB and circle CA.
3 Draw circle EA with center at point E and passing through point A.
4 Draw circle DA with center at point D and passing through point A.
5 Let point C' be the unlabeled intersection of circles DA and EA.
6 Draw line AC'.

As one can see, the preceding six steps are obviously steps that can be accomplished with either a straight-edge (step 6) or a compass (steps 1 – 5). Now, in order to prove that point C' is the inverse of point C, first measure m_1, the distance from point A to point B, then measure m_2, the distance from point A to point C, and finally, measure m_3, the distance from point A to point C'. Make the following two calculations, $m_4 = m_1^2$ and $m_5 = m_2 \cdot m_3$. You will find that $m_4 = m_5$ and that by dragging point C, or point B, or both, all the values of the quantities change but m_4 will remain equal to m_5.

A2. Verification of Formulas in the Text

The Preface alluded to the fact that almost all the derived formulas in the text have been verified by a process that uses Geometer's Sketchpad (GSP). This section explains that verification process. All of the formulas presented with the exception of areas, arc lengths, and volumes have been verified by this method. It should be emphasized that this verification process does not prove that the formula is necessarily correct; however, the nature of the process is such that it gives a very high degree of confidence that the stated formula is correct. The verification process requires the curve, whose formulas are to be verified, to be represented in parametric form, i.e., $x = f(t)$ and $y = g(t)$.

The first step in the process is to represent the parameter t by some GSP variable such as a dynamic point or angle. For example, if the parameter t for the curve under consideration is defined as ranging over, say, the values $0 \leq t \leq 2\pi$ (many of the curves in the text are defined similarly), then let t be represented as the central angle of a circle where one of the non-vertex points comprising the angle can be dragged around the circle thereby varying the value of t. If, on the other hand, t is defined as ranging over the values $-\infty < t < +\infty$, then let t be represented by the y-coordinate of a variable point on a vertical line. By dragging the point up and down the line, the y-coordinate assumes the appropriate values. The next step in the process is to represent any parameters in $f(t)$ and $g(t)$ (other than t) as GSP constants. For example, if a and b are constants, the parametric representation of the curve may more accurately be denoted by $x = f(a, b, t)$ and $y = g(a, b, t)$. It is the a and b that we wish to denote as GSP constants. We accomplish this by letting a and b (and any other constants in the parametric representation of the curve) be represented as the x-coordinates of points on the x-axis.

Let us pause in our explanation of this verification process and illustrate what has been said so far with an example. The example we will use is the very first curve in the text, namely, the Cissoid of Diocles. Recall that the parametric representation for the Cissoid of Diocles was given as $x = 2a\sin^2 t$ and $y = 2a\tan t \cdot \sin^2 t$ for $-\pi/2 < t < \pi/2$, where a is the only constant. Here are the corresponding GSP steps.

1. Create x-y axes with point A as the origin and B as the unit point of the x-axis.
2. In the lower right corner of the screen, draw circle CD with center at point C and passing through point D.
3. Let E be a random point on the circumference of circle CD
4. Draw line segments CD and CE.
5. Measure $\angle$DCE in radians.
6. Relabel $\angle$DCE as t.

To perform step 6, use the text tool and double click on $\angle$DCE to obtain the Format Measurement window. Select Text Format and type the letter t to replace Angle[DCE] and click on OK. Having done this, note what happens when you drag point E around circle CD. The process of dragging point E around circle CD causes $\angle$DCE (and therefore t) to vary over the values $-\pi$ to $+\pi$ at least within the granularity of GSP, and of course, this range includes $-\pi/2 < t < \pi/2$, the interval specified for the curve in its parametric representation.

7. Let F be a random point on the positive x-axis.
8. Measure the coordinates of point F.
9. Extract the x-coordinate of point F and relabel it as a.

To perform step 9, open the Calculator and then click on the x-coordinate of point F. This will enter x_F into the Calculator. Now click on OK and x_F will be written to the screen. Use the text tool to double click on x_F to obtain the Format Measurement window. Select Text Format and type the letter a to replace x[F] and click on OK. We now have the parameter *a* represented and although we do not intend to vary it, its value can be changed by dragging it along the *x*-axis.

The next step is to do a GSP calculation of $f(t)$ and $g(t)$ and then to plot the results as *x* and *y*, respectively, that is plot (f, g). Then, trace the point (f, g) and animate the construction so that the variable point, the point representing the parameter *t*, assumes all of its allowable values.

10 Calculate $2 \cdot a \cdot \sin^2 t$.
11 Calculate $2 \cdot a \cdot \tan t \cdot \sin^2 t$.
12 Let point G be the result of plotting step 10 as *x* and step 11 as *y*.
13 Trace point G and change its color.
14 Animate point E around circle CD.

The resulting trace should be the curve under consideration and if it is, this is a pretty good indication that the parametric representation of the curve is correct. That is, we have verified the parametric representation; however, we are not done yet.

The next step is to attempt to verify the equation of the curve's tangent. In the text, the equation of the curve's tangent is given in the *y*-intercept form. That is, in the form $y = mx + c$, where *m* is the slope of the tangent line (and the slope of the curve at that point of tangency) and *c* is the point where the tangent line intersects the *y*-axis. We know all of these values except *c;* that is, we know the slope *m* as it is given as y', the first derivative of *y* with respect to *x*, and we know the values of *y* and *x*, which are simply $f(t)$ and $g(t)$. So, $c = g(t) - y' \cdot f(t)$ and by making this calculation for *c* and plotting the point $(0,c)$ we can construct a straight line between $(0,c)$ and the point that is tracing the curve (f, g), and that straight line should be the tangent and it should remain tangent as the animation is executed. If it does remain tangent as the animation executes, it is a pretty good indication that not only is the equation of the tangent correct, but so is the slope, i.e., the calculation of y'. Furthermore, if the slope is correct, this is a good indication that $\dot{y}$ and $\dot{x}$ are also both correct since $y' = \dot{y}/\dot{x}$. Hence we have verified the equation of the tangent, the formula denoting y', and the formulas denoting $\dot{y}$ and $\dot{x}$. So, continuing our Cissoid of Diocles example, we have from the text that the equation of the curve at the point *t* is $2 \cdot y = \tan t \cdot \sec^2 t\,(1 + 2\cos^2 t) \cdot x - 2a\tan^3 t$. Further, the equation of the slope is given by $y' = ½ \tan t \cdot \sec^2 t\,(1 + 2\cos^2 t)$. If we now calculate *c*, we find that $c = -a \cdot \tan^3 t$.

15 Calculate $c = -a \cdot \tan^3 t$.
16 Let point H be the result of plotting 0.000 as *x* and $-a \cdot \tan^3 t$ as *y*.
17 Draw line GH.
18 Make line GH thick and change its color.
19 Rerun the animation.

One way to obtain the quantity 0.000 is to simply open the Calculator, type in that value, and click OK.

The next step in our verification process is to verify the radial distance, that is, the distance between the point (f, g) and the origin. First, measure the coordinates of the point (f, g) and then take the square root of the sum of each coordinate squared (note that the measure of the coordinates is the same as the two values calculated in steps 10 and 11); that is, calculate $\sqrt{f^2+g^2}$. Then, make a second calculation, that is, calculate the formula given in the text as the radial distance, i.e., r. Finally, make a third calculation which is the result of the first calculation minus the absolute value of the second calculation, that is, $\sqrt{f^2+g^2}-|r|$. This quantity should be zero and remain zero as the animation is executed. If it does, the formula for the radial distance is verified. In our continuing example, these steps would be

20 Measure the coordinates of point G.
21 Calculate $(x_G^2 + y_G^2)^{½}$.
22 Calculate $2 \cdot a \sin t \cdot \tan t$.
23 Calculate $(x_G^2 + y_G^2)^{½} - |2 \cdot a \sin t \cdot \tan t|$.
24 Rerun the animation.

We're not done yet! The next step is to verify the distance from the origin to the curve's tangent. This process is very similar to that of verifying the radial distance, however, we must first construct the distance to the tangent; that is, construct a perpendicular to the curve's tangent through the origin and then locate the point where that perpendicular and the tangent intersect, then measure the coordinates of that intersection point and proceed as we did in the radial distance verification.

25 Construct perpendicular P_1 to line GH through point A.
26 Let point I be the intersection of perpendicular P_1 and line GH.
27 Measure the coordinates of point I.
28 Calculate $(x_I^2 + y_I^2)^{½}$.
29 Calculate $-2 \cdot a \cdot \sin^3 t / (1 + 3 \cdot \cos^2 t)^{½}$.
30 Calculate $(x_I^2 + y_I^2)^{½} - |-2 \cdot a \cdot \sin^3 t / (1 + 3 \cdot \cos^2 t)^{½}|$.
31 Rerun the animation.

We're not done yet! Next, let us verify the coordinates of the center of curvature, given in the text as α and β. This is a relatively simple process. We already have the tangent to the curve at point t. Construct the curve's normal at that point, calculate the values of α and β, and then plot α and β as x and y, respectively. If the point that is the result of plotting α and β lies on the normal to the curve and stays on the normal to the curve as the animation executes, then that is a pretty good indication that the formulas for the center of curvature are correct. Furthermore, verifying the formulas for the center of curvature also automatically verifies the formula for y' and y'' since neither α nor β could have been calculated correctly (in all probability) without a correct formula for y' or y'', i.e., recall that

$$\alpha = x - \frac{y'\left[1+(y')^2\right]}{y''} \text{ and } \beta = y + \frac{1+(y')^2}{y''}.$$

Continuing our example,

32 Construct perpendicular P_2 to line GH through point G.
33 Calculate $\alpha = -⅓\cdot a\cdot \sin^2 t\cdot \sec^4 t\cdot (1 + 5\cdot \cos^2 t)$.
34 Calculate $\beta = 8\cdot a\cdot \tan t / 3$.
35 Let point J be the result of plotting α and β as x and y, respectively.
36 Rerun the animation and observe whether point J remains on the normal.

We're not done yet! It is now a relatively simple matter to verify the formula for the radius of curvature. Recall that the radius of curvature, ρ, is the distance from the center of curvature to the curve. This distance is the square root of the sum of the difference between their respective coordinates squared, that is, $\sqrt{(\alpha - f)^2 + (\beta - g)^2}$. If we then compare this calculation with that of the formula in the text for the radius of curvature (as we did in the case of the radial distance and distance to the tangent), and find that the comparison calculation is zero as the animation executes, well *voila*, we have verified the radius of curvature. Further, since the radius of curvature is a function of $\dot{x}, \ddot{x}, \dot{y},$ and $\ddot{y}$ (i.e., the radius of curvature requires these quantities for its own calculation), we have also verified them. Continuing our example,

37 Measure the coordinates of point J.
38 Measure the coordinates of point G.
39 Calculate $[(x_J{}^2 - x_G{}^2)^2 + (y_J{}^2 - y_G{}^2)^2]^{1/2}$.
40 Calculate $\rho = ⅓\cdot a\cdot \sin t\cdot \sec^4 t\cdot (1 + 3\cdot \cos^2 t)^{3/2}$.
41 Calculate $[(x_J{}^2 - x_G{}^2)^2 + (y_J{}^2 - y_G{}^2)^2]^{1/2} - |\rho|$.
42 Rerun the animation.

We're not done yet! The angle between the tangent and the radius vector, the so-called tangential-radial angle, can also be verified. This is done by measuring that angle on the display, calculating its tangent, and comparing that calculation with the formula in the text. If their difference is zero and continues to be zero as the animation executes it's a pretty good indication that the formula in the text is correct. First, we must first find some way to designate the appropriate angle. We can do this by drawing a line between the origin and the point (f, g) and then placing a random point on this line on the opposite side of the curve's normal from the origin. Similarly, we must also place a random point on the curve's tangent but, again, on the opposite side of the curve's normal from the origin. The angle formed from these two random points with point (f, g) as the vertex is the desired angle. Therefore, continuing our example,

43 Draw line AG.
44 Let point K be a random point on line AG but on the opposite side of P_2 from point A.
45 Let point L be a random point on line GH but on the opposite side of P_2 from point A.
46 Measure ∠KGL in radians.
47 Relabel ∠KGL as psi.
48 Calculate tan(psi).
49 Calculate $\sin t\cdot \cos t / (1 + \cos^2 t)$.
50 Calculate $|\tan(\text{psi})| - |\sin t\cdot \cos t / (1 + \cos^2 t)|$.
51 Rerun the animation.

We're done! At least done with this example; one more example ought to make this verification technique pretty clear. We will use the Folium of Descartes as the second example. Recall that the parametric equations for the Folium of Descartes are

$$(x, y) = \frac{3at}{1+t^3}(1, t) \quad -\infty < t < +\infty .$$

Note that this parametric equation is defined over the interval $-\infty$ to $+\infty$ whereas in the previous example the equations were defined over the interval $-\pi/2$ to $+\pi/2$. This makes a big difference in the way we define the variable *t*. We cannot define *t* as the central angle of a circle (as was done in the previous example) because as the angle varies it will not assume all of the values necessary to trace the Folium of Descartes. We therefore define it as the *y*-coordinate of a variable point on a vertical line. As the point is dragged up and/or down the line, the *y*-coordinate, and therefore *t*, will assume the required values. This alternate approach to defining the parameter *t* is the only difference in the verification technique between the previous example and this example.

----- Verifying the parametric equations -----

1 Create axes with point A as the origin and B as the unit point of the *x*-axis.
2 In the lower right, draw circle CD with center at point C and passing through point D.
3 Let E be a random point on the circumference of circle CD.
4 Let F be a random point on the *x*-axis.
5 Construct perpendicular P_1 to the *x*-axis through point F.
6 Draw line CE.
7 Let point G be the intersection of line CE and perpendicular P_1.
8 Measure the coordinates of point G.
9 Extract the *y*-coordinate of point G and relabel it as *t*.
10 Let H be a random point on the positive *x*-axis.
11 Extract the *x*-coordinate of point H and relabel it as *a*.
12 Calculate $x = 3at / (1 + t^3)$.
13 Calculate $y = 3at^2 / (1 + t^3)$.
14 Let point I be the result of plotting (x, y).
15 Trace point I and change its color.
16 Animate point E around circle CD.

----- Verifying the equation of the tangent, $y', \dot{x}$, and $\dot{y}$ -----

17 Calculate $c = -3at^2 / (1 - 2t^3)$.
18 Let point J be the result of plotting $(0, c)$.
19 Draw line IJ.
20 Make line IJ thick and change its color.
21 Rerun the animation and observe the motion of line IJ as a tangent.

----- Verifying the formula for the radial distance -----

22 Measure the coordinates of point I.
23 Calculate $(x_I^2 + y_I^2)^{1/2}$.
24 Calculate $r = 3at\,(1 + t^2)^{1/2} / (1 + t^3)$.
25 Calculate $(x_I^2 + y_I^2)^{1/2} - |r|$.
26 Rerun the animation and observe that result of step 25 remains zero.

----- Verifying the distance to the curve's tangent -----

27 Construct perpendicular P_2 to line IJ through point A.
28 Let point K be the intersection of perpendicular P_2 and line IJ.
29 Measure the coordinates of point K.
30 Calculate $(x_K^2 + y_K^2)^{1/2}$.

31 Calculate $p = -3at^2 / (t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1)^{1/2}$.
32 Calculate $(x_K^2 + y_K^2)^{1/2} - |p|$.
33 Rerun the animation and observe that result of step 32 remains zero.

----- Verifying the coordinates of the center of curvature and y''-----

34 Construct perpendicular P_3 to line IJ through point I.
35 Calculate $\alpha = -3at^3 (8 - 15t - 12t^3 + 6t^4 + 6t^6 - 6t^7 - t^9) / [2(1 + t^3)^4]$.
36 Calculate $\beta = 3a (1 + 6t^2 - 6t^3 - 6t^5 + 12t^6 + 15t^8 - 8t^9) / [2(1 + t^3)^4]$.
37 Let point L be the result of plotting (α, β).
38 Rerun the animation and observe that point L is on P_3 and remains on P_3.

----- Verifying the radius of curvature, $\ddot{x}$, and $\ddot{y}$ -----

39 Measure the coordinates of point L.
40 Measure the coordinates of point I.
41 Calculate $[(x_L - x_I)^2 + (y_L - y_I)^2]^{1/2}$.
42 Calculate $\rho = 3a (1 + 4t^2 - 4t^3 - 4t^5 + 4t^6 + t^8)^{3/2} / [2(1 + t^3)^4]$.
43 Calculate $[(x_L - x_I)^2 + (y_L - y_I)^2]^{1/2} - |\rho|$.
44 Rerun the animation and observe that result of step 43 remains zero.

----- Verifying the tangential-radial angle -----

45 Let point M be a random point on line IJ but on the opposite side of P_3 from point A.
46 Draw line AI.
47 Let point N be a random point on line AI but on the opposite side of P_3 from point A.
48 Measure $\angle$MIN in radians.
49 Relabel $\angle$MIN as psi.
50 Calculate tan(psi).
51 Calculate $t(1 + t^3) / (1 + 2t^2 - 2t^3 - t^5)$.
52 Calculate $|\tan(\text{psi})| - |t(1 + t^3) / (1 + 2t^2 - 2t^3 - t^5)|$.
53 Rerun the animation and observe that result of step 52 remains zero.